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THE COMPONENT SIZES OF A CRITICAL RANDOM GRAPH WITH GIVEN DEGREE SEQUENCE

BY ADRIEN JOSEPH

Université Pierre et Marie Curie

Consider a critical random multigraph $\mathcal{G}_n$ with n vertices constructed by the configuration model such that its vertex degrees are independent random variables with the same distribution ν (criticality means that the second moment of ν is finite and equals twice its first moment). We specify the scaling limits of the ordered sequence of component sizes of $\mathcal{G}_n$ as n tends to infinity in different cases. When ν has finite third moment, the components sizes rescaled by $n^{-2/3}$ converge to the excursion lengths of a Brownian motion with parabolic drift above past minima, whereas when ν is a power law distribution with exponent $\gamma \in (3, 4)$, the components sizes rescaled by $n^{-(\gamma-2)/(\gamma-1)}$ converge to the excursion lengths of a certain nontrivial drifted process with independent increments above past minima. We deduce the asymptotic behavior of the component sizes of a critical random simple graph when ν has finite third moment.

1. Introduction.

1.1. *Overview.* The classical random graph model $G(n, p)$ has received a lot of attention since its introduction by Erdős and Rényi [12], especially because of the existence of a phase transition. In this model, a graph on n labeled vertices is constructed randomly by joining any pair of vertices by an edge with probability p, independently of the other pairs. For large n, the structure of this random graph depends on the value of np: for $p \sim c/n$ with $c < 1$, the largest connected component contains $O(\ln n)$ vertices, whereas when $p \sim c/n$ with $c > 1$, the largest component has $\Theta(n)$ vertices while the second largest component has $O(\ln n)$ vertices. The cases $c < 1$ and $c > 1$ are called subcritical and supercritical, respectively. Much attention has been devoted to the critical case $p \sim 1/n$. When p is exactly equal to $1/n$, the largest components of $G(n, p)$ have sizes of order $n^{2/3}$.

Molloy and Reed [20] showed that a random graph with a given degree sequence exhibits a similar phase transition. More precisely, for each $n \geq 1$, let $\mathbf{d}^{(n)} = (d_i^{(n)})_{1\leq i\leq n}$ be a nonincreasing sequence of positive integers such that $\sum_{i=1}^n d_i^{(n)}$ is even. Let $G(n, \mathbf{d}^{(n)})$ be a random simple graph on n labeled vertices with degree sequence $\mathbf{d}^{(n)}$, uniformly chosen among all possibilities (tacitly

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assuming that there exists any such graph). We suppose throughout the overview that there exists a probability distribution $(\nu_k)_{k\geq 1}$ such that for each k, $\#\{i : d_i^{(n)} = k\}/n \to \nu_k$ as $n \to \infty$. Let $\omega(n) = d_1^{(n)}$ be the largest degree in the graph. Under some further strong conditions on the sequences $\mathbf{d}^{(n)}$, Molloy and Reed proved that if $Q = \sum_{k=1}^{\infty} k(k-2)\nu_k < 0$ and $\omega(n) \leq n^{1/8-\varepsilon}$ for some $\varepsilon > 0$, then with probability tending to 1, the size of the largest component of $G(n, \mathbf{d}^{(n)})$ is $O(\omega^2(n) \ln n)$, whereas if $Q > 0$ and $\omega(n) \leq n^{1/4-\varepsilon}$ for some $\varepsilon > 0$, then with probability tending to 1, the size of the largest component is $\Theta(n)$, and if additionally Q is finite, the size of the second largest component is $O(\ln n)$.

More recently, the near-critical behavior of such graphs has been studied. When $Q = 0$, the structure of $G(n, \mathbf{d}^{(n)})$ depends on how fast the quantity

$$\alpha_n = \sum_{k=1}^{\infty} k(k-2) \frac{\#\{i : d_i^{(n)} = k\}}{n} = \sum_{i=1}^{n} \frac{d_i^{(n)}(d_i^{(n)} - 2)}{n}$$

converges to 0; see Kang and Seierstad [19]. Requiring a fourth moment condition, Janson and Luczak [18] proved that if $n^{1/3}\alpha_n \to \infty$, then the size of the largest component of $G(n, \mathbf{d}^{(n)})$ divided by $n\alpha_n$ converges in probability to $\frac{2\mu}{\beta}$, while the size of the second largest component of $G(n, \mathbf{d}^{(n)})$ divided by $n\alpha_n$ converges in probability to 0, where $\mu = \sum_{k=1}^{\infty} k\nu_k$ and $\beta = \sum_{k=3}^{\infty} k(k-1)(k-2)\nu_k \in (0, \infty)$. Furthermore, they noticed that their results can also be applied to some other random graph models by conditioning on the vertex degrees, provided that the random graph conditioned on the degree sequence has a uniform distribution over all possibilities. This is the case for $G(n, p)$ with $np \to 1$ and $n^{1/3}(np - 1) \to \infty$. Note that if $n^{1/3}(np - 1) = O(1)$, it is well known that the largest component and the second largest component both have sizes of the same order $n^{2/3}$, so that their results do not hold.

A major difficulty when dealing with the natural random graph $G(n, \mathbf{d}^{(n)})$ is that, despite its straightforward definition, it cannot be constructed via an easy algorithm. To circumvent that obstacle, it is convenient to work with *multigraphs*, in which multiple edges and loops are allowed, using the explicit procedure provided by the *configuration model*, which was introduced by Bender and Canfield [4] and later studied by Bollobás [9] and Wormald [25]. See also Molloy and Reed [20, 21], Kang and Seierstad [19], Bertoin and Sidoravicius [6], van der Hofstad [24] and Hatami and Molloy [14]. Specifically, take a set of $d_i^{(n)}$ half-edges for the vertex with label i, $i \in \{1, \ldots, n\}$, and combine the half-edges into pairs by a uniformly random matching of the set of all half-edges. Observing that every simple graph $G(n, \mathbf{d}^{(n)})$ may be constructed through the same number, $d_1^{(n)}! \cdots d_n^{(n)}!$, of pairing of half-edges, we get that conditional on being a (simple) graph, the multigraph obtained by the configuration model has the same distribution as $G(n, \mathbf{d}^{(n)})$. That is why we shall first deal with multigraphs. We shall then see how to derive results for simple graphs.

1.2. *The present model.* The present work is devoted to studying $G(n, \mathbf{d}^{(n)})$ for a family of degree sequences that are, in a certain sense, "inside the critical window." We suppose that we are given a probability distribution $\nu = (\nu_k)_{k \geq 1}$ with finite second moment such that $\nu_2 < 1$ and $\sum_{k=1}^{\infty} k(k-2)\nu_k = 0$. Let D be a random variable with distribution ν. The multigraph $\mathcal{G}_n$ consisting of n vertices is defined by the configuration model as follows. Let $D_1, D_2, \dots, D_n$ be n independent copies of D. Condition on $\sum_{i=1}^{n} D_i$ being even. Take a set of D_i half-edges for each vertex, and combine the half-edges into pairs by a uniformly random matching of the set of all half-edges. We denote by $\mathcal{G}_n$ the random multigraph this construction leads to.

Let $\mathcal{C}_n^{\nu}$ be the ordered sequence of component sizes of $\mathcal{G}_n$. We aim at specifying the asymptotics of $\mathcal{C}_n^{\nu}$ in two different settings. First, we shall study the case when ν has finite third moment. We shall prove that $n^{-2/3}\mathcal{C}_n^{\nu}$ then converges in distribution (with respect to a certain topology that will be detailed below) as $n \to \infty$ to the ordered sequence of the excursion lengths of a Brownian motion with parabolic drift; see Theorem 2.1 below for the precise statement. This should be viewed as an extension of Aldous's well-known result for the critical behavior of Erdős–Renyi random graphs; see [1]. Next the case when ν is a power law distribution with exponent $\gamma \in (3, 4)$ will be studied. We shall show that $n^{-(\gamma-2)/(\gamma-1)}\mathcal{C}_n^{\nu}$ converges in distribution as $n \to \infty$ to the ordered sequence of the excursion lengths of a certain nontrivial drifted process with independent increments; see Theorem 8.3 below.

Similar results have already been obtained for different random graph models. For example, Turova [23] and Bhamidi, van der Hofstad and van Leeuwaarden [7, 8] studied special cases of rank-1 inhomogeneous random graphs constructed as follows. Let F be a distribution function on $[0, \infty)$ and $w_1, w_2, \dots, w_n$ be defined by $w_i = [1-F]^{-1}(i/n)$. Consider a simple graph on n labeled vertices such that an edge joins the vertices i and j $(i \neq j)$ with probability $1 - \exp(-w_i w_j / l_n)$, where $l_n = \sum_{i=1}^{n} w_i$, different edges being independent. Denoting by W a r.v. with distribution function F, suppose that $\mathbb{E}[W^2] < \infty$. The criticality of the model occurs when $\mathbb{E}[W^2] = \mathbb{E}[W]$. As in the present work, two different settings have been considered. In the case $\mathbb{E}[W^3] < \infty$, Turova [23] and Bhamidi, van der Hofstad and van Leeuwaarden [7] separately showed that the ordered sequence of component sizes of the inhomogeneous random graph with n vertices once rescaled by $n^{-2/3}$ converges in distribution as $n \to \infty$ to the ordered sequence of the excursion lengths of a Brownian motion with parabolic drift, thus extending the results of Aldous [1]. As for the power law distribution case, Bhamidi, van der Hofstad and van Leeuwaarden [8] proved that if there exist $\gamma \in (3, 4)$ and $c > 0$ such that $1 - F(x) \sim_{x \to \infty} cx^{1-\gamma}$, the ordered sequence of component sizes of the inhomogeneous random graph with n vertices once rescaled by $n^{-(\gamma-2)/(\gamma-1)}$ then converges in distribution as $n \to \infty$ to hitting times of a thinned Lévy process. This convergence is related to certain cases of the results obtained by Aldous and Limic in [2].

We too shall be interested in random simple graphs. Specifically, let $\mathcal{SG}_n$ be the random simple graph consisting of n vertices such that, conditionally on the degree sequence $(D_1, \dots, D_n)$, it is uniformly distributed over all simple graphs with this degree sequence. Denoting by $\boldsymbol{\mathcal{D}}^{(n)}$ the ordered sequence of $(D_1, \dots, D_n)$, $\mathcal{SG}_n$ has the same distribution as $G(n, \boldsymbol{\mathcal{D}}^{(n)})$. The random simple graph $\mathcal{SG}_n$ may also be viewed as the multigraph $\mathcal{G}_n$ conditioned to be simple. When ν has finite third moment, we shall be able to prove that the ordered sequence $\boldsymbol{\mathcal{SC}}_n^\nu$ of component sizes of the graph $\mathcal{SG}_n$ has the same asymptotic behavior as $\boldsymbol{\mathcal{C}}_n^\nu$; see Theorem 2.2 below. We refer to Britton, Deijfen and Martin-Löf [11] for an understanding of the link between inhomogeneous random graphs and $\mathcal{SG}_n$.

The paper is organized as follows. In Sections 2, 3, 4, 5, 6 and 7, we deal with the finite third moment case. Apart from Section 7, the main techniques developed there are used in Section 8, where the power law distribution case is studied. Section 7, devoted to $\boldsymbol{\mathcal{SC}}_n^\nu$, is specific to the finite third moment case. The main results will be stated in Section 2. In Section 3, following the ideas of Aldous [1], we shall observe that the study may be reduced to the understanding of a walk defined via an algorithmic procedure related to the configuration model. Thanks to [1], convergence of that walk turns out to be sufficient. Such convergence will be obtained in Section 5 using standard methodology from stochastic process theory; see, for example, the CLT for continuous-time martingale. A key technique to obtain martingales is Poissonization. Basically, instead of considering multigraphs with exactly n vertices, we shall deal with multigraphs with Poisson(n) vertices. This will be fully explained in Section 4. Our approach also relies on size-biased ordering. Finally, in Section 6, we shall be interested in the number of cycles in the multigraph $\mathcal{G}_n$. To conclude, in Section 8, we shall study $\boldsymbol{\mathcal{C}}_n^\nu$ when ν is a power law distribution with exponent in $(3, 4)$. We shall follow the same strategy, except we shall apply results of Aldous and Limic [2]. The final Appendix puts together technical lemmas.

2. Formulation of the main results in the finite third moment setting.

In the first sections of the paper, we suppose that ν satisfies

$$\sum_{k=1}^{\infty} k(k-2)\nu_k = 0, \qquad \sum_{k=1}^{\infty} k^3 \nu_k < \infty \text{ and } \nu_2 < 1. \tag{2.1}$$

The more general power law distribution case will be studied in Section 8.

Let

$$\mu = \sum_{k=1}^{\infty} k\nu_k \quad \text{and} \quad \beta = \sum_{k=3}^{\infty} k(k-1)(k-2)\nu_k.$$

Observe that $\beta > 0$. Define the Brownian motion with parabolic drift

$$W^\nu(t) = \sqrt{\frac{\beta}{\mu}} W(t) - \frac{\beta}{2\mu^2} t^2, \qquad t \geq 0,$$

where $(W(t), t \geq 0)$ is a standard Brownian motion. The reflected process indexed by the nonnegative half-line is

$$R^\nu(t) = W^\nu(t) - \min_{0 \leq s \leq t} W^\nu(s), \qquad t \geq 0.$$

An interval $\gamma = [l(\gamma), r(\gamma)]$ is an *excursion interval* of R^ν if $R^\nu(l(\gamma)) = R^\nu(r(\gamma)) = 0$ and $R^\nu(t) > 0$ on $l(\gamma) < t < r(\gamma)$. The excursion has length $|\gamma| = r(\gamma) - l(\gamma)$. Aldous observed in [1] that we can a.s. order excursions by length, that is, the set of excursions of R^ν may be written $\{\gamma_j, j \geq 1\}$ so that the lengths $|\gamma_j|$ are decreasing. In the notation of [1], define $l^2_\searrow$ as the set of infinite sequences $x = (x_1, x_2, \ldots)$ with $x_1 \geq x_2 \geq \cdots \geq 0$ and $\sum_i x_i^2 < \infty$, endowed with the Euclidean metric. Aldous showed in [1], Lemma 25, that $\mathbb{E}[\sum_{j \geq 1} |\gamma_j|^2] < \infty$. In particular $(|\gamma_j|, j \geq 1)$ a.s. belongs to $l^2_\searrow$. On the other hand, we may regard the finite sequence $\mathcal{C}_n^\nu$ as a random element of $l^2_\searrow$ by appending zero entries.

Our main result describes the component sizes of $\mathcal{G}_n$ for large n; it mirrors that of Aldous [1] for the critical random graph.

THEOREM 2.1. *Suppose ν satisfies (2.1). Let $\mathcal{C}_n^\nu$ be the ordered sequence of component sizes of $\mathcal{G}_n$. Then*

$$n^{-2/3} \mathcal{C}_n^\nu \xrightarrow[n \to \infty]{(d)} (|\gamma_j|, j \geq 1)$$

with respect to the $l^2_\searrow$ topology.

We shall observe that Theorem 2.1 is a direct corollary of a simpler result, namely Theorem 3.1; see the remark after its statement.

REMARK 2.1. Suppose $\nu_2 = 1$, that is, $D \equiv 2$. Then the components of $\mathcal{G}_n$ are cycles. It is well known that the distribution of cycle lengths is given by the Ewens's sampling formula ESF(1/2), and thus the size of the largest component divided by n converges in distribution to a nondegenerate distribution on $[0, 1]$; see [3], Lemma 5.7. This is also the case for the kth largest component, where k is a fixed positive integer. That is why the assumption $\nu_2 < 1$ made in (2.1) is crucial.

Note that in our setting,

$$\liminf_{n \to \infty} \mathbb{P}(\mathcal{G}_n \text{ is a simple graph}) > 0; \tag{2.2}$$

see Bollobás [10], Janson [17]. Recall that $\mathcal{SG}_n$ is the random simple graph such that, conditioned on the degree sequence $(D_1, \ldots, D_n)$, it is uniformly distributed over all graphs with this degree sequence. But it is also the multigraph $\mathcal{G}_n$ conditioned on being simple. That is why authors usually first focus on $\mathcal{G}_n$ to then deduce results for $\mathcal{SG}_n$ using (2.2); see, for instance, Pittel [22], Janson [16], Janson and Luczak [18]. In the finite third moment setting, we shall be able to set up this strategy; we shall prove an analogous result of Theorem 2.1:

THEOREM 2.2. *Suppose ν satisfies* (2.1). *Let $\boldsymbol{\mathcal{SC}}_n^\nu$ be the ordered sequence of component sizes of $\mathcal{SG}_n$. Then*

$$n^{-2/3}\boldsymbol{\mathcal{SC}}_n^\nu \xrightarrow[n\to\infty]{(d)} (|\gamma_j|, j \geq 1)$$

with respect to the $l^2_\searrow$ topology.

As before, we shall derive Theorem 2.2 from a simpler result stated in Theorem 3.2.

REMARK 2.2. Consider the case when ν is the Poisson distribution with parameter 1 [observe though that $\mathbb{P}(D = 0) > 0$, so strictly speaking, it is out of our setting, but our result still holds as vertices with degree 0 play no role]. Then, for large integers n, $\mathcal{SG}_n$ is an approximation of the Erdős–Renyi random graph $G(n, 1/n)$. Now, in that case, $\mu = \beta = 1$, so the process W^ν is the Brownian motion with drift $-t$ at time t, which also describes the asymptotic component sizes of $G(n, 1/n)$; see [1].

3. The depth-first search.

3.1. *An algorithmic construction of $\mathcal{G}_n$.* We start by describing a convenient algorithm to construct a multigraph distributed as $\mathcal{G}_n$. Suppose that $\sum_{i=1}^n D_i$ is even. We partition the set of half-edges into three subsets: the set $\mathcal{S}$ of sleeping half-edges, the set $\mathcal{A}$ of active half-edges and the set $\mathcal{D}$ of dead half-edges. $\mathcal{S} \cup \mathcal{A}$ is the set of living half-edges. Initially, all the half-edges are sleeping.

Pick a sleeping half-edge uniformly at random, and let v_1 denote the vertex it is attached to. Declare all the half-edges attached to v_1 active. While $\mathcal{A} \neq \varnothing$, proceed as follows:

- Let i be the largest integer k such that there exists an active half-edge attached to v_k.
- Consider an active half-edge l attached to v_i.
- Kill l, that is, remove it from $\mathcal{A}$, and place it into $\mathcal{D}$.
- Choose uniformly at random a living half-edge r and pair l to it.
- If r is sleeping, let v_{j+1} denote the vertex it is attached to, where j is the number of vertices which were found before the discovery of the vertex attached to r. Then declare all the half-edges attached to v_{j+1} except r active.
- Kill r.

Iterate until $\mathcal{A} = \varnothing$. At that step, the first component has been totally explored. If $\mathcal{S} \neq \varnothing$, proceed similarly with the remaining living vertices until all the half-edges have been killed. Then consider the multigraph with vertex set $\{v_i, 1 \leq i \leq n\}$ such that for all $1 \leq i, j \leq n$, the vertex v_i is joined by k edges to the vertex v_j if and only if k half-edges of v_i have been paired to k other half-edges of v_j during the

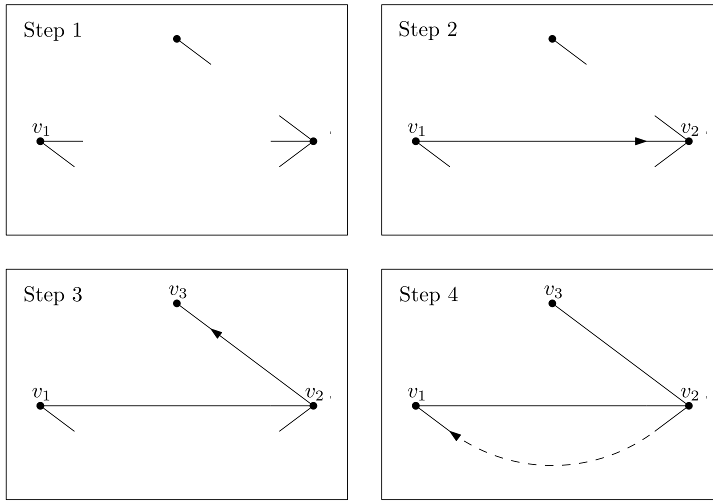

FIG. 1. *A realization of the algorithm constructing $\mathcal{G}_3$. The dashed oriented edge of the last picture contains a cycle half-edge at its origin*: v_2 *has a cycle half-edge. By definition*, $W_3(0) = 0$, $W_3(1) = 0$, $W_3(2) = -1$ *and* $W_3(3) = -2$.

procedure. It is easily seen this multigraph is distributed as $\mathcal{G}_n$ and its vertices have been ordered via a depth-first search. See Figure 1 above for a simple illustration.

Also note that, by construction, the order in which the components appear in the depth-first search is size-biased order.

3.2. *The depth-first walk.* We now explain how the information on the component sizes may be encoded in a walk constructed via the depth-first search which, as we shall see, is related to the process W^{ν}. We first need the notion of *cycle half-edge*.

DEFINITION 3.1. A half-edge l is called a *cycle half-edge* if there exists a half-edge r such that:

- l was killed before r;
- l was paired to r;
- r was active when l was paired to it.

Let us now define the walk associated to the depth-first search which will encode all the information that we need to study the component sizes. Write $(\widehat{D}_i, i \in \{1, 2, \ldots, n\})$ the sequence of the degrees of the vertices of $\mathcal{G}_n$ ordered by their appearances in the depth-first search: for every $i \in \{1, \ldots, n\}$,

$$\widehat{D}_i = \text{degree of } v_i.$$

Define the depth-first walk $(W_n(i), 0 \le i \le n)$ by letting for all $i \in \{0, \dots, n\}$,

$$W_n(i) = \sum_{j=1}^{i} \big(\widehat{D}_j - 2 - 2\#\{\text{cycle half-edges attached to } v_j\}\big). \tag{3.1}$$

Note that since the cycle half-edges attached to v_j always appear after v_j has been discovered, the number of them is not measurable with respect to the first j steps of the process.

Order the components $\mathcal{C}(n,1), \mathcal{C}(n,2), \dots$ according to the depth-first search. Let

$$\zeta(n,k) = \sum_{j=1}^{k} \big|\mathcal{C}(n,j)\big|,$$

$$\zeta^{-1}(n,i) = \min\{k : \zeta(n,k) \ge i\},$$

so that $\zeta^{-1}(n,i)$ is the index of the component containing v_i. It is easily seen that

$$W_n\big(\zeta(n,k)\big) = -2k \quad \text{and} \quad W_n(i) \ge -2k - 1 \qquad \text{for all } \zeta(n,k) \le i < \zeta(n,k+1). \tag{3.2}$$

It follows that we can recover component sizes and indices from the walk via

$$\zeta(n,k) = \min\{i : W_n(i) = -2k\},$$

$$\big|\mathcal{C}(n,j)\big| = \zeta(n,j) - \zeta(n,j-1),$$

$$\zeta^{-1}(n,i) = 1 - \left\lceil \min_{j<i} \frac{W_n(j)}{2} \right\rceil.$$

3.3. *Weak convergence on every finite interval of the depth-first walk.* Let X_n, $n \ge 1$, and X be $\mathbb{R}$-valued Càdlàg processes defined on $[0,\infty)$. For every $t > 0$, denote by $\mathbb{D}([0,t])$ the space of all $\mathbb{R}$-valued càdlàg functions defined on $[0,t]$ endowed with the Skorokhod topology. Throughout this work, we say that X_n converges in distribution to X with respect to the Skorokhod topology on every finite interval as $n \to \infty$ if for every $t > 0$ and every bounded, continuous function f defined on $(\mathbb{D}([0,t]), \mathbb{R})$,

$$\mathbb{E}\big[f(X_n)\big] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\big[f(X)\big]$$

(here, we write X_n and X for their restrictions to the interval $[0,t]$).

Our main result relates the walk to the process W^ν:

THEOREM 3.1. *Suppose ν satisfies* (2.1). *Rescale the depth-first walk W_n by defining for every $t \in [0, n^{1/3}]$*

$$\overline{W}_n(t) = n^{-1/3} W_n\big(\lfloor tn^{2/3} \rfloor\big).$$

Then

$$\overline{W}_n \xrightarrow[n\to\infty]{(d)} W^\nu$$

with respect to the Skorokhod topology on every finite interval.

To see how Theorem 2.1 follows from Theorem 3.1, we refer to Section 3.4 of the remarkable paper [1] of Aldous.[1] Intuitively, the result should be clear from property (3.2) of depth-first walk. Component sizes are indeed encoded as lengths of path segments above past even minima; these converge to lengths of excursions of W^ν above past minima, which are just lengths of excursions of the reflected process $(W^\nu(t) - \min_{0\le s\le t} W^\nu(s), t \ge 0)$ above 0. Similarly, Theorem 2.2 is proven as soon as the following result is shown:

THEOREM 3.2. *If ν satisfies (2.1), then the rescaled walk $\overline{W}_n$ conditioned on the event $\{\mathcal{G}_n$ is simple$\}$ converges in distribution to W^ν with respect to the Skorokhod topology on every finite interval as $n \to \infty$.*

The next three sections are devoted to the proof of Theorem 3.1. Section 4 will introduce the method. In Section 5, we shall be interested in the depth-first walk $(\sum_{j=1}^{i}(\widehat{D}_j - 2), 0 \le i \le n)$. It is easier to study the latter than the walk W_n since it ignores cycle half-edges, and its law only depends on the sequence $(\widehat{D}_j, 1 \le j \le n)$, which has the law of the size-biased ordering of n independent copies of D. Let

$$\bar{s}_n(t) = n^{-1/3} \sum_{1\le j\le tn^{2/3}} (\widehat{D}_j - 2), \qquad t \in [0, n^{1/3}].$$

We shall show that the walk $\bar{s}_n$ converges in distribution to W^ν as $n \to \infty$. In Section 6, we shall see that the difference between the two rescaled depth-first walks $\overline{W}_n$ and $\bar{s}_n$ is so small that in the limit, these processes have the same behavior. The combination of the two remarks yields Theorem 3.1. As for Theorem 3.2, it will be proved in Section 7.

4. Poissonization. As mentioned above, in this section, we forget the contribution of the cycle half-edges to the depth-first walk W_n (we shall see in Section 6 that there are indeed few cycle half-edges up to time $tn^{2/3}$ for every fixed $t > 0$), and we only focus on the simpler walk $(\sum_{j=1}^{i}(\widehat{D}_j - 2), 0 \le i \le n)$.

It is easily seen that the configuration model defining $\mathcal{G}_n$ induces a degree-biased ordering of its vertices: conditionally on the degrees $D_1, \dots, D_n$, the sequence $(\widehat{D}_1, \dots, \widehat{D}_n)$ has the law of a size-biased reordering of the real numbers $D_1, \dots, D_n$. Conditionally on $D_1 = d_1, \dots, D_n = d_n$, a convenient way to order

[1]Recall that components appeared in size-biased order in the depth-first walk.

the vertices of $\mathcal{G}_n$ in a degree-biased fashion is to assign an exponential clock with parameter d_i to the vertex i, $i \in \{1, 2, \dots, n\}$, and to order the vertices according to the times the clocks they are attached to ring.

4.1. *Heuristics.* We are able to sample at the same time both the degrees of the vertices of $\mathcal{G}_n$ and their reordering in a size-biased way via a clever Point point process. The only drawback of this approach is that the total number of vertices of the obtained multigraph is not exactly n but is a Poisson variable with parameter n (so that to actually obtain $\mathcal{G}_n$, one has to condition the total number of vertices to be equal to n). Let us be more precise.

Consider a Poisson point process $\Pi_n^{(0)}$ on $\mathbb{N}^* = \{1, 2, \dots\}$ with parameter $n\nu$. The total number of its atoms is a Poisson variable with parameter n, and conditionally on this number, the atoms of $\Pi_n^{(0)}$ are i.i.d. with distribution ν. Assigning to each of them an exponential clock with appropriate parameter would order them in a size-biased fashion.

We could have done those two operations directly by defining more carefully the Poisson point process; indeed define $\Pi_n^{(1)}$ as a Poisson point process on $(0, \infty) \times \mathbb{N}^*$ with intensity $\pi_n^{(1)}$, where

$$\pi_n^{(1)}(\mathrm{d}t, k) = n\nu_k k e^{-kt}\,\mathrm{d}t.$$

Sort the atoms of $\Pi_n^{(1)}$ in increasing order of their t-components:

$$\Pi_n^{(1)} = \{(t_1^{(1)}, k_1^{(1)}), \dots, (t_{N^{(1)}}^{(1)}, k_{N^{(1)}}^{(1)})\}$$

(we drop the dependency on n in the notations of $t_i^{(1)}$, $k_i^{(1)}$ and $N^{(1)}$). Then

$$\{k_1^{(1)}, \dots, k_{N^{(1)}}^{(1)}\} \stackrel{(d)}{=} \Pi_n^{(0)}.$$

Moreover, since $t_i^{(1)}$ corresponds to the exponential clock of $k_i^{(1)}$, the sequence $(k_1^{(1)}, \dots, k_{N^{(1)}}^{(1)})$ has the law as the size-biased reordering of the real numbers $k_1^{(1)}, \dots, k_{N^{(1)}}^{(1)}$. Consequently, conditionally on $N^{(1)} = m$, $(k_1^{(1)}, \dots, k_m^{(1)})$ has the same distribution as the random vector $(\widehat{D}_1, \dots, \widehat{D}_m)$.

As mentioned in the introduction of this section, we are interested in the walk $(\sum_{j=1}^{i}(\widehat{D}_j - 2), 0 \le i \le n)$, which may be viewed as a function having discontinuities at integer-valued times. We see that we cannot reasonably approximate this function by $(\sum_{(s,k)\in\Pi_n^{(1)}}(k-2)\mathbf{1}_{s\le t}, t \ge 0)$, which has discontinuities at $t_1^{(1)}, \dots, t_{N^{(1)}}^{(1)}$; the sequence $(t_1^{(1)}, t_2^{(1)} - t_1^{(1)}, \dots, t_{N^{(1)}}^{(1)} - t_{N^{(1)}-1}^{(1)})$ has no chance to look like $(1, \dots, 1)$, partly because for every $i \in \{1, \dots, N^{(1)} - 1\}$, $t_i^{(1)} - t_{i-1}^{(1)}$ is stochastically dominated by $t_{i+1}^{(1)} - t_i^{(1)}$ (with convention $t_0^{(1)} = 0$). We should thus transform the t-components of the atoms: an atom (t, k) should be replaced by $(\phi_n(t), k)$, where ϕ_n is an concave increasing function, so that conditionally on

$N^{(1)}$, $\phi_n(t_1^{(1)}), \phi_n(t_2^{(1)}) - \phi_n(t_1^{(1)}), \dots, \phi_n(t_{N^{(1)}}^{(1)}) - \phi_n(t_{N^{(1)}-1}^{(1)})$ are i.i.d. and close to 1. We shall show in the next section that there exists such a function ϕ_n and that, conditionally on $N^{(1)} \geq i$, $\phi_n(t_i^{(1)}) - \phi_n(t_{i-1}^{(1)})$ is an exponential variable with parameter 1; see Lemma 4.1 below.

4.2. *Toward the definition of* Π_n. It turns out that the function ϕ_n is $n(1-\mathcal{L})$, where $\mathcal{L}$ is the Laplace transform of ν,

$$\mathcal{L}(t) = \sum_{k \in \mathbb{N}^*} e^{-kt} \nu_k, \qquad t \geq 0.$$

Indeed, write ψ for the inverse of $1 - \mathcal{L}$ and consider a Poisson point process Π_n on $(0, n) \times \mathbb{N}^*$ with intensity π_n, where

$$\pi_n(\mathrm{d}t, k) = \nu_k k e^{-k\psi(t/n)} \psi'(t/n)\, \mathrm{d}t.$$

Recall that the k-components of the atoms of Π_n should be viewed as degrees whereas the t-components should be seen as time. [Note that Π_n could have been defined as $\Pi_n = \{(\tilde{t}_1, \tilde{k}_1), \dots, (\tilde{t}_N, \tilde{k}_N)\}$, where N is a Poisson variable with parameter n and $(\tilde{t}_i, \tilde{k}_i)_{i \geq 1}$ is a sequence of i.i.d. r.v. with distribution $\frac{\pi_n}{n}$ *independent* of N.] Sort the atoms of Π_n in increasing order of their t-components,

$$\Pi_n = \{(t_1, k_1), \dots, (t_N, k_N)\}$$

(here again, we drop the dependency on n in the notations). Then, by standard properties of Poisson point processes,

$$((t_1, k_1), \dots, (t_N, k_N)) \overset{(d)}{=} ((\phi_n(t_1^{(1)}), k_1^{(1)}), \dots, (\phi_n(t_{N^{(1)}}^{(1)}), k_{N^{(1)}}^{(1)})),$$

where $\phi_n = n(1-\mathcal{L}) = n\psi^{-1}$. In particluar $\{k_1, \dots, k_N\}$ has the same distribution as $\Pi_n^{(0)}$.

As before, conditionally on $N = m$, $(k_1, \dots, k_m)$ has the same law as the random vector $(\widehat{D}_1, \dots, \widehat{D}_m)$. This in particular holds for $m = n$. Since N is a Poisson variable with parameter n and, as we shall soon see, we are only interested in what happens up to time $O(n^{2/3})$, we shall study the process Π_n without the latter conditioning. We thus get a Markovian process. Let us prove that conditionally on $N \geq i$, $t_i - t_{i-1}$ is an exponential variable with parameter 1.

LEMMA 4.1. *The point process* $\{t_1, \dots, t_N\}$ *is Poisson point process on* $(0, n)$ *with intensity* $\mathrm{d}t$.

PROOF. Define p as the projection $p : (t, k) \mapsto t$. Then, by standard properties of Poisson point processes, $\{t_1, \dots, t_N\} = p(\Pi_n)$ is a Poisson point process on $(0, n)$ with intensity $\pi_n^{(p)}$ characterized by the following:

$$\text{for every Borel subset } A \text{ of } (0, n), \qquad \pi_n^{(p)}(A) = \pi_n(p^{-1}(A)).$$

Therefore, for every Borel subset A of $(0,n)$,

$$\pi_n^{(p)}(A) = \int_A \sum_{k\in\mathbb{N}^*} \nu_k k e^{-k\psi(t/n)} \psi'(t/n)\,\mathrm{d}t$$
$$= \int_A (\psi^{-1})'(\psi(t/n))\psi'(t/n)\,\mathrm{d}t = \int_A \mathrm{d}t,$$

which proves the result. □

4.3. *Keys points of the section.* Let us sum up the points that will be used in the sequel.

PROPOSITION 4.2. *Let n be a positive integer, $(e_i)_{i\geq 1}$ be a sequence of independent exponential variables with parameter* 1 *and $(U_i)_{i\geq 1}$ be a sequence of independent random variables uniformly distributed on* $(0,1)$ independent *of $(e_i)_{i\geq 1}$. Define*

$$N = \max\left\{i \geq 0 : \sum_{j=1}^{i} e_j < n\right\},$$

$$t_i = \sum_{j=1}^{i} e_j, \qquad i \geq 1,$$

$$k_i = \sum_{j\in\mathbb{N}^*} j\mathbf{1}_{d_{i,j-1}<U_i<d_{i,j}}, \qquad 1 \leq i \leq N,$$

$$\Pi_n = \{(t_1,k_1),\dots,(t_N,k_N)\},$$

where for every integer i and j such that $1 \leq i \leq N$ and $j \geq 1$, $d_{i,j} = \sum_{l=1}^{j} \nu_l l e^{-l\psi(t_i/n)} \psi'(t_i/n)$. Then:

- *Π_n is a Poisson point process on $(0,n)\times\mathbb{N}^*$ with intensity π_n, where*
$$\pi_n(\mathrm{d}t,k) = \nu_k k e^{-k\psi(t/n)} \psi'(t/n)\,\mathrm{d}t.$$
- *For every positive integer m, conditionally on $N=m$, $(k_1,\dots,k_m)$ has the same law as the random vector $(\widehat{D}_1,\dots,\widehat{D}_m)$.*

In the sequel, N, $(t_i)_{i\geq 1}$, $(k_i)_{1\leq i\leq N}$ and Π_n will always refer to those just-defined quantities.

5. Convergence of the walk $\bar{s}_n$. It should now be natural to introduce the process $(S_n(t))_{t\geq 0}$ defined as the sum of the k-components of the atoms of Π_n minus 2 with t-components less than or equal to t:

$$S_n(t) = \sum_{(s,k)\in\Pi_n} (k-2)\mathbf{1}_{s\leq t} = \sum_{1\leq j\leq N} (k_j-2)\mathbf{1}_{t_j\leq t}.$$

We can now state the key result of the present work:

PROPOSITION 5.1. *Rescale S_n by defining $\overline{S_n}(t) = n^{-1/3} S_n(tn^{2/3})$. Then*

$$\overline{S_n} \xrightarrow[n\to\infty]{(d)} W^{\nu}$$

with respect to the Skorokhod topology on every finite interval.

PROOF. We follow the ideas of Aldous [1]. Let

$$A_n(t) = \int_{(0,n)\times\mathbb{N}^*} \pi_n(\mathrm{d}s, k)(k-2)\mathbf{1}_{s\le t}, \qquad t \ge 0,$$

be the continuous bounded variation process such that

$$M_n(t) = S_n(t) - A_n(t), \qquad t \ge 0,$$

is a martingale. Observe that A_n is deterministic. Just as we rescaled S_n to form $\overline{S_n}$, write $\overline{A}_n$ and $\overline{M}_n$ for the correspondingly rescaled versions of A_n and M_n. Proposition 5.1 is shown as soon the following two results are established:

$$\forall t_0 > 0 \qquad \lim_{n\to\infty} \sup_{t\le t_0} \left| \overline{A}_n(t) + \frac{\beta}{2\mu^2} t^2 \right| = 0$$

and

$$\overline{M}_n \xrightarrow[n\to\infty]{(d)} \sqrt{\frac{\beta}{\mu}} B$$

with respect to the Skorokhod topology on every finite interval, where B denotes a standard Brownian motion. We postpone their proofs to the Appendix; see Lemmas A.1 and A.2. The following estimate on $\mathcal{L}$:

$$\mathcal{L}''(t) = \mathbb{E}[D^2] - \mathbb{E}[D^3]t + \underset{t\to 0}{o}(t) \tag{5.1}$$

is a key ingredient in the proofs. □

We now give a key consequence of Proposition 5.1 concerning the depth-first walk $\bar{s}_n$.

COROLLARY 5.2. *The rescaled depth-first walk $\bar{s}_n$ converges in distribution to W^{ν} with respect to the Skorokhod topology on every finite interval as $n \to \infty$.*

PROOF. Denote by $\widetilde{S}_n$ the process

$$\overline{S_n}(n^{-2/3} t_{\lfloor un^{2/3} \rfloor}), \qquad u \ge 0.$$

Applying Propositions 4.2 and 5.1, $\widetilde{S}_n$ converges in distribution to W^{ν} with respect to the Skorokhod topology on every finite interval as $n \to \infty$. Now, for every $u \ge 0$,

$$\widetilde{S}_n(u) = n^{-1/3} \sum_{1\le j\le un^{2/3}} (k_j - 2).$$

Since conditionally on $N = n$, $(k_1, \dots, k_n)$ has the same law as the random vector $(\widehat{D}_1, \dots, \widehat{D}_n)$ (see Proposition 4.2), we get that for every $u > 0$ and every bounded, continuous function f defined on $(\mathbb{D}([0, u]), \mathbb{R})$,

$$\mathbb{E}[f(\widetilde{S}_n)|N = n] = \mathbb{E}[f(\bar{s}_n)]$$

(here, we write $\widetilde{S}_n$ and $\bar{s}_n$ for their restrictions to the interval $[0, u]$). We thus just need to see why

$$\mathbb{E}[f(\widetilde{S}_n)|N = n] - \mathbb{E}[f(\widetilde{S}_n)] \underset{n\to\infty}{\longrightarrow} 0.$$

Now, conditionally on $N = n$, by Proposition 4.2, the sequence $(t_1, \dots, t_{\lfloor un^{2/3}\rfloor})$ has the same distribution as $(nV_1, nV_2, \dots, nV_{\lfloor un^{2/3}\rfloor})$, where $0 < V_1 < \cdots < V_n < 1$ is the ordered statistics of the family of n i.i.d. variables uniformly distributed on $(0, 1)$. In other words, the distribution of the random vector $(t_1, \dots, t_{\lfloor un^{2/3}\rfloor})$ under the event $\{N = n\}$ is exactly the distribution (without conditioning) of

$$\frac{n}{t_{n+1}}(t_1, t_2, \dots, t_{\lfloor un^{2/3}\rfloor})$$

(moreover, the latter random vector is independent of t_{n+1}). Applying Proposition 4.2, we thus deduce that the conditional distribution of $\widetilde{S}_n$ under $\{N = n\}$ is asymptotically close to the distribution of $\widetilde{S}_n$. We get the result by applying the dominated convergence theorem (recall that f is bounded and continuous). $\square$

6. Study of the cycle half-edges. In this section, we turn our attention to the cycle half-edges. In Section 6.1, we shall prove that there are few cycle half-edges in $\mathcal{G}_n$; see Lemma 6.1 below. We shall then show in Section 6.2 how to derive Theorem 3.1 from Corollary 5.2 and Lemma 6.1.

6.1. *Upper bound of the number of cycle half-edges.* In this section, we prove the following result:

LEMMA 6.1. *Let $t > 0$ and $M > 0$. Introduce the event*

$$E_n(t, M) = \Big\{\max_{i\le t}\Big\{\bar{s}_n(i) - \min_{k\le i}\bar{s}_n(k)\Big\} \le M\Big\}.$$

Then we have

$$\limsup_{n\to\infty}\mathbb{E}[\#\{\text{cycle half-edges attached to } v_i, i \le tn^{2/3}, \text{ in } \mathcal{G}_n\}\mathbf{1}_{E_n(t,M)}] < \infty.$$

PROOF. We first study the number of active half-edges, given they contribute to the appearance of cycle half-edges.

We claim that when a half-edge of v_i is about to be paired (in the algorithmic construction of $\mathcal{G}_n$ described in Section 3.1), the number $\#\mathcal{A}$ of active half-edges is less than or equal to $2 + s_n(i) - \min_{j\le i} s_n(j)$, where s_n denotes the walk

$(\sum_{j\le i}(\widehat{D}_j-2), 0\le i\le n)$. To see why this claim is true, first notice that it suffices to prove it only for the first component. Then observe that the claim is true if v_i has just been discovered (this can be shown by induction: this is true for the first vertex v_1 and, when $i>1$, the number of active half-edges that the discovery of v_i creates is $-1+\text{degree of } v_i-1$, which is exactly the increment of s_n). On the other hand, if v_i had already been discovered before, the number of active half-edges present when a new half-edge of v_i is about to be paired is less than the number of active half-edges present when the first half-edge of v_i was about to be paired (due to our choice of the depth-first search; we go back to v_i only when the vertices v_j, $j>i$, have all been fully explored). As seen in the first alternative, that last number is at most $2+s_n(i)-\min_{j\le i}s_n(j)$. This completes the proof of the claim.

Consequently, under the event $E_n(t,M)$, during the first $\lfloor tn^{2/3}\rfloor$ steps, $\#\mathcal{A}$ is always less than or equal to $2+Mn^{1/3}$.

For every deterministic sequence $(x_1,\dots,x_n)$ of positive integers such that $\sum_{i=1}^n x_i$ is even, conditionally on the event $(\widehat{D}_1,\dots,\widehat{D}_n)=(x_1,\dots,x_n)$, one has

$$\begin{aligned}
&\mathbb{E}\big[\#\{\text{cycle half-edges attached to } v_i, i\le tn^{2/3}\}\mathbf{1}_{E_n(t,M)}|\widehat{D}_1=x_1,\dots,\widehat{D}_n=x_n\big]\\
&\quad=\mathbb{E}\Bigg[\sum_{i=1}^{tn^{2/3}}\sum_{k=1}^{\widehat{D}_i}\mathbf{1}_{\{\text{the } k\text{th half-edge of } v_i \text{ is a cycle half-edge}\}}\mathbf{1}_{E_n(t,M)}\Bigg|\\
&\qquad\qquad\qquad\qquad\widehat{D}_1=x_1,\dots,\widehat{D}_n=x_n\Bigg]\\
&\quad\le\sum_{i=1}^{tn^{2/3}}\sum_{k=1}^{x_i}\mathbb{P}\big(\text{the } k\text{th half-edge of } v_i \text{ is a cycle half-edge}|\\
&\qquad\qquad\qquad\widehat{D}_1=x_1,\dots,\widehat{D}_n=x_n \text{ and } E_n(t,M)\big)\\
&\quad\le\sum_{i=1}^{tn^{2/3}}x_i\frac{Mn^{1/3}+2}{\sum_{m=1}^n x_m-\sum_{m=1}^{tn^{2/3}}x_m}\\
&\quad\le\sum_{i=1}^{tn^{2/3}}x_i\frac{Mn^{1/3}+2}{n-tn^{2/3}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbb{E}\big[\#\{\text{cycle half-edges attached to } v_i, i\le tn^{2/3}, \text{ in } \mathcal{G}_n\}\mathbf{1}_{E_n(t,M)}\big]\\
&\quad\le\frac{Mn^{1/3}+2}{n-tn^{2/3}}\mathbb{E}\Bigg[\sum_{i=1}^{tn^{2/3}}\widehat{D}_i\Bigg]\\
&\quad\le\frac{Mn^{1/3}+2}{n-tn^{2/3}}tn^{2/3}\mathbb{E}[\widehat{D}_1].
\end{aligned}$$

Note that $\mathbb{E}[\widehat{D}_1] \le \sum_{k=1}^{\infty} k \frac{k\nu_k}{\mu}$. Hence

$$\limsup_{n\to\infty} \mathbb{E}[\#\{\text{cycle half-edges attached to } v_i, i \le tn^{2/3}\}\mathbf{1}_{E_n(t,M)}] \le 2Mt,$$

which completes the proof of Lemma 6.1. □

REMARK 6.1. We can prove that in fact, for every $t > 0$,

$$\limsup_{n\to\infty} \mathbb{E}[\#\{\text{cycle half-edges attached to } v_i, i \le tn^{2/3}, \text{ in } \mathcal{G}_n\}] < \infty.$$

REMARK 6.2. We stress that a consequence of [6], Theorem 1, is that the expected total number of half-edges present in a component containing a cycle half-edge is $o(n)$. This also holds for the subcritical regime.

6.2. *End of the proof of Theorem* 3.1. In this section, we prove Theorem 3.1. We keep the notation of Section 6.1. Let $t > 0$. Applying Corollary 5.2 and the Portmanteau theorem, it suffices to prove that for every bounded, Lipschitz function f defined on $(\mathbb{D}([0,t]), \mathbb{R})$, $\mathbb{E}[f(\overline{W}_n)] - \mathbb{E}[f(\bar{s}_n)]$ tends to 0 as $n \to \infty$. Let f be such a function. There exists $K > 0$ such that for every $w, w' \in (\mathbb{D}([0,t]), \mathbb{R})$, $|f(w)| \le K$ and $|f(w) - f(w')| \le K\|w - w'\|$. Let $M > 0$. One has

$$\begin{aligned}
&|\mathbb{E}[f(\overline{W}_n)] - \mathbb{E}[f(\bar{s}_n)]| \\
&\quad = \mathbb{E}[|f(\overline{W}_n) - f(\bar{s}_n)|\mathbf{1}_{E_n(t,M)}] + \mathbb{E}[|f(\overline{W}_n) - f(\bar{s}_n)|(1 - \mathbf{1}_{E_n(t,M)})] \\
&\quad \le \mathbb{E}[K\|\overline{W}_n - \bar{s}_n\|\mathbf{1}_{E_n(t,M)}] + \mathbb{E}[2K(1 - \mathbf{1}_{E_n(t,M)})] \\
&\quad \le 2Kn^{-1/3}\mathbb{E}[\#\{\text{cycle half-edges attached to } v_i, i \le tn^{2/3}\}\mathbf{1}_{E_n(t,M)}] \\
&\qquad + 2K\mathbb{P}\Big(\max_{i\le t}\Big\{\bar{s}_n(i) - \min_{k\le i}\bar{s}_n(k)\Big\} \ge M\Big).
\end{aligned}$$

Lemma 6.1 ensures that

$$\lim_{n\to\infty} n^{-1/3}\mathbb{E}[\#\{\text{cycle half-edges attached to } v_i, i \le tn^{2/3}\}\mathbf{1}_{E_n(t,M)}] = 0.$$

Moreover, applying Corollary 5.2 and the Portmanteau theorem,

$$\limsup_{n\to\infty} \mathbb{P}\Big(\max_{i\le t}\Big\{\bar{s}_n(i) - \min_{k\le i}\bar{s}_n(k)\Big\} \ge M\Big) \le \mathbb{P}\Big(\max_{s\le t}\Big\{W^\nu(s) - \min_{u\le s} W^\nu(u)\Big\} \ge M\Big).$$

Therefore, for every $M > 0$,

$$\limsup_{n\to\infty} |\mathbb{E}[f(\overline{W}_n)] - \mathbb{E}[f(\bar{s}_n)]| \le 2K\mathbb{P}\Big(\max_{s\le t}\Big\{W^\nu(s) - \min_{u\le s} W^\nu(u)\Big\} \ge M\Big).$$

Now, the continuity of W^ν implies that

$$\lim_{M\to\infty} \mathbb{P}\Big(\max_{s\le t}\Big\{W^\nu(s) - \min_{u\le s} W^\nu(u)\Big\} \ge M\Big) = 0.$$

Hence

$$\lim_{n\to\infty} \mathbb{E}[f(\overline{W}_n)] - \mathbb{E}[f(\bar{s}_n)] = 0.$$

Theorem 3.1 is therefore proved.

7. Study of the random simple graph $\mathcal{SG}_n$. The setting of this section is the same as before: ν is supposed to satisfy (2.1). We intend to show Theorem 3.2 (recall that, as before, Theorem 2.2 follows from Theorem 3.2). The proof is divided into two steps. First (see Lemma 7.1 below) we shall prove that, with probability tending to 1, the possible loops and multiple edges in $\mathcal{G}_n$ arrive only after the first $\lfloor n^{3/4} \rfloor$ vertices have been explored during the depth-first search. We shall then deduce that the walk $\overline{W}_n$ conditioned on the event $\{\mathcal{G}_n$ is simple$\}$ has the same asymptotic behavior as the walk $\overline{W}_n$; see Section 7.2.

7.1. *Time arrival of loops and multiple edges.* In this section, we prove the following result:

LEMMA 7.1. *Let $T(n)$ be the minimal index of a vertex of $\mathcal{G}_n$ having a loop or a multiple edge*:

$$T(n) = \inf\{i \in \{1, \dots, n\} : v_i \text{ has a loop or a multiple edge}\}.$$

Then

$$\lim_{t\to\infty} \mathbb{P}\big(T(n) > n^{3/4}\big) = 1.$$

Observe that $T(n) = \infty$ if and only if $\mathcal{G}_n$ is simple.

PROOF OF LEMMA 7.1. Obviously, it suffices to show that

$$\lim_{n\to\infty} \mathbb{E}\big[\#\big\{\text{loops or multiple edges attached to } v_i,\, i \le n^{3/4}, \text{ in } \mathcal{G}_n\big\}\big] = 0.$$

Let us establish this assertion. We proceed the same way as in the proof of Lemma 6.1. For every deterministic sequence $(x_1, \dots, x_n)$ of positive integers such that $\sum_{i=1}^n x_i$ is even, conditionally on the event $(\widehat{D}_1, \dots, \widehat{D}_n) = (x_1, \dots, x_n)$, one has

$$\begin{aligned}
&\mathbb{E}\big[\#\big\{\text{loops or multiple edges attached to } v_i,\, i \le n^{3/4}\big\} | \widehat{D}_1 = x_1, \dots, \widehat{D}_n = x_n\big] \\
&\quad \le \sum_{i=1}^{n^{3/4}} \sum_{k=1}^{x_i} \mathbb{P}\big(\text{the } k\text{th half-edge of } v_i \text{ creates a loop or a multiple edge} | \\
&\qquad\qquad \widehat{D}_1 = x_1, \dots, \widehat{D}_n = x_n\big).
\end{aligned}$$

Now, the kth half-edge of a vertex with degree x_i ($i \le n^{3/4}$) creates a loop with probability at most $\frac{x_i - k}{n - n^{3/4}}$. It creates a multiple edge with probability at most $\frac{k-1}{n-n^{3/4}}$. Consequently

$$\begin{aligned}
&\mathbb{E}\big[\#\big\{\text{loops or multiple edges attached to } v_i,\, i \le n^{3/4}\big\} | \widehat{D}_1 = x_1, \dots, \widehat{D}_n = x_n\big] \\
&\quad \le \sum_{i=1}^{n^{3/4}} \sum_{k=1}^{x_i} \frac{x_i}{n - n^{3/4}}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}[\#\{\text{loops or multiple edges attached to } v_i,\, i \le n^{3/4}, \text{ in } \mathcal{G}_n\}] \\
&\quad\le \frac{1}{n-n^{3/4}}\mathbb{E}\left[\sum_{i=1}^{n^{3/4}} \widehat{D}_i^2\right] \\
&\quad\le \frac{n^{3/4}}{n-n^{3/4}}\mathbb{E}[\widehat{D}_1^2].
\end{aligned}$$

Since ν has finite third moment, this completes the proof. □

7.2. *End of the proof of Theorem* 3.2. Let $t>0$. Let f be a bounded, continuous function defined on $(\mathbb{D}([0,t]),\mathbb{R})$. It suffices to prove that

$$\lim_{n\to\infty}\mathbb{E}[f(\overline{W}_n)|T(n)=\infty]=\mathbb{E}[f(W^\nu)],$$

where $f(\overline{W}_n)$ denotes the image of the restriction of the walk $\overline{W}_n$ to $[0,t]$ by f. Let us show that result. Observe that the event

$$\{\text{neither loop nor multiple edge is attached to } v_i,\, i>n^{3/4}\}$$

is asymptotically independent of the r.v. $f(\overline{W}_n)$

$$\begin{aligned}
&\mathbb{E}[f(\overline{W}_n)\mathbf{1}_{\text{neither loop nor multiple edge is attached to } v_i, i>n^{3/4}}] \\
&\quad\underset{n\to\infty}{\sim}\mathbb{E}[f(\overline{W}_n)]\mathbb{P}(\text{neither loop nor multiple edge is attached to } v_i,\, i>n^{3/4}).
\end{aligned}$$

Deciding whether or not a loop or a multiple edge is created after step $n^{3/4}$ does indeed not depend on the first $tn^{2/3}$ steps.[2]

Now, by Theorem 3.1,

$$\lim_{n\to\infty}\mathbb{E}[f(\overline{W}_n)]=\mathbb{E}[f(W^\nu)].$$

Moreover,

$$\begin{aligned}
0 &\le \frac{\mathbb{P}(\text{neither loop nor multiple edge is attached to } v_i,\, i>n^{3/4})}{\mathbb{P}(T(n)=\infty)}-1 \\
&= \frac{\mathbb{P}(\text{neither loop nor multiple edge is attached to } v_i,\, i>n^{3/4} \text{ and } T(n)\le n^{3/4})}{\mathbb{P}(\mathcal{G}_n \text{ is a simple graph})} \\
&\le \frac{\mathbb{P}(T(n)\le n^{3/4})}{\mathbb{P}(\mathcal{G}_n \text{ is a simple graph})}.
\end{aligned}$$

[2]To make this argument rigorous, consider the Poissonian model introduced in Section 4; independence is then straightforward, and the fact that the two models are asymptotically close has already been seen.

According to Lemma 7.1 and equation (2.2),

$$\lim_{n\to\infty} \frac{\mathbb{P}(T(n) \le n^{3/4})}{\mathbb{P}(\mathcal{G}_n \text{ is a simple graph})} = 0.$$

Consequently,

$$\mathbb{P}\big(\text{neither loop nor multiple edge is attached to } v_i,\, i > n^{3/4}\big) \underset{n\to\infty}{\sim} \mathbb{P}\big(T(n) = \infty\big).$$

We finally obtain

$$\mathbb{E}\big[f(\overline{W}_n)\mathbf{1}_{\text{neither loop nor multiple edge is attached to } v_i, i > n^{3/4}}\big] \\ \underset{n\to\infty}{\sim} \mathbb{E}\big[f\big(W^\nu\big)\big]\mathbb{P}\big(T(n) = \infty\big).$$

Recalling Lemma 7.1 and equation (2.2) again (just proceed as before), this proves that

$$\mathbb{E}\big[f(\overline{W}_n)\mathbf{1}_{T(n)=\infty}\big] \underset{n\to\infty}{\sim} \mathbb{E}\big[f\big(W^\nu\big)\big]\mathbb{P}\big(T(n) = \infty\big),$$

completing the proof of Theorem 3.2.

8. The power law distribution setting. In this section, we do not suppose the finiteness of the moment of order 3 for distribution ν, and rather we replace assumption (2.1) by

$$\sum_{k=1}^{\infty} k(k-2)\nu_k = 0 \quad \text{and} \quad \nu_k \underset{k\to\infty}{\sim} ck^{-\gamma}, \tag{8.1}$$

where $c > 0$ and $\gamma \in (3, 4)$. This implies that (5.1) has to be replaced by

$$\mathcal{L}''(t) = 2\mu - \frac{c\Gamma(4-\gamma)}{\gamma - 3} t^{\gamma-3} + \underset{t\to 0}{o}\big(t^{\gamma-3}\big). \tag{8.2}$$

We are interested in the component sizes of the multigraph constructed the same way as before. To have a good idea of what the order of the component sizes should be, we adopt the same strategy, using Poisson calculus; see Section 8.1. We shall then show in Section 8.2 how to deduce the asymptotic behavior of the component sizes of $\mathcal{G}_n$ in our new situation. In Section 8.3 we shall state some open problems.

8.1. *The Poissonian argument*. Taking the same notation as in Section 5, we here consider the process $(S_n(t))_{t\ge 0}$ defined by

$$S_n(t) = \sum_{(s,k)\in\Pi_n} (k-2)\mathbf{1}_{s\le t}.$$

Recall that Π_n is a Poisson point process on $(0, n) \times \mathbb{N}^*$ with intensity π_n, where $\pi_n(\mathrm{d}t, k) = \nu_k k e^{-k\psi(t/n)}\psi'(t/n)\,\mathrm{d}t$. We intend to prove the following result:

THEOREM 8.1. *Rescale S_n by defining $\overline{S_n}(t) = n^{-1/(\gamma-1)} S_n(tn^{(\gamma-2)/(\gamma-1)})$. Then*

$$\overline{S_n} \xrightarrow[n\to\infty]{(d)} X^\nu + A^\nu$$

with respect to the Skorokhod topology on every finite interval, where

$$A_t^\nu = -\frac{c\Gamma(4-\gamma)}{(\gamma-3)(\gamma-2)\mu^{\gamma-2}} t^{\gamma-2}, \qquad t \ge 0,$$

and X^ν is the unique process with independent increments such that for every $t \ge 0$ and $u \in \mathbb{R}$,

$$\mathbb{E}[\exp(iuX_t^\nu)] = \exp\left(\int_0^t \mathrm{d}s \int_0^\infty \mathrm{d}x\,(e^{iux} - 1 - iux)\frac{c}{\mu}\frac{1}{x^{\gamma-1}} e^{-xs/\mu}\right).$$

PROOF. As before, let

$$A_n(t) = \int_{(0,n)\times\mathbb{N}^*} \pi_n(\mathrm{d}s, k)(k-2)\mathbf{1}_{s\le t}, \qquad t \ge 0$$

be the deterministic continuous bounded variation function such that

$$M_n(t) = S_n(t) - A_n(t), \qquad t \ge 0$$

is a martingale. Just as we rescaled S_n to form $\overline{S_n}$ in Theorem 8.1, write $\overline{A_n}$ and $\overline{M_n}$ for the correspondingly rescaled versions of A_n and M_n. Note that (5.1) was the only ingredient of the proof of Lemma A.1. Since in our setting equation (8.2) holds, we can perform the same elementary calculations and then find that for every $t > 0$,

$$\lim_{n\to\infty} \sup_{s\le t} \left|\overline{A_n}(s) - A_s^\nu\right| = 0.$$

To complete the proof of Theorem 8.1, it thus suffices to show that

$$\overline{M_n} \xrightarrow[n\to\infty]{(d)} X^\nu$$

with respect to the Skorokhod topology on every finite interval. We postpone the proof of that result to the Appendix; see Lemmas B.1 and B.2. □

8.2. *The main result.* Repeating exactly what we did in Section 6, we deduce from Theorem 8.1 the following key result. As before, the walk defined via (3.1) is denoted by W_n.

COROLLARY 8.2. *Rescale the depth-first walk W_n by defining for every $t \in [0, n^{1/(\gamma-1)}]$,*

$$\overline{W_n}(t) = n^{-1/(\gamma-1)} W_n(\lfloor tn^{(\gamma-2)/(\gamma-1)} \rfloor).$$

Then

$$\overline{W}_n \xrightarrow[n\to\infty]{(d)} X^\nu + A^\nu$$

with respect to the Skorokhod topology on every finite interval.

We now give an analogous result of Theorem 2.1 in the present setting. Let R^ν be the reflected process defined by

$$R_t^\nu = X_t^\nu + A_t^\nu - \inf_{0\le s\le t}\{X_s^\nu + A_s^\nu\}, \qquad t \ge 0.$$

We define excursion intervals and excursion lengths of R^ν as in Section 2.

THEOREM 8.3. *Suppose ν satisfies* (8.1). *Then a.s. the set of excursions of R^ν may be written $\{\gamma_j, j \ge 1\}$ so that the lengths $|\gamma_j|$ are decreasing and*

$$\sum_{j\ge1} |\gamma_j|^2 < \infty,$$

and letting $\mathcal{C}_n^\nu$ be the ordered sequence of component sizes of $\mathcal{G}_n$,

$$n^{-(\gamma-2)/(\gamma-1)}\mathcal{C}_n^\nu \xrightarrow[n\to\infty]{(d)} \big(|\gamma_j|, j \ge 1\big)$$

with respect to the $l^2_\searrow$ topology.

Contrary to the finite third moment case, Theorem 8.3 cannot been seen as a straightforward consequence of Corollary 8.2; the analogy of Section 3.4 of [1] does not exist here. The following lemma (which uses Corollary 8.2) will nonetheless enable us to get Theorem 8.3. We refer to Section 3 for the definitions of $\zeta(n,k)$ and $\mathcal{C}(n,k)$.

LEMMA 8.4. *For every positive integer n, let $\Xi^{(n)}$ be the point process*

$$\Xi^{(n)} = \big\{\big(n^{-(\gamma-2)/(\gamma-1)}\zeta(n, k-1), n^{-(\gamma-2)/(\gamma-1)}\mathcal{C}(n,k)\big) : k \ge 1\big\},$$

and let $\Xi^{(\infty)}$ be the point process

$$\Xi^{(\infty)} = \big\{\big(l(\gamma), |\gamma|\big) : \gamma \text{ is an excursion of } R^\nu\big\}.$$

Then $\Xi^{(n)}$ converges vaguely in distribution to $\Xi^{(\infty)}$ as $n \to \infty$.[3] *Moreover, $\Xi^{(\infty)}$ satisfies the following three points*:

(1) $\sup\{s : (s, y) \in \Xi^{(\infty)}$ *for some* $y\} = \infty$ *a.s.*;
(2) *if* $(s, y) \in \Xi^{(\infty)}$, *then* $\sum_{(s',y')\in\Xi^{(\infty)} : s'<s} y' = s$ *a.s.*;
(3) $\max\{y : (s, y) \in \Xi^{(\infty)}$ *for some* $s > s_0\} \xrightarrow{p} 0$ *as* $s_0 \to \infty$.

[3]Vague convergence of counting measures on $[0, \infty) \times (0, \infty)$ is considered here.

PROOF. Observe that the component sizes of the multigraph $\mathcal{G}_n$, in the order of appearance in depth-first walk, are size-biased ordered. Following the *proof* of [2], Proposition 17, Lemma 8.4 thus derives from Corollary 8.2 and forthcoming Lemma B.3 stated in the Appendix. □

PROOF OF THEOREM 8.3. Applying [2], Proposition 17 (see also [1], Proposition 15 and Lemma 25), Lemma 8.4 ensures that a.s. the set of excursions of R^ν can be written $\{\gamma_j, j \geq 1\}$ so that the lengths $|\gamma_j|$ are decreasing and

$$\sum_{j\geq 1} |\gamma_j|^2 < \infty.$$

By [2], Proposition 17, another consequence of Lemma 8.4 is that

$$n^{-(\gamma-2)/(\gamma-1)} \mathcal{C}_n^\nu \xrightarrow[n\to\infty]{(d)} (|\gamma_j|, j \geq 1)$$

with respect to the $l^2_\searrow$ topology. □

8.3. *Open questions*. The argument used to prove Lemma 7.1 does not work in our present setting. Observe though that (2.2) still holds here. That is why we believe that the following result is true:

CONJECTURE 8.5. *Suppose ν satisfies* (8.1). *Let $\mathcal{SC}_n^\nu$ be the ordered sequence of component sizes of $\mathcal{SG}_n$ and $(|\gamma_j|, j \geq 1)$ be the ordered sequence of the excursion lengths of R^ν. Then*

$$n^{-(\gamma-2)/(\gamma-1)} \mathcal{SC}_n^\nu \xrightarrow[n\to\infty]{(d)} (|\gamma_j|, j \geq 1)$$

with respect to the $l^2_\searrow$ topology.

As before, Conjecture 8.5 would be proven as soon as the following result is shown:

CONJECTURE 8.6. *If ν satisfies* (8.1), *then the rescaled walk $\overline{W}_n$ conditioned on the event $\{\mathcal{G}_n$ is simple$\}$ converges in distribution to $X^\nu + A^\nu$ with respect to the Skorokhod topology on every finite interval as $n \to \infty$.*

APPENDIX A: THE FINITE THIRD MOMENT SETTING

In this section, we complete the proof of Proposition 5.1 by showing two technical results.

LEMMA A.1. *For every $t_0 > 0$,*

$$\lim_{n\to\infty} \sup_{t\leq t_0} \left| \overline{A}_n(t) + \frac{\beta}{2\mu^2} t^2 \right| = 0.$$

PROOF. By definition,

$$A_n(t) = \int_0^t \sum_{k\in\mathbb{N}^*} (k^2 - 2k) e^{-k\psi(s/n)} \psi'(s/n)\nu_k \,\mathrm{d}s$$
$$= \int_0^t \bigl(a_n(s) - 2\bigr)\,\mathrm{d}s,$$

where

$$a_n(s) = \frac{\mathcal{L}''(\psi(s/n))}{-\mathcal{L}'(\psi(s/n))}.$$

Since $\mathbb{E}[D^2] = 2\mathbb{E}[D]$, $a_n(s)$ tends to 2 as $n \to \infty$. Moreover, it is easily seen by approximating $\psi(s/n)$ by $\frac{s}{\mu n}$ that $a_n(s) - 2$ is approximatively $-\frac{\beta}{\mu^2}\frac{s}{n}$. Let us be more precise. Recalling (5.1), in the neighborhood of $t = 0$,

$$\mathcal{L}''(t) = 2\mu - (\beta + 4\mu)t + o(t) \quad \text{and} \quad \mathcal{L}'(t) = -\mu + 2\mu t + o(t).$$

Therefore

$$\frac{\mathcal{L}''(t) + 2\mathcal{L}'(t)}{-\mathcal{L}'(t)} = -\frac{\beta}{\mu}t + o(t),$$

that is, there exists a function $\varepsilon^{(1)}(\cdot)$ tending to 0 at 0 such that

$$\frac{\mathcal{L}''(t)}{-\mathcal{L}'(t)} - 2 = -\frac{\beta}{\mu}t + t\varepsilon^{(1)}(t).$$

Now, $\psi(t) = \frac{t}{\mu} + o(t)$ so that there exists a function $\varepsilon^{(2)}(\cdot)$ tending to 0 at 0 such that

$$\psi(t) = \frac{t}{\mu} + t\varepsilon^{(2)}(t).$$

We deduce that

$$\begin{aligned}
a_n(s) - 2 &= -\frac{\beta}{\mu}\psi\left(\frac{s}{n}\right) + \psi\left(\frac{s}{n}\right)\varepsilon^{(1)}\left(\psi\left(\frac{s}{n}\right)\right) \\
&= -\frac{\beta}{\mu}\left(\frac{s}{\mu n} + \frac{s}{n}\varepsilon^{(2)}\left(\frac{s}{n}\right)\right) \\
&\quad + \left(\frac{s}{\mu n} + \frac{s}{n}\varepsilon^{(2)}\left(\frac{s}{n}\right)\right)\varepsilon^{(1)}\left(\frac{s}{\mu n} + \frac{s}{n}\varepsilon^{(2)}\left(\frac{s}{n}\right)\right) \\
&= -\frac{\beta}{\mu^2}\frac{s}{n} + \frac{s}{n}\left\{-\frac{\beta}{\mu}\varepsilon^{(2)}\left(\frac{s}{n}\right) + \frac{1}{\mu}\varepsilon^{(1)}\left(\frac{s}{\mu n} + \frac{s}{n}\varepsilon^{(2)}\left(\frac{s}{n}\right)\right)\right. \\
&\qquad \left. + \varepsilon^{(2)}\left(\frac{s}{n}\right)\varepsilon^{(1)}\left(\frac{s}{\mu n} + \frac{s}{n}\varepsilon^{(2)}\left(\frac{s}{n}\right)\right)\right\}.
\end{aligned}$$

Defining

$$\varepsilon : t \mapsto -\frac{\beta}{\mu}\varepsilon^{(2)}(t) + \frac{1}{\mu}\varepsilon^{(1)}\left(\frac{t}{\mu} + t\varepsilon^{(2)}(t)\right) + \varepsilon^{(2)}(t)\varepsilon^{(1)}\left(\frac{t}{\mu} + t\varepsilon^{(2)}(t)\right),$$

we finally get

$$a_n(s) - 2 = -\frac{\beta}{\mu^2}\frac{s}{n} + \frac{s}{n}\varepsilon\left(\frac{s}{n}\right)$$

with $\varepsilon(\cdot)$ tending to 0 at 0. Thus, for every $t \in [0, t_0 n^{2/3}]$,

$$\left|A_n(t) + \frac{\beta}{\mu^2}\frac{t^2}{2n}\right| \le \frac{1}{n}\int_0^t s\left|\varepsilon\left(\frac{s}{n}\right)\right| \mathrm{d}s \le \frac{1}{n}\int_0^{t_0 n^{2/3}} s\left|\varepsilon\left(\frac{s}{n}\right)\right| \mathrm{d}s.$$

As a result, for every $\eta > 0$, there exists an integer $n_0(\eta)$ such that for every integer $n \ge n_0(\eta)$,

$$\sup_{t \le t_0 n^{2/3}}\left|A_n(t) + \frac{\beta}{2\mu^2}\frac{t^2}{n}\right| \le \frac{1}{n}\int_0^{t_0 n^{2/3}} s\eta \,\mathrm{d}s = \frac{t_0^2}{2}\eta n^{1/3},$$

which proves Lemma A.1. □

LEMMA A.2. *$\overline{M}_n \xrightarrow[n\to\infty]{(d)} \sqrt{\frac{\beta}{\mu}}B$ with respect to the Skorokhod topology on every finite interval, where B denotes a standard Brownian motion.*

PROOF. We want to apply the functional CLT for continuous-time martingales. Since A_n is continuous, and S_n only jumps at points t_j, M_n is a purely discontinuous martingale, so that $[M_n]_t = \sum_{s \le t} \Delta M_n(s)^2$ and its predictable projection

$$\langle M_n\rangle(t) = \int_{(0,n)\times\mathbb{N}^*} \pi_n(\mathrm{d}s, k)(k-2)^2 \mathbf{1}_{s \le t}, \qquad t \ge 0,$$

is the continuous, increasing process such that $M_n^2 - \langle M_n\rangle$ is a martingale. Observe that $\langle M_n\rangle$ is deterministic. Define $\langle \overline{M}_n\rangle(t) = n^{-2/3}\langle M_n\rangle(tn^{2/3})$. Applying [13], Theorem 7.1.4(b), the following two points imply Lemma A.2:

$$\text{(A.1)} \qquad \forall t_0 > 0 \qquad \langle \overline{M}_n\rangle(t_0) \xrightarrow[n\to\infty]{} \frac{\beta}{\mu}t_0$$

and

$$\text{(A.2)} \qquad \lim_{n\to\infty} \mathbb{E}\Big[\sup_{t \le t_0}\big|\overline{M}_n(t) - \overline{M}_n(t-)\big|^2\Big] = 0.$$

Let us establish (A.1). First note that

$$\begin{aligned}\langle M_n\rangle(t) &= \int_0^t \sum_{k\in\mathbb{N}^*} k(k-2)^2 e^{-k\psi(s/n)}\psi'(s/n)\nu_k \,\mathrm{d}s \\ &= \int_0^t b_n(s)\,\mathrm{d}s,\end{aligned}$$

where

$$b_n(s) = \frac{\mathcal{L}^{(3)} + 4\mathcal{L}'' + 4\mathcal{L}'}{\mathcal{L}'} \circ \psi\left(\frac{s}{n}\right).$$

Since $\psi(t)$ tends to 0 as $t \to 0$ and

$$\lim_{t\to 0} \frac{\mathcal{L}^{(3)}(t) + 4\mathcal{L}''(t) + 4\mathcal{L}'(t)}{\mathcal{L}'(t)} = \frac{-(\beta + 4\mu) + 8\mu - 4\mu}{-\mu} = \frac{\beta}{\mu},$$

there exists a function $\varepsilon(\cdot)$ tending to 0 at 0 such that

$$b_n(s) = \frac{\beta}{\mu} + \varepsilon\left(\frac{s}{n}\right).$$

We deduce that

$$\left|\langle M_n\rangle\big(t_0 n^{2/3}\big) - \frac{\beta}{\mu} t_0 n^{2/3}\right| \le \int_0^{t_0 n^{2/3}} \left|\varepsilon\left(\frac{s}{n}\right)\right| \mathrm{d}s.$$

Hence, for every $\eta > 0$, there exists an integer $n_1(\eta)$ such that for every integer $n \ge n_1(\eta)$,

$$\left|\langle M_n\rangle\big(t_0 n^{2/3}\big) - \frac{\beta}{\mu} t_0 n^{2/3}\right| \le \eta t_0 n^{2/3},$$

which proves (A.1).

We next turn our attention to (A.2). Note that $M_n(t) - M_n(t-) = S_n(t) - S_n(t-)$, so

$$\begin{aligned}\sup_{t \le t_0 n^{2/3}} \big|M_n(t) - M_n(t-)\big|^2 &= \sup\big\{(k-2)^2 : (s,k) \in \Pi_n \text{ and } s \le t_0 n^{2/3}\big\} \\ &\le \sup\big\{k^2 : (s,k) \in \Pi_n \text{ and } s \le t_0 n^{2/3}\big\}.\end{aligned}$$

Let L_n denote $\sup\{k : (s,k) \in \Pi_n \text{ and } s \le t_0 n^{2/3}\}$ (we drop the dependency on t_0 in the notation). We have

$$\mathbb{E}\big[L_n^2\big] = \sum_{k=1}^{\lfloor n^{1/3}\rfloor - 1} \mathbb{P}(L_n \ge \sqrt{k}) + \sum_{k \ge n^{1/3}} \mathbb{P}(L_n \ge \sqrt{k}) \le n^{1/3} + \sum_{k \ge n^{1/3}} \mathbb{P}(L_n \ge \sqrt{k}).$$

Now, for every $m \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}(L_n \ge m) &= 1 - \mathbb{P}(L_n < m) \\ &= 1 - \mathbb{P}\big(\Pi_n\big(\big[0, t_0 n^{2/3}\big] \times \{m, m+1, \ldots\}\big) = 0\big) \\ &= 1 - \exp\big(-\pi_n\big(\big[0, t_0 n^{2/3}\big] \times \{m, m+1, \ldots\}\big)\big) \\ &\le \pi_n\big(\big[0, t_0 n^{2/3}\big] \times \{m, m+1, \ldots\}\big)\end{aligned}$$

$$= \sum_{l \geq m} \nu_l \int_0^{t_0 n^{2/3}} \mathrm{d}s\, l e^{-l\psi(s/n)} \psi'(s/n)$$

$$= n \sum_{l \geq m} \nu_l \bigl(1 - e^{-l\psi(t_0 n^{-1/3})}\bigr)$$

$$\leq n\psi\bigl(t_0 n^{-1/3}\bigr) \sum_{l \geq m} l \nu_l.$$

As a result,

$$\mathbb{E}[L_n^2] \leq n^{1/3} + \sum_{k \geq n^{1/3}} n\psi\bigl(t_0 n^{-1/3}\bigr) \sum_{l \geq \sqrt{k}} l \nu_l$$

$$= n^{1/3} + n\psi\bigl(t_0 n^{-1/3}\bigr) \sum_{l \geq n^{1/6}} l \nu_l \sum_{k = \lfloor n^{1/3} \rfloor}^{l^2} 1.$$

We deduce that for every integer n,

$$n^{-2/3}\mathbb{E}\Bigl[\sup_{t \leq t_0 n^{2/3}} \bigl|M_n(t) - M_n(t-)\bigr|^2\Bigr] \leq n^{-1/3} + n^{1/3}\psi\bigl(t_0 n^{-1/3}\bigr) \sum_{l \geq n^{1/6}} l^3 \nu_l.$$

Now, $n^{1/3}\psi(t_0 n^{-1/3})$ tends to $\frac{t_0}{\mu}$ and since $\mathbb{E}[D^3]$ is finite, $\sum_{l \geq n^{1/6}} l^3 \nu_l$ tends to 0. Equation (A.2) is therefore proved. □

APPENDIX B: THE POWER LAW DISTRIBUTION SETTING

B.1. End of the proof of Theorem 8.1. This section is organized as follows. In Lemma B.1 we shall study the martingale $\overline{M}_n^{(1)}$ related to the small jumps of $\overline{M}_n$. Then, in Lemma B.2, we shall be interested in the martingale $\overline{M}_n^{(2)}$ which counts the big jumps. The fact that $\overline{M}_n = \overline{M}_n^{(1)} + \overline{M}_n^{(2)}$ converges to X^ν, which is the sum of the limits of $\overline{M}_n^{(1)}$ and $\overline{M}_n^{(2)}$, stems from the independence of $\overline{M}_n^{(1)}$ and $\overline{M}_n^{(2)}$ (since they never jump simultaneously). To ease notation, let

$$a = \frac{1}{\gamma - 1}.$$

LEMMA B.1. *The martingale $\overline{M}_n^{(1)}$ defined for every $t \geq 0$ by*

$$\overline{M}_n^{(1)}(t) = \sum_{(s,k) \in \Pi_n} \mathbf{1}_{k < n^a}(k-2) n^{-a} \mathbf{1}_{s \leq t n^{1-a}}$$

$$- \int_{(0,n) \times \mathbb{N}^*} \pi_n(\mathrm{d}s, k) \mathbf{1}_{k < n^a}(k-2) n^{-a} \mathbf{1}_{s \leq t n^{1-a}}$$

converges in distribution with respect to the Skorokhod topology on every finite interval as $n \to \infty$ to a process $(X_t^{(1)})_{t\geq 0}$ with independent increments characterized by: *for every $t \geq 0$ and $u \in \mathbb{R}$*,

$$\mathbb{E}[\exp(iuX_t^{(1)})] = \exp\biggl(\int_0^t \mathrm{d}s \int_0^1 \mathrm{d}x\,(e^{iux} - 1 - iux)\frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xs/\mu}\biggr).$$

PROOF. First observe that the process $X^{(1)}$ may be defined as the limit for the metric induced by the norm

$$\|Y\| = \mathbb{E}[\sup\{Y_s^2 : 0 \leq s \leq t\}]^{1/2}$$

of the Cauchy family

$$t \mapsto \sum_{s\leq t} \mathbf{1}_{\Delta_s > \varepsilon}\Delta_s - \int_0^t \mathrm{d}s \int_\varepsilon^1 \mathrm{d}x\, x\frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xs/\mu}$$

as ε tends to 0, Δ being a Poisson point process with intensity $\mathbf{1}_{x\in(0,1)}\nu(\mathrm{d}s, \mathrm{d}x)$ where

$$\nu(\mathrm{d}s, \mathrm{d}x) = \frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xs/\mu}\,\mathrm{d}s\,\mathrm{d}x.$$

To prove Lemma B.1, we rely on [15], Theorem VII.3.7. Dealing with small jumps of the martingale $\overline{M}_n$ indeed enables us to work with "square-integrable" processes [note that $\int_0^t \int_{\mathbb{R}} x^2 \mathbf{1}_{x\in(0,1)}\nu(\mathrm{d}s, \mathrm{d}x) < \infty$].

Taking the same notation as in [15], we first have to compute the characteristics (B^n, C^n, ν^n) of $\overline{M}_n^{(1)}$, which are defined via the following equation: for every $t \geq 0$ and $u \in \mathbb{R}$,

$$\begin{aligned}&\mathbb{E}[\exp(iu\overline{M}_n^{(1)}(t))]\\ &\quad = \exp\biggl(iuB^n(t) - \frac{1}{2}u^2C^n(t) + \int_0^t \int_{-n^{-a}}^1 (e^{iux} - 1 - iux)\nu^n(\mathrm{d}s, \mathrm{d}x)\biggr).\end{aligned}$$

The exponential formula for Poisson point processes yields

$$\begin{aligned}&\mathbb{E}[\exp(iu\overline{M}_n^{(1)}(t))]\\ &\quad = \exp\biggl\{n\sum_{k<n^a}\nu_k(1 - e^{-k\psi(tn^{-a})})(e^{iu(k-2)n^{-a}} - 1 - iu(k-2)n^{-a})\biggr\}.\end{aligned}$$

Consequently, $B^n = C^n = 0$ and

$$\nu^n(\mathrm{d}s, \mathrm{d}x) = \mathrm{d}s \sum_{k<n^a}\delta_{(k-2)n^{-a}}(\mathrm{d}x)n^{1-a}k\nu_k\psi'(sn^{-a})e^{-k\psi(sn^{-a})}.$$

According to [15], Theorem VII.3.7, Lemma B.1 will be proved as soon as we have shown that for every $t \geq 0$,

$$\text{(B.1)}\qquad \int_0^t \int_{-n^{-a}}^1 x^2\nu^n(\mathrm{d}s, \mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \int_0^t \int_0^1 x^2\nu(\mathrm{d}s, \mathrm{d}x),$$

and for every $t \geq 0$ and $g \in C_2(\mathbb{R}_+)$,

$$\int_0^t \int_{-n^{-a}}^1 g(x)\nu^n(\mathrm{d}s, \mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \int_0^t \int_0^1 g(x)\nu(\mathrm{d}s, \mathrm{d}x), \tag{B.2}$$

where $C_2(\mathbb{R}_+)$ is the set of all continuous bounded functions $\mathbb{R}_+ \to \mathbb{R}$ which are 0 on a neighborhood 0 and have a limit at infinity.

Let us establish (B.1). Elementary calculations yield

$$\int_0^t \int_{-n^{-a}}^1 x^2\nu^n(\mathrm{d}s, \mathrm{d}x) = n^{1-2a} \sum_{k<n^a} (k-2)^2 \nu_k \big(1 - e^{-k\psi(tn^{-a})}\big).$$

A difficulty stems from the lack of good estimates for ν_k when k is small. That is why we write

$$\begin{aligned}\int_0^t \int_{-n^{-a}}^1 x^2\nu^n(\mathrm{d}s, \mathrm{d}x) = n^{1-2a} &\sum_{k\in\mathbb{N}^*} (k-2)^2 \nu_k \big(1 - e^{-k\psi(tn^{-a})}\big)\\ &- n^{1-2a} \sum_{k\geq n^a} (k-2)^2 \nu_k \big(1 - e^{-k\psi(tn^{-a})}\big).\end{aligned}$$

It is easy to see that the first term in the difference tends to $\frac{c\Gamma(4-\gamma)}{(\gamma-3)\mu^{\gamma-3}} t^{\gamma-3}$. As for the second, recalling that $\nu_k \sim ck^{-\gamma}$,

$$\begin{aligned}&n^{1-2a} \sum_{k\geq n^a} (k-2)^2 \nu_k \big(1 - e^{-k\psi(tn^{-a})}\big)\\ &\quad \underset{n\to\infty}{\sim} n^{1-2a} \int_{n^a}^\infty \mathrm{d}x\, x^2 c x^{-\gamma} \big(1 - e^{-x\psi(tn^{-a})}\big).\end{aligned}$$

A change of variable and an application of the dominated convergence theorem [recall that $\psi(x) = \frac{x}{\mu} + o(x)$] yield

$$n^{1-2a} \sum_{k\geq n^a} (k-2)^2 \nu_k \big(1 - e^{-k\psi(tn^{-a})}\big) \underset{n\to\infty}{\longrightarrow} \int_1^\infty \mathrm{d}x\, c \frac{1 - e^{-xt/\mu}}{x^{\gamma-2}}.$$

Noticing that

$$\frac{c\Gamma(4-\gamma)}{(\gamma-3)\mu^{\gamma-3}} t^{\gamma-3} = \int_0^\infty \mathrm{d}x\, c \frac{1 - e^{-xt/\mu}}{x^{\gamma-2}},$$

and we finally get

$$\int_0^t \int_{-n^{-a}}^1 x^2\nu^n(\mathrm{d}s, \mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \int_0^1 \mathrm{d}x\, c \frac{1 - e^{-xt/\mu}}{x^{\gamma-2}},$$

which proves (B.1).

We now turn our attention to (B.2). Let $\varepsilon \in (0,1)$ and $g:[\varepsilon,1]\to\mathbb{R}$ be a continuous function. Then

$$\int_0^t\int_{-n^{-a}}^1 g(x)\nu^n(\mathrm{d}s,\mathrm{d}x) = n\sum_{\varepsilon n^a<k<n^a} g\left(\frac{k-2}{n^a}\right)\nu_k\left(1-e^{-k\psi(tn^{-a})}\right).$$

Proceeding as before, we obtain

$$\int_0^t\int_{-n^{-a}}^1 g(x)\nu^n(\mathrm{d}s,\mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \int_\varepsilon^1 \mathrm{d}x\, g(x)c\frac{1-e^{-xt/\mu}}{x^\gamma},$$

completing the proof of Lemma B.1. $\square$

In order to finish the proof Theorem 8.1, we now show the convergence of the martingale related to the big jumps.

LEMMA B.2. *The martingale $\overline{M}_n^{(2)}$ defined for every $t\geq 0$ by*

$$\overline{M}_n^{(2)}(t) = \sum_{(s,k)\in\Pi_n} \mathbf{1}_{k\geq n^a}(k-2)n^{-a}\mathbf{1}_{s\leq tn^{1-a}} - \int_{(0,n)\times\mathbb{N}^*}\pi_n(\mathrm{d}s,k)\mathbf{1}_{k\geq n^a}(k-2)n^{-a}\mathbf{1}_{s\leq tn^{1-a}}$$

converges in distribution with respect to the Skorokhod topology on every finite interval as $n\to\infty$ to a process $(X_t^{(2)})_{t\geq 0}$ with independent increments characterized by: for every $s,t\geq 0$, $u\in\mathbb{R}$,

$$\mathbb{E}\left[\exp\left(iuX_t^{(2)}\right)\right] = \exp\left(\int_0^t \mathrm{d}s\int_1^\infty \mathrm{d}x\left(e^{iux}-1-iux\right)\frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xs/\mu}\right).$$

PROOF. The existence of $X^{(2)}$ is easily obtained as the sum of

$$B_t^\nu = -\int_0^t \mathrm{d}s\int_1^\infty \mathrm{d}x\, x\frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xs/\mu}, \qquad t\geq 0, \tag{B.3}$$

and the partial sum of the jumps of a Poisson point process with intensity $\mathbf{1}_{x\geq 1}\nu(\mathrm{d}s,\mathrm{d}x)$ [recall that $\nu(\mathrm{d}s,\mathrm{d}x) = \frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xs/\mu}\,\mathrm{d}s\,\mathrm{d}x$]. Let us see how Lemma B.2 derives from [15], Theorem VII.3.4.

As before, we first have to compute the characterics (B^n, C^n, ν^n) of $\overline{M}_n^{(2)}$, which are now defined via the equation: for every $s,t\geq 0$, $u\in\mathbb{R}$,

$$\mathbb{E}\left[\exp\left(iu\overline{M}_n^{(2)}(t)\right)\right] = \exp\left(iuB^n(t)-\frac{1}{2}u^2C^n(t)+\int_0^t\int_{1-2n^{-a}}^\infty \left(e^{iux}-1\right)\nu^n(\mathrm{d}s,\mathrm{d}x)\right).$$

The exponential formula for Poisson point processes yields

$$\mathbb{E}[\exp(iu\overline{M}_n^{(2)}(t))] = \exp\Big\{-iun^{1-a}\sum_{k\geq n^a}(k-2)\nu_k(1-e^{-k\psi(tn^{-a})}) \\ + n\sum_{k\geq n^a}\nu_k(1-e^{-k\psi(tn^{-a})})(e^{iu(k-2)n^{-a}}-1)\Big\}.$$

Consequently, $C^n = 0$,

$$B^n(t) = -n^{1-a}\sum_{k\geq n^a}(k-2)\nu_k(1-e^{-k\psi(tn^{-a})})$$

and

$$\nu^n(\mathrm{d}s,\mathrm{d}x) = \mathrm{d}s\sum_{k\geq n^a}\delta_{(k-2)n^{-a}}(\mathrm{d}x)k\nu_k n^{1-a}\psi'(sn^{-a})e^{-k\psi(sn^{-a})}.$$

According to [15], Theorem VII.3.4, Lemma B.2 will be proved as soon as we have shown that for every $t \geq 0$,

$$\sup_{s\leq t}|B^n(t) - B_t^\nu| \underset{n\to\infty}{\longrightarrow} 0, \tag{B.4}$$

and for every $t \geq 0$ and $g \in C_2(\mathbb{R}_+)$,

$$\int_0^t\int_{1-2n^{-a}}^\infty g(x)\nu^n(\mathrm{d}s,\mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \int_0^t\int_1^\infty g(x)\nu(\mathrm{d}s,\mathrm{d}x). \tag{B.5}$$

Equation (B.5) can be shown exactly the same way as (B.2), and to prove (B.4), it suffices to compare the series to the corresponding integrals as we did above. □

B.2. Completion of the proof of Lemma 8.4. In this section, we give the missing elements in the proof of Lemma 8.4. This is provided by Lemma B.3.

LEMMA B.3. *The following four assertions hold*:

(1) $X_t^\nu + A_t^\nu \overset{p}{\to} -\infty$ *as* $t \to \infty$;
(2) $\sup\{|\gamma| : \gamma$ *is an excursion of* R^ν *s.t.* $l(\gamma) \geq t\} \overset{p}{\to} 0$ *as* $t \to \infty$;
(3) *The set* $\{t : R_t^\nu = 0\}$ *contains no isolated points a.s.*;
(4) *For every* $t > 0$, $\mathbb{P}(R_t^\nu = 0) = 0$.

PROOF OF LEMMA B.3(1). By Lemma B.1,

$$\mathbb{E}[(X_t^{(1)})^2] = \frac{c}{\mu}\int_0^t \mathrm{d}s\int_0^1 \mathrm{d}x\frac{1}{x^{\gamma-3}}e^{-xs/\mu} \leq ct\int_0^{1/t}\mathrm{d}x\frac{1}{x^{\gamma-3}} + c\int_{1/t}^\infty \mathrm{d}x\frac{1}{x^{\gamma-2}},$$

so that

$$\mathbb{E}[(X_t^{(1)})^2] \leq \frac{c}{(\gamma-3)(4-\gamma)}t^{\gamma-3}.$$

Applying Markov's inequality, we deduce that

$$t^{-(\gamma-3)} X_t^{(1)} \xrightarrow[t\to\infty]{p} 0. \tag{B.6}$$

Letting $\eta = (\gamma - 3)/2$, this implies that $t^{-(1+\eta)} X_t^{(1)} \xrightarrow{p} 0$ as $t \to \infty$. Then notice that $X_t^{(2)}$ is less than $\sum_{s\le t} \Delta_s$, where Δ is a Poisson point process with intensity $\mathbf{1}_{x\ge 1}\nu(\mathrm{d}s, \mathrm{d}x)$ [recall that $\nu(\mathrm{d}s, \mathrm{d}x) = \frac{c}{\mu}\frac{1}{x^{\gamma-1}} e^{-xs/\mu}\,\mathrm{d}s\,\mathrm{d}x$]. Now $\mathbb{E}[\sum_{s\le t}\Delta_s] = \frac{c}{\mu}\int_0^t \mathrm{d}s \int_1^\infty \mathrm{d}x \frac{1}{x^{\gamma-2}} e^{-xs/\mu} \le \frac{c}{\mu(\gamma-3)} t$. Consequently, by Markov's inequality, $t^{-(1+\eta)} \sum_{s\le t}\Delta_s \xrightarrow{p} 0$ as $t\to\infty$. Since $t^{-(1+\eta)} A_t^\nu \to -\infty$ as $t \to \infty$, property (1) is proved. □

PROOF OF LEMMA B.3(2). Restate (2) as follows: for every $\varepsilon > 0$,

$$\text{number of }\big(\text{excursion of } R^\nu \text{ with length } > 2\varepsilon\big) < \infty \qquad \text{a.s.}$$

Fix $\varepsilon > 0$ and define events $C_n = \{\sup_{s\in[(n-1)\varepsilon, n\varepsilon]} (X^\nu_{(n+1)\varepsilon} + A^\nu_{(n+1)\varepsilon} - X_s^\nu - A_s^\nu) > 0\}$. It is easily seen that it suffices to show that $\mathbb{P}(C_n \text{ infinitely often}) = 0$. By (B.6), it is enough to prove that

$$\sum_{n\ge 1+s_0/\varepsilon} \mathbb{P}\big(C_n \cap C^{s_0}\big) < \infty \qquad \text{for every large } s_0, \tag{B.7}$$

where $C^{s_0} = \{\sup_{t\ge s_0} t^{-(\gamma-3)} |X_t^{(1)}| \le \delta\}$ for some positive (small) constant $\delta > 0$ to be chosen later. Now

$$\begin{aligned} C_n \subset \Bigg\{ &\sup_{s\in[(n-1)\varepsilon,n\varepsilon]} \big(X^{(2)}_{(n+1)\varepsilon} - X_s^{(2)}\big) \\ &\ge \frac{c\Gamma(4-\gamma)}{(\gamma-3)(\gamma-2)\mu^{\gamma-2}} \varepsilon^{\gamma-2}\big((n+1)^{\gamma-2} - n^{\gamma-2}\big) \\ &\qquad - \sup_{s\in[(n-1)\varepsilon,n\varepsilon]} \big(X^{(1)}_{(n+1)\varepsilon} - X_s^{(1)}\big)\Bigg\}. \end{aligned}$$

For every n larger than $1 + s_0/\varepsilon$, on C^{s_0}, we have

$$\sup_{s\in[(n-1)\varepsilon,n\varepsilon]} \big(X^{(1)}_{(n+1)\varepsilon} - X_s^{(1)}\big) \le 2\delta\varepsilon^{\gamma-3}(n+1)^{\gamma-3} \le 2\delta\varepsilon^{\gamma-3} 2^{\gamma-3} n^{\gamma-3}.$$

Consequently, for every n larger than $1 + s_0/\varepsilon$,

$$\begin{aligned} C_n \cap C^{s_0} \subset \Bigg\{ &\sup_{s\in[(n-1)\varepsilon,n\varepsilon]} \big(X^{(2)}_{(n+1)\varepsilon} - X_s^{(2)}\big) \\ &\ge \left(\frac{c\Gamma(4-\gamma)}{(\gamma-3)\mu^{\gamma-2}} \varepsilon^{\gamma-2} - \delta\varepsilon^{\gamma-3} 2^{\gamma-2}\right) n^{\gamma-3}\Bigg\}. \end{aligned}$$

Taking $\delta = \varepsilon \frac{c\Gamma(4-\gamma)}{(\gamma-3)\mu^{\gamma-2}2^{\gamma-1}}$, and denoting $\frac{c\Gamma(4-\gamma)}{2(\gamma-3)\mu^{\gamma-2}}\varepsilon^{\gamma-2}$ by ρ, we thus have for every n large enough,

$$C_n \cap C^{s_0} \subset \Bigl\{ \sup_{s\in[(n-1)\varepsilon, n\varepsilon]} \bigl(X^{(2)}_{(n+1)\varepsilon} - X^{(2)}_s\bigr) \geq \rho n^{\gamma-3}\Bigr\}.$$

Now, considering a Poisson point process Δ with intensity $\mathbf{1}_{x\geq 1}\nu(\mathrm{d}s, \mathrm{d}x)$, where $\nu(\mathrm{d}s, \mathrm{d}x) = \frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xs/\mu}\,\mathrm{d}s\,\mathrm{d}x$, observe that

$$\begin{aligned}
&\mathbb{P}\Bigl(\sup_{s\in[(n-1)\varepsilon, n\varepsilon]} \bigl(X^{(2)}_{(n+1)\varepsilon} - X^{(2)}_s\bigr) \geq \rho n^{\gamma-3}\Bigr) \\
&\quad\leq \mathbb{P}\Bigl(\sum_{s\in[(n-1)\varepsilon,(n+1)\varepsilon]} \Delta_s \geq \rho n^{\gamma-3}\Bigr) \\
&\quad\leq \rho^{-1} n^{-\gamma+3}\mathbb{E}\Bigl[\sum_{s\in[(n-1)\varepsilon,(n+1)\varepsilon]} \Delta_s\Bigr] \\
&\quad= \rho^{-1} n^{-\gamma+3}\frac{c}{\mu}\int_{(n-1)\varepsilon}^{(n+1)\varepsilon} \mathrm{d}s \int_1^{\infty} \mathrm{d}x\, x\frac{1}{x^{\gamma-1}}e^{-xs/\mu}.
\end{aligned}$$

We deduce that for every n larger than $2 + s_0/\varepsilon$,

$$\mathbb{P}\bigl(C_n \cap C^{s_0}\bigr) \leq \frac{2\varepsilon c}{\rho\mu} n^{-\gamma+3}\int_1^{\infty} \mathrm{d}x\, x^{2-\gamma} e^{-nx\varepsilon/(2\mu)} \leq \frac{4c}{\rho} n^{-\gamma+2} e^{-n\varepsilon/(2\mu)},$$

which proves (B.7) and completes the proof of assertion (2). □

PROOF OF LEMMA B.3(3). To show property (3), we first consider the case $t = 0$. We aim at showing that $\inf\{s > 0 : X^{\nu}_s + A^{\nu}_s < 0\} = 0$ a.s. To do so, we shall in fact prove an analogous result for a certain Lévy process, which will be obtained by using standard properties of Lévy processes. We shall deduce property (3) by comparing our process X^{ν} with the studied Lévy process.

Observe that for every $s \in [0,\infty)$ and $x \in (0,\infty)$, $\frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xs/\mu} \leq \frac{c}{\mu}\frac{1}{x^{\gamma-1}}$. Recalling the two remarks situated at the beginning of the proofs of Lemmas B.1 and B.2 (we described there a way to define $X^{(1)}$ and $X^{(2)}$), we can couple the process X^{ν} and construct a stable process L with index $\gamma - 2$ with no negative jumps such that

$$\forall s \geq 0, \forall u \in \mathbb{R} \qquad \mathbb{E}\bigl[\exp(iuL_s)\bigr] = \exp\Bigl(s\int_0^{\infty} \mathrm{d}x\bigl(e^{iux} - 1 - iux\bigr)\frac{c}{\mu}\frac{1}{x^{\gamma-1}}\Bigr)$$

satisfying

$$\forall s \geq 0 \qquad X^{\nu}_s \leq L_s + \frac{c}{\mu}\int_0^{s} \mathrm{d}r \int_0^{\infty} \mathrm{d}x \frac{1}{x^{\gamma-2}}\bigl(1 - e^{-xr/\mu}\bigr),$$

that is,

$$\forall s \geq 0 \qquad X_s^\nu \leq L_s + \frac{c\Gamma(4-\gamma)}{(\gamma-3)(\gamma-2)\mu^{\gamma-2}} s^{\gamma-2}.$$

Consequently $X^\nu + A^\nu \leq L$. Since $\inf\{s > 0 : L_s < 0\} = 0$ a.s., with probability 1, 0 is not an isolated point of the set $\{t : R_t^\nu = 0\}$.

This is now standard to get assertion (3); see, for instance, [5], Proposition VI.4, or the end of the proof of assertion (d) of [2], Proposition 14. □

PROOF OF LEMMA B.3(4). Here again, we shall use a coupling argument. Indeed, imagine we are able to prove that for a certain process $(Q_s)_{s\in[0,t]}$,

$$\mathbb{P}(Q_t = \inf\{Q_s : s \in [0,t]\}) = 0,$$

and for every $s \in [0,t]$,

$$X_t^\nu + A_t^\nu - (X_s^\nu + A_s^\nu) \geq Q_t - Q_s.$$

Then, with probability one,

$$\sup\{X_t^\nu + A_t^\nu - (X_s^\nu + A_s^\nu) : s \in [0,t]\} \geq \sup\{Q_t - Q_s : s \in [0,t]\} > 0,$$

establishing assertion (4). Let us prove that such a coupling exists.

We have to bound the increments of $X^\nu + A^\nu$ from below. We first focus on $X^{(1)}$. Let $t \in (0,\infty)$. Arguing as before (just recall the remark made at the beginning of Lemma B.1), since for every $s \in [0,t]$ and $x \in (0,1)$, $\frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xs/\mu} \geq \frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xt/\mu}$, we can construct a Lévy process $(Q_s^{(1)})_{s\in[0,t]}$ such that

$$\mathbb{E}[\exp(iuQ_s^{(1)})] = \exp\left(s\int_0^1 \mathrm{d}x\,(e^{iux} - 1 - iux)\frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xt/\mu}\right) \qquad \forall s \in [0,t], \forall u \in \mathbb{R}$$

satisfying

$$X_t^{(1)} - X_s^{(1)} \geq Q_t^{(1)} - Q_s^{(1)} + \frac{c}{\mu}\int_s^t \mathrm{d}r \int_0^1 \mathrm{d}x\,\frac{1}{x^{\gamma-2}}(e^{-xt/\mu} - e^{-xr/\mu}) \qquad \forall s \in [0,t].$$

Since for every $a, b \in (0,\infty)$ such that $a < b$, $e^{-a} - e^{-b} \leq b - a$, we have for every $s \in [0,t]$

$$X_t^{(1)} - X_s^{(1)} \geq Q_t^{(1)} - Q_s^{(1)} - \frac{c}{2(4-\gamma)\mu^2}(t-s)^2.$$

Recalling the definition of B^ν [see (B.3)], we deduce that for every $s \in [0,t]$,

$$X_t^\nu - X_s^\nu \geq Q_t^{(1)} - Q_s^{(1)} - \frac{c}{2(4-\gamma)\mu^2}(t-s)^2 + B_t^\nu - B_s^\nu.$$

We easily deduce that there exists $C > 0$ (only depending on t) such that for every $s \in [0, t]$,

$$X_t^\nu + A_t^\nu - (X_s^\nu + A_s^\nu) \geq Q_t^{(1)} - Q_s^{(1)} - C(t - s).$$

Consequently,

$$\begin{aligned}\sup\{X_t^\nu + A_t^\nu - (X_s^\nu + A_s^\nu) : s \in [0, t]\} \\ \geq \sup\{Q_t^{(1)} - Ct - (Q_s^{(1)} - Cs) : s \in [0, t]\}.\end{aligned}$$

Now, applying [5], Theorem VII.2 and page 158, to the Lévy process $(Q_s^{(1)} - Cs)_{s\in[0,t]}$, we have

$$\mathbb{P}(Q_t^{(1)} - Ct = \inf\{Q_s^{(1)} - Cs : s \in [0, t]\}) = 0.$$

We deduce that

$$\mathbb{P}(X_t^\nu + A_t^\nu = \inf\{X_s^\nu + A_s^\nu : s \in [0, t]\}) = 0,$$

which is assertion (4). □

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REFERENCES

[1] ALDOUS, D. (1997). Brownian excursions, critical random graphs and the multiplicative coalescent. *Ann. Probab.* **25** 812–854. MR1434128

[2] ALDOUS, D. and LIMIC, V. (1998). The entrance boundary of the multiplicative coalescent. *Electron. J. Probab.* **3** 59 pp. (electronic). MR1491528

[3] ARRATIA, R., BARBOUR, A. D. and TAVARÉ, S. (2003). *Logarithmic Combinatorial Structures: A Probabilistic Approach*. European Mathematical Society (EMS), Zürich. MR2032426

[4] BENDER, E. A. and CANFIELD, E. R. (1978). The asymptotic number of labeled graphs with given degree sequences. *J. Combin. Theory Ser. A* **24** 296–307. MR0505796

[5] BERTOIN, J. (1996). *Lévy Processes. Cambridge Tracts in Mathematics* **121**. Cambridge Univ. Press, Cambridge. MR1406564

[6] BERTOIN, J. and SIDORAVICIUS, V. (2009). The structure of typical clusters in large sparse random configurations. *J. Stat. Phys.* **135** 87–105. MR2505727

[7] BHAMIDI, S., VAN DER HOFSTAD, R. and VAN LEEUWAARDEN, J. S. H. (2010). Scaling limits for critical inhomogeneous random graphs with finite third moments. *Electron. J. Probab.* **15** 1682–1703. MR2735378

[8] BHAMIDI, S., VAN DER HOFSTAD, R. and VAN LEEUWAARDEN, J. S. H. (2012). Novel scaling limits for critical inhomogeneous random graphs. *Ann. Probab.* **40** 2299–2361. MR3050505

[9] BOLLOBÁS, B. (1980). A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *European J. Combin.* **1** 311–316. MR0595929

[10] BOLLOBÁS, B. (2001). *Random Graphs*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **73**. Cambridge Univ. Press, Cambridge. MR1864966

[11] BRITTON, T., DEIJFEN, M. and MARTIN-LÖF, A. (2006). Generating simple random graphs with prescribed degree distribution. *J. Stat. Phys.* **124** 1377–1397. MR2266448

[12] ERDŐS, P. and RÉNYI, A. (1960). On the evolution of random graphs. *Magy. Tud. Akad. Mat. Kut. Intéz. Közl.* **5** 17–61. MR0125031

[13] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes*: *Characterization and Convergence*. Wiley, New York. MR0838085

[14] HATAMI, H. and MOLLOY, M. (2010). The scaling window for a random graph with a given degree sequence. In *Proceedings of the Twenty-First Annual ACM–SIAM Symposium on Discrete Algorithms* 1403–1411. SIAM, Philadelphia, PA. MR2809753

[15] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877

[16] JANSON, S. (2008). The largest component in a subcritical random graph with a power law degree distribution. *Ann. Appl. Probab.* **18** 1651–1668. MR2434185

[17] JANSON, S. (2009). The probability that a random multigraph is simple. *Combin. Probab. Comput.* **18** 205–225. MR2497380

[18] JANSON, S. and LUCZAK, M. J. (2009). A new approach to the giant component problem. *Random Structures Algorithms* **34** 197–216. MR2490288

[19] KANG, M. and SEIERSTAD, T. G. (2008). The critical phase for random graphs with a given degree sequence. *Combin. Probab. Comput.* **17** 67–86. MR2376424

[20] MOLLOY, M. and REED, B. (1995). A critical point for random graphs with a given degree sequence. *Random Structures Algorithms* **6** 161–179. MR1370952

[21] MOLLOY, M. and REED, B. (1998). The size of the giant component of a random graph with a given degree sequence. *Combin. Probab. Comput.* **7** 295–305. MR1664335

[22] PITTEL, B. G. (2008). On the largest component of a random graph with a subpower-law degree sequence in a subcritical phase. *Ann. Appl. Probab.* **18** 1636–1650. MR2434184

[23] TUROVA, T. S. (2013). Diffusion approximation for the components in critical inhomogeneous random graphs of rank 1. *Random Structures Algorithms* **43** 486–539. MR3124693

[24] VAN DER HOFSTAD, R. (2013). Critical behavior in inhomogeneous random graphs. *Random Structures Algorithms* **42** 480–508. MR3068034

[25] WORMALD, N. C. (1978). Some problems in the enumeration of labelled graphs. Ph.D. thesis, Newcastle Univ.

LABORATOIRE DE PROBABILITÉS ET MODÉLES ALÉATOIRES
UNIVERSITÉ PIERRE ET MARIE CURIE
4 PLACE JUSSIEU TOUR 16-26
75005 PARIS
FRANCE
E-MAIL: adrien.joseph@normalesup.org