On implicit and explicit discretization schemes for parabolic SPDEs in any dimension

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Abstract

We study the speed of convergence of the explicit and implicit space-time discretization schemes of the solution $u(t,x)$ to a parabolic partial differential equation in any dimension perturbed by a space-correlated Gaussian noise. The coefficients only depend on $u(t,x)$ and the influence of the correlation on the speed is observed.

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1 Introduction

Discretization schemes for parabolic SPDEs driven by the space-time white noise have been considered by several authors. I. Gyöngy and D. Nualart [9] and [10], have studied implicit time discretization schemes for the heat equation in dimension 1. J. Printems [15] has studied several time discretization schemes (implicit and explicit Euler schemes as well as the Crank-Nicholson one) for Hilbert-valued parabolic SPDEs, such as the Burgers equation on $[0,1]$, introduced several notions of order of convergence in order to deal with coefficients with polynomial growth and proved convergence in the Hilbert space norm. This work has been completed by E. Hausenblas [11], who studied several schemes for quasi-linear equations driven by a nuclear noise, and taking values in a Hilbert or a Banach space X. Several approximation procedures (such as the Galerkin approximation, finite difference methods or wavelets approximations) were considered, but the coefficients of the SPDE were supposed to depend on the whole function $u(t,.)$ in X, and not only on (t,x). Notice that, unlike [11], the coefficients considered in this paper do not depend on the whole function $u(s,.)$.

I. Gyöngy [7] has studied the strong speed of convergence in the norm of uniform convergence over the space variable for a space finite-difference scheme u^n with mesh $1/n$ for the parabolic SPDE with homogeneous Dirichlet's boundary conditions. He has also studied the speed of convergence of an implicit (resp. explicit) finite-difference discretization scheme $u^{n,m}$ (resp.

u^n_m) with time mesh T/m and space mesh $1/n$ for the solution u to the following parabolic SPDE in dimension 1 driven by the space-time white noise W:

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + \sigma(t,x,u(t,x))\frac{\partial^2 W}{\partial t \partial x} + b(t,x,u(t,x))\,, \\ u(t,0) = u(t,1) = 0\,, \end{array} \right. \tag{1.1}$$

with the initial condition u_0. He has proved that, if the coefficients $\sigma(t,x,.)$ and $b(t,x,.)$ satisfy the usual Lipschitz property uniformly in (t,x) and if the functions $\sigma(t,x,y)$ and $b(t,x,y)$ are 1/4-Hölder continuous in t and 1/2-Hölder continuous in x uniformly with respect to the other variables, then for $t \in]0,T]$, $p \in [1,+\infty[$, $0 < \beta < \frac{1}{4}$ and $0 < \gamma < \frac{1}{2}$ one has:

$$\sup_{x\in[0,1]} \mathbb{E}\left(|u^{n,m}(t,x) - u(t,x)|^p\right) \le K(t)\left(m^{-\beta p} + n^{-\gamma p}\right)\,. \tag{1.2}$$

Furthermore, if $u_0 \in \mathcal{C}^3([0,1])$, then (1.2) holds on $[0,T]$ with $\beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$ and with a constant K which does not depend on t. A similar result holds for the explicit scheme u^n_m if $\frac{n^2 T}{m} \le q < \frac{1}{2}$.

A. Debussche and J. Printems [5] have implemented simulations of a discretization scheme for the KDV equation, and C. Cardon-Weber [2] has studied explicit and implicit discretization schemes for the function-valued solution to the stochastic Cahn-Hilliard equation in dimension $d \le 3$ when the driving noise is the space-time white noise. The polynomial growth of the drift term made her require the diffusion coefficient σ to be bounded, and she proved convergence in probability (respectively in L^p with a given rate of a localized version) of the scheme.

In the present paper, we deal with a d-dimensional version of (1.1). As it is well-known, we can no longer use the space-time white noise for the perturbation; indeed, in dimension $d \ge 2$, the Green function associated with $\frac{\partial}{\partial t} - \Delta$ with the homogeneous Dirichlet boundary conditions on $[0,1]^d$ is not square integrable. Thus, we replace W by some Gaussian process F which is white in time and has a space correlation given by a Riesz potential $f(r) = r^{-\alpha}$, i.e., such that if A and B are bounded Borel subsets of $\mathbb{R}^d$, $E\big(F(s,A)\,F(t,B)\big) = (s\wedge t)\,\int_A dx \int_B dy |x-y|^{-\alpha}$ for some $\alpha \in]0, 2\wedge d[$. See e.g. [12], [4], [14] and [3] for more general results concerning necessary and sufficient conditions on the covariance of the Gaussian noise F ensuring the existence of a function-valued solution to (1.1) with F instead of W.

The aim of this paper is threefold. We at first study the speed of convergence of space and space-time finite discretization implicit (resp. explicit) schemes in dimension $d \ge 1$, i.e., on the grid $(\frac{iT}{m}, (\frac{j_k}{n}, 1 \le k \le d))$, $0 \le i \le m$, $0 \le j_k \le n$ and extended to $[0,T]\times[0,1]^d$ by linear interpolation. As in [7] and [8], the processes u^n and $u^{n,m}$ (resp. u^n_m) have an evolution formulation written in terms of approximations $(G_d)^n$, $(G_d)^{n,m}$ and $(G_d)^n_m$ of the Green function G_d, while u is solution of an evolution equation defined in terms of G_d. These evolution equations involve stochastic integrals with respect to the worthy martingale-measure defined by F (see e.g. [18] and [4]). As usual, the speed of convergence is given by the norm of the differences of stochastic integrals; more precisely, the optimal speed of convergence for the implicit scheme is the norm of the difference $G_d(.,x,.) - (G_d)^{n,m}(.,x,.)$ in $L^2([0,T],\mathcal{H}_d)$, where $\mathcal{H}_d$ is the Reproducing Kernel Hilbert Space defined by the covariance function. More precisely, if φ and ψ are continuous functions on $Q = [0,1]^d$, set

$$< \varphi, \psi >_{\mathcal{H}_d} = \int_Q \int_Q \varphi(x)\, f(|x-y|)\, \psi(y) dx\, dy\,. \tag{1.3}$$

We denote by $\mathcal{H}_d$ the completion of this pre-Hilbert space; note that $\mathcal{H}_d$ elements which are not functions and that a function φ belongs to $\mathcal{H}_d$ if and only if $\int_Q \int_Q |\varphi(y)|\, f(|y-z|)\, |\varphi(z)|\, dy\, dz < +\infty$. However, unlike in [7] and [8], the functions $\varphi_j(x) = \sqrt{2}\,\sin(j\pi x)$, $j \ge 1$ and $\varphi_j(\kappa_n(x))$,

$1 \leq j \leq n$, where $\kappa_n(y) = [ny]\, n^{-1}$ are not an orthonormal family of $\mathcal{H}_1$. Thus, even in dimension $d = 1$, the use of the Parseval identity has to be replaced by more technical computations based on Abel's summation method. Similar results could be obtained for a more general space covariance, provided that it is absolutely continuous and that its density f satisfies some integrability property at the origin (see e.g. [4], [14]). However, the speed of convergence would depend on integrals including f, which would make the results less transparent than that stated in the case of Riesz potentials. The key technical lemmas, giving upper estimates of $\|G_d(.,x,.) - (G_d)^n(.,x,.)\|_{L^2([0,\infty[,\mathcal{H}_d)}$ and $\|(G_d)^n(.,x,.) - (G_d)^{n,m}(.,x,.)\|_{L^2([0,T],\mathcal{H}_d)}$ (resp. $\|(G_d)^n(.,x,.) - (G_d)^n_m(.,x,.)\|_{L^2([0,T],\mathcal{H}_d)}$), are proved in section 4.

We describe the discretization schemes in any dimension $d \geq 1$ and introduce some notations in section 2. In section 3, an argument similar to that in [7] shows that for $0 < \alpha < d \wedge 2$, and $p \in [1, +\infty[$, if u_0 is regular enough, then

$$\sup_{(t,x)\in[0,+\infty[\times Q} \mathbb{E}(\|u(t,x) - u^n(t,x)\|^{2p}) \leq C_{p,\alpha}\, n^{-(2-\alpha)p}\,, \tag{1.4}$$

and extending [8] we prove in section 4 that

$$\sup_{(t,x)\in[0,T]\times Q} \mathbb{E}(\|u^n(t,x) - u^{n,m}(t,x)\|^{2p}) \leq C_{p,\alpha}\, m^{-(1-\frac{\alpha}{2})p}\,. \tag{1.5}$$

If $d = 1$, as $\alpha \nearrow 1$ the space density becomes more and more degenerate and the speed of convergence approaches that obtained by Gyöngy for the space-time white noise.

In dimension $d \geq 2$, the proof depends on the product form of the Green function and its approximations, as well as of upper estimates of $|x-y|^{-\alpha}$ in terms of $\prod_{i=1}^d |x_i - y_i|^{-\alpha_i}$ for some well-chosen α_i. Thus, estimates of the $\mathcal{H}_d$-norm of the differences of $G_d(s,x,.) - (G_d)^n(s,x,.)$, $(G_d)^n(s,x,.) - (G_d)^{n,m}(s,x,.)$ and $(G_d)^n(s,x,.) - (G_d)^n_m(s,x,.)$ in dimension $d \geq 2$ depend on bounds of the $\mathcal{H}_1$-norm of similar differences as well as of $\mathcal{H}_r$-norms of $G(s,x,.)$, $G^n(s,x,.)$ and $G^{n,m}(s,x,.)$ for $r < d$.

Section 5 contains some numerical results. For $T = 1$, we have implemented in C the (more stable) implicit discretization scheme for affine coefficients $\sigma(t,y,u) = \sigma_1\, u + \sigma_2$ and $b(t,x,u) = b_1\, u + b_2$ and for $\sigma(t,y,u) = b(t,y,u) = a + b\, \cos(u)$. We have studied the "experimental" speed of convergence with respect to one mesh, when the other one is fixed and gives a "much smaller" theoretical error. The second moments are computed by Monte-Carlo approximations. These implementations have been done in dimension $d = 1$ for the space-time white noise W and the colored noise F. As expected, the observed speeds are better than the theoretical ones, and decrease with α. For example, choosing N and M "large" with $M \geq N^2$ and considering "small" divisors n of N, we have computed the observed linear regression coefficient and drawn the curves of $\sup_{x\in[0,1]} \ln(E(|u^{n,M}(1,x) - u^{N,M}(1,x)|^2))$ as a function of $\ln(n)$ for various values of α.

Note that all the results of this paper remain true if in (1.1) we replace the homogeneous Dirichlet boundary conditions $u(t,x) = 0$ for $x \in \partial Q$ by the homogeneous Neumann ones $\frac{\partial u}{\partial x}(t,x) = 0$ for $x \in \partial Q$. In this last case, the eigenfunctions of $\frac{\partial}{\partial t} - \Delta$ in dimension one is $\varphi_0(x) = 1$ and for $j \geq 1$, $\varphi_j(x) = \sqrt{2}\, \cos(j\pi x)$. Since the upper estimates of the partial sums $\sum_{j=1}^K \varphi_j(x)$ used in the Abel transforms still hold in the case of Neumann's conditions, the crucial result is proved in a similar way in this case, and the speed of convergence is preserved.

2 Formulation of the problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $Q = [0,1]^d$ for some integer $d \geq 1$ and let $F = \big(F(\varphi)\,,\, \varphi \in \mathcal{D}(\mathbb{R}_+ \times Q)\big)$ be an $L^2(P)$-valued centered Gaussian process, which is white in time but has a space correlation defined as follows: given φ and ψ in $\mathcal{D}(\mathbb{R}_+ \times Q)$, the covariance functional of $F(\varphi)$ and $F(\psi)$ is

$$J(\varphi, \psi) = E\big(F(\varphi)\, F(\psi)\big) = \int_0^{+\infty} dt \int\int_{(Q-Q)^*} \varphi(t,y) f(y-z) \psi(t,z) dy dz, \tag{2.1}$$

where $(Q-Q)^* = \{y - z \,:\, y, z \in Q,\, y \neq z\}$ and $f : (Q-Q)^* \to [0, +\infty[$ is a continuous function. The bilinear form J defined by (2.1)is non-negative definite if and only if f is the Fourier transform of a non-negative tempered distribution μ on Q. Then F defines a martingale-measure (still denoted by F), which allows to use stochastic integrals (see [18]). In the sequel, we suppose that for $z \in \mathbb{R}^d$, $z \neq 0$, $f(z) = |z|^{-\alpha}$, where $|z|$ denotes the Euclidean norm of the vector z. Since $x^2 + y^2 \geq 2xy$, if $\alpha_j = \alpha 2^{-j}$ for $1 \leq j < d$ and $\alpha_d = \alpha 2^{-d+1}$, there exists a positive constant C such that for any $z = (z_1, \cdots, z_d) \in \mathbb{R}^d$,

$$f(z) \leq C \prod_{1 \leq j \leq d} f_{\alpha_j}(z_j), \tag{2.2}$$

where $f_\alpha(\zeta) = |\zeta|^{-\alpha}$ for any $\zeta \in \mathbb{R}$, $\zeta \neq 0$. To lighten the notations, for this choice of f and $\varphi \in \mathcal{H}_d$ set

$$\|\varphi\|^2_{(\alpha)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi(y)|\, |y - z|^{-\alpha}\, |\varphi(z)|\, dy dz\,. \tag{2.3}$$

For any $t \geq 0$, we denote by $\mathcal{F}_t$ the sigma-algebra generated by $\{F([0,s] \times A) : 0 \leq s \leq t\,,\, A \subset Q\}$. Let $\sigma : [0, +\infty[\times Q \times \mathbb{R} \to \mathbb{R}$ and $b : [0, +\infty[\times Q \times \mathbb{R} \to \mathbb{R}$. Suppose that there exists a positive constant C such that for $s, t \in [0, \infty[$, $x, y \in Q$, $r, v \in \mathbb{R}$, the linear growth condition (2.4) and either Lipschitz condition (2.5), (2.6) or (2.7) hold

$$|\sigma(t,x,r)| + |b(t,x,r)| \leq C\,(1 + |r|)\,, \tag{2.4}$$

and for $D(s,t,x,y,r,v) = |\sigma(s,x,r) - \sigma(t,y,v)| + |b(s,x,r) - b(t,y,v)|$

$$D(t,t,x,x,r,v) \leq C|r - v|, \tag{2.5}$$

$$D(t,t,x,y,r,v) \leq C(|x-y|^{1-\frac{\alpha}{2}} + |r-v|), \tag{2.6}$$

$$D(s,t,x,y,r,v) \leq C(|t-s|^{\frac{1}{2}-\frac{\alpha}{4}} + |x-y|^{1-\frac{\alpha}{2}} + |r-v|). \tag{2.7}$$

For any function u_0 which vanishes on the boundary of Q, let $u(t,x)$ denote the solution to the parabolic SPDE, which is similar to (1.1)

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \sigma(t,x,u(t,x)) \frac{\partial^2 F}{\partial t \partial x} + b(t,x,u(t,x))\,, \\ u(t,x) = 0 \quad \text{for} \quad x \in \partial Q\,, \end{cases} \tag{2.8}$$

with initial condition $u(0,x) = u_0(x)$. Let $\mathbb{N}^*$ denote the set of strictly positive integers. For any $j \in \mathbb{N}^*$ and $\xi \in \mathbb{R}$, set $\varphi_j(\xi) = \sqrt{2}\sin(j\pi\xi)$ and for $\underline{k} = (k_1, \cdots, k_d) \in \mathbb{N}^{*d}$, set

$$|\underline{k}| = \sum_{j=1}^d k_j, \quad \varphi_{\underline{k}}(x) = \prod_{j=1}^d \varphi_{k_j}(x_{k_j}) \quad \text{for} \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Let $G_d(t,x,y)$ denote the Green function associated with $\frac{\partial}{\partial t} - \Delta$ on Q and homogeneous Dirichlet boundary conditions; then for $t>0$, $x,y\in Q$, $G_d(t,x,y) = \sum_{\underline{k}\in\mathbb{N}^{*d}} \exp(-|\underline{k}|^2\pi^2 t)\varphi_{\underline{k}}(x)\varphi_{\underline{k}}(y)$ and

$$|G_d(t,x,y)| \le C t^{-\frac{d}{2}} \exp\big(-c\frac{|x-y|^2}{t}\big). \tag{2.9}$$

When $d=1$, set $G_1=G$. These upper estimates are classical when the domain Q has a smooth boundary under either homogeneous Neumann's or Dirichlet's boundary conditions (see e.g. [6], [13]). A simple argument shows that they can be extended for these homogeneous conditions on the set $Q=[0,1]^d$; see e.g. [1] for the similar case of the parabolic operator $\frac{\partial}{\partial t}+\Delta^2$ on $[0,\pi]^d$. The equation (2.8) makes sense in the following evolution formulation (see e.g. [18] for $d=1$):

$$\begin{aligned} u(t,x) = \int_Q G_d(t,x,y)\, u_0(y)\, dy + \int_0^t \int_Q G_d(t-s,x,y) \\ \times [\sigma(s,y,u(s,y))\, F(ds,dy) + b(s,y,u(s,y))\, dsdy]\ . \end{aligned} \tag{2.10}$$

We also consider the parabolic SPDE with the homogeneous boundary conditions $\frac{\partial u}{\partial x}(t,x)=0$ for $x\in\partial Q$. Then the functions $(\varphi_j\,;\, j\ge 1)$ are replaced by $\varphi_0(\xi)=1$ and $\varphi_j(\xi)=\sqrt{2}\cos(j\pi\xi)$ for $\xi\in\mathbb{R}$ and $j\ge 1$. All the other formulations remain true with $\underline{k}\in\mathbb{N}^d$ instead on $\mathbb{N}^{*d}$.

2.1 Space discretization scheme

As in [7], we at first consider a finite space discretization scheme, replacing the Laplacian by its discretization on the grid $\frac{\underline{k}}{n} = (\frac{k_1}{n},\cdots,\frac{k_d}{n})$, where $k_j\in\{0,\cdots,n\}$, $1\le j\le d$. In dimension 1, we proceed as in [7], and consider the $(n-1)\times(n-1)$-matrix D_n associated with the homogeneous Dirichlet boundary conditions and defined by $D_n(i,i)=-2$, $D_n(i,j)=1$ if $|i-j|=1$ and $D_n(i,j)=0$ for $|i-j|\ge 2$; then $\frac{\partial^2 u(t,x)}{\partial x^2}$ is replaced by $n^2\, D_n\vec{u}_n(t,.)$, where $\vec{u}^n(t)$ denotes the $(n-1)$-dimensional vector of an approximate solution defined on the grid $j/n, 1\le j\le n$. In arbitrary dimension, we proceed as in [2] and define $D_n^{(d)}$ by induction. Let $D_n^{(1)}=D_n$ and suppose that $D_n^{(d-1)}$ has been defined as a $(n-1)^{d-1}\times(n-1)^{d-1}$ matrix. Let Id_k denotes the $k\times k$ identity matrix and given a $(n-1)^{d-1}\times(n-1)^{d-1}$ matrix A, let $diag(A)$ denote the $(n-1)^d\times(n-1)^d$ matrix with $d-1$ diagonal blocs equal to A; let $D_n^{(d)}$ denote the $(n-1)^d\times(n-1)^d$-matrix $D_n^{(d)}$ defined by

$$D_n^{(d)} = diag(D_n^{(d-1)}) + \begin{pmatrix} -2Id_{n^{d-1}} & Id_{n^{d-1}} & 0 & \cdots & 0 \\ Id_{n^{d-1}} & -2Id_{n^{d-1}} & Id_{n^{d-1}} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & Id_{n^{d-1}} & -2Id_{n^{d-1}} & Id_{n^{d-1}} \\ 0 & \cdots & 0 & Id_{n^{d-1}} & -2Id_{n^{d-1}} \end{pmatrix}.$$

Let $\vec{u}^n(t)$ denote the $(n-1)^d$dimensional vector defined by $\vec{u}^n(t)_{\underline{\mathbf{k}}} = u_n(t,\underline{\mathbf{x}_\mathbf{k}})$, with $\underline{\mathbf{x}_\mathbf{k}} = (x_{k_1},\cdots,x_{k_d})$, where k_j is the unique integer such that $x_{k_j}=\frac{k_j-1}{n}$ and $k_j\in\{1,\cdots,n-1\}$ is such that $\underline{\mathbf{k}} = (k_d-1)(n-1)^{d-1}+\cdots+(k_2-1)(n-1)+k_1$. Let $\mathcal{L}=\{\underline{\mathbf{x}_\mathbf{k}} : \underline{\mathbf{k}}\in\{1,\cdots,(n-1)^d\}\}$, $\Box\underline{\mathbf{x}_\mathbf{k}}$ be the lattice parallepiped of diagonal $\underline{\mathbf{x}_\mathbf{k}}=(x_{k_1},\cdots,x_{k_d})$ and $(x_{k_1}+\frac{1}{n},\cdots,x_{k_d}+\frac{1}{n})$, and set $F^n(t,\underline{\mathbf{x}_\mathbf{k}}) = \int_{\Box\underline{\mathbf{x}_\mathbf{k}}} dF(t,x)$. Given a function $h:[0,+\infty[\times Q\times\mathbb{R}\to\mathbb{R}$, and $\vec{u}\in\mathbb{R}^r$, let $h(t,x,\vec{u}) = (h(t,x,u_1),\cdots,h(t,x,u_r))$. Then $\vec{u}^n(t)$ is solution to the following equation

$$d\vec{u}^n(t) = n^2\, D_n^{(d)}\vec{u}^n(t)\, dt + n\,\sigma(t,x,\vec{u}^n(t))\, dF(t,.), +b(t,x,\vec{u}^n(t)), \tag{2.11}$$

$\vec{u}^n(0) = \big(u_0\left(\frac{j}{n}\right), j \in \{1, \cdots, n-1\}^d\big)$. We then complete $u^n(t,.)$ from the lattice $\mathcal{L}$ to Q as follows. If $d=1$, set $u^n(t,0) = u^n(t,1) = 0$, $u^n(t, \frac{j}{n}) = \vec{u}^n(t)_j$, $\kappa_n(y) = [ny]/n$, $\varphi_j^n(i/n) = \varphi_j(i/n)$ for $0 \le i \le n$, and for $x \in]i/n, (i+1)/n[$, $0 \le i < n$, $1 \le j \le n-1$, let $\varphi_j^n(x) = \varphi_j\left(\frac{i}{n}\right) + (nx - i)\left[\varphi_j\left(\frac{i+1}{n}\right) - \varphi_j\left(\frac{i}{n}\right)\right]$, and let

$$\lambda_j^n = -4\sin^2\left(\frac{j\pi}{2n}\right) n^2 = -j^2\pi^2 c_n^j \quad \text{with} \quad c_n^j = \sin^2\left(\frac{j\pi}{2n}\right)\left(\frac{j\pi}{2n}\right)^{-2} \in \left[\frac{4}{\pi^2}, 1\right],$$

denote the eigenvalues of $n^2\, D_n = n^2\, D_n^{(1)}$; then for $t > 0$, $x, y \in [0,1]$,

$$(G_1)^n(t,x,y) = \sum_{j=1}^{n-1} \exp(\lambda_j^n\, t)\varphi_j^n(x)\, \varphi_j(\kappa_n(y))\,. \tag{2.12}$$

In dimension $d \ge 2$, we also complete the solution $u^n(t,x)$ from $x \in \mathcal{L}$, defined as $u^n(t, \underline{\mathbf{x}}_{\mathbf{k}}) = \vec{u}^n(t)$ to $x \in Q$ by linear interpolation, interpolating inductively on the points (x,y) for $x \in \mathbb{R}^i$ and $y = (k_{i+1}/n, \cdots, k_d/n)$. The eigenvalues and eigenvectors of $n^2\, D_n^{(d)}$ are $\lambda_{\underline{\mathbf{k}}}^n = \sum_{j=1}^d \lambda_{k_j}^n$, and $\varphi_{\underline{\mathbf{k}}}\left(\frac{k_1\pi}{n}, \cdots, \frac{k_d\pi}{n}\right)$. For $t > 0$, x and $y \in Q$ if $\kappa_n(y) = (\kappa_n(y_1), \cdots, \kappa_n(y_d))$, let

$$(G_d)^n(t,x,y) = \sum_{\underline{\mathbf{k}} \in \{1,\cdots,(n-1)^d\}} \exp(\lambda_{\underline{\mathbf{k}}}^n t)\varphi_{\underline{\mathbf{k}}}^n(x)\varphi_{\underline{\mathbf{k}}}^n(\kappa_n(y)). \tag{2.13}$$

When $d=1$, simply set $G_1 = G$ and $(G_1)^n = G^n$. Then the linear interpolation of $u^n(t,.)$ from the lattice $\mathcal{L}$ to $Q = [0,1]^d$ is solution to the evolution equation

$$\begin{aligned} u^n(t,x) = &\int_Q (G_d)^n(t,x,y)\, u_0(\kappa_n(y))\, dy + \int_0^t \int_Q (G_d)^n(t-s,x,y) \\ &\times \Big[\sigma(s,\kappa_n(y), u^n(s,\kappa_n(y))F(ds,dy) + b(s,\kappa_n(y), u^n(s,\kappa_n(y))\, dsdy\Big]. \end{aligned} \tag{2.14}$$

The $n \times n$ matrix $D_n = D_n^{(1)}$ associated with the homogeneous Neumann boundary conditions is defined by $D_n(1,1) = D_n(n,n) = -1$, $D_n(1,2) = D_n(n,n-1) = 1$ and for $2 \le i \le n-1$ and $1 \le j \le n$, $D_n(i,i) = -2$, $D_n(i,j) = 1$ if $|j-i| = 1$ and $D_n(i,j) = 0$ for $|j-i| \ge 2$. The inductive procedure used to construct $D_n^{(d)}$ is similar to the previous one, replacing 1 by Id_{n^d}. Then the eigenvalues of $n^2\, D_n$ are $\lambda_j^n = -4\, n^2\, \sin^2(\frac{j\pi}{2n}) = -j^2\, \pi^2\, \tilde{c}_n^j$ with $\tilde{c}_n^j \in [\frac{2}{\pi^2}, 1]$. The corresponding normed eigenvectors $(e_j, 0 \le j \le n-1)$ are again evaluations of φ_j. More precisely, $e_j(k) = \frac{1}{\sqrt{n}}\, \varphi_j(\frac{2k-1}{2n})$ for $0 \le j \le n-1$ and $1 \le k \le n$. The eigenvalues $\lambda_n^{\underline{\mathbf{k}}}$ and the eigenfunctions $\varphi_{\underline{\mathbf{k}}}$ of $n^2\, D_n^{(d)}$ are defined in a way similar to the Dirichlet case, taking sums over $\underline{\mathbf{k}} \in \{1, \cdots, n^d\}$. Formulas similar to (2.12) and (2.13) still hold and (2.14) is unchanged.

2.2 Implicit space-time discretization scheme

We now introduce a space-time discretization scheme. Given $T > 0$, $n, m \ge 1$ we use the space mesh $1/n$ and the time mesh T/m, set $t_i = iTm^{-1}$ for $0 \le i \le m$ and replace the time derivative by a backward difference. Thus for $d = 1$, in the case of Dirichlet's homogeneous boundary conditions, set $\vec{u}_0 = (u_0(j/n), 1 \le j \le n-1)$ and for $i \le m$, set $\vec{u}_i = (u^{n,m}(iTm^{-1}, jn^{-1}), 1 \le j \le n-1)$, and for $g = \sigma$ and $g = b$ let $g(t_i, ., \vec{u}_i) = g((t_i, jn^{-1}, (u^{n,m}(t_i, jn^{-1}))\,, 1 \le j \le n-1)$. Let $\square_{n,m}F(t_i,.)$ denote the $(n-1)$-dimensional Gaussian vector of space-time increments of F on the space-time grid, i.e., for $1 \le j \le n-1$, set

$$\square_{n,m}F(t_i,j) = nmT^{-1}[F(t_{i+1},(j+1)n^{-1}) - F(t_i,(j+1)n^{-1}) - F(t_{i+1},jn^{-1}) + F(t_i,jn^{-1})];$$

then for every $0 \le i < m$

$$\vec{u}_{i+1} = \vec{u}_i + n^2 \frac{T}{m} D_n \vec{u}_{i+1} + Tm^{-1} \left[\sigma(t_i, ., \vec{u}_i) \Box_{n,m} F(t_i, .) + b(t_i, ., \vec{u}_i) \right] . \tag{2.15}$$

Since $Id - Tm^{-1} D_n$ is invertible,

$$\vec{u}_{i+1} = (Id - \frac{T}{m} D_n)^{-(i+1)} \vec{u}_0 + \sum_{k=0}^{i} (Id - \frac{T}{m} D_n)^{-(i-k-1)} \left[\sigma(t_k, ., \vec{u}_k) \Box_{n,m} F(t_k, .) + b(t_k, ., \vec{u}_k) \right] . \tag{2.16}$$

If $d \ge 2$, we set $\Box_{n,m} F(t_i, \underline{\mathbf{x}}_{\underline{\mathbf{k}}}) = n^d m T^{-1} \int_{t_i}^{t_{i+1}} \int_{\Box \underline{\mathbf{x}}_{\underline{\mathbf{k}}}} dF(t,x)$, and for homogeneous Dirichlet's (resp. Neumann's) boundary conditions, define similarly $\vec{u}_{i+1}$ as the $(n-1)^d$-dimensional (resp. n^d-dimensional) vector such that (2.16) holds with $D_n^{(d)}$ instead of D_n. We only describe the scheme in the case of Dirichlet's conditions; the case of Neumann's conditions is obviously dealt with by obvious changes. The process $u^{n,m}$ is defined on the space-time lattice $\mathcal{L}_T = \big\{ (t_i, \underline{\mathbf{x}}_{\underline{\mathbf{k}}}) : 0 \le i \le m \,, \underline{\mathbf{k}} \in \{1, \cdots, (n-1)^d\} \big\}$ as $(u^{n,m}(t_i, \underline{\mathbf{x}}_{\underline{\mathbf{k}}}) \,, 0 \le i \le m \,, \underline{\mathbf{k}} \in \{1, \cdots, (n-1)^d\}) = \vec{u}_i$; it is then extended to the time lattice (t_i, x), $0 \le i \le m$, $x \in Q$ as in the previous subsection, and then extended to $[0,T] \times Q$ by time linear interpolation. Since $\lambda_{\underline{\mathbf{k}}} = \sum_{i=1}^d \lambda_{k_i}^n$ and $\varphi_{\underline{\mathbf{k}}}(\underline{\mathbf{x}}_{\underline{\mathbf{k}}})$ are the eigenvalues and eigenvectors of $D_n^{(d)}$, if

$$(G_d)^{n,m}(t,x,y) = \sum_{\underline{\mathbf{k}} \in \{1,\cdots,(n-1)^d\}} \big(1 - Tm^{-1} \lambda_{\underline{\mathbf{k}}} \big)^{-[mtT^{-1}]} \; \varphi_{\underline{\mathbf{k}}}^n(x) \; \varphi_{\underline{\mathbf{k}}}(\kappa_n(y)) \,, \tag{2.17}$$

then for $t = iTm^{-1}$, $1 \le i \le m$, if for $s \in [0,T]$, one sets $\Lambda_m(s) = [msT^{-1}] m^{-1}$ one has:

$$\begin{aligned} u^{n,m}(t,x) = & \int_Q (G_d)^{n,m}(t,x,y) u_0(\kappa_n(y)) dy + \int_0^t \int_Q (G_d)^{n,m}(t - s + \frac{T}{m}, x, y) \\ & \times \Big[\sigma(\Lambda_m(s), \kappa_n(y), u^{n,m}(\Lambda_m(s), \kappa_n(y))) F(ds,dy) + b(\Lambda_m(s), \kappa_n(y), u^{n,m}(\Lambda_m(s), \kappa_n(y))) dy ds \Big]. \end{aligned} \tag{2.18}$$

Again for $d = 1$, let $G^{n,m} = (G_1)^{n,m}$.

2.3 Explicit schemes

For $T > 0$, a space mesh n^{-1} and a time mesh Tm^{-1}, we now replace the time derivative by a forward difference. Thus if u_m^n denotes the approximating process defined for $t = t_i = iTm^{-1}$ and $x_{k_j} \in \{1, \cdots, n-1\}$, setting $\vec{u}_i = u_m^n(t_i, .)$, we have

$$\vec{u}_{i+1} = \vec{u}_i + n^2 T m^{-1} D_n^{(d)} \vec{u}_i + Tm^{-1} \left[\sigma(t_i, ., \vec{u}_i) \Box_{n,m} F(t_i, .) + b(t_i, ., \vec{u}_i) \right] . \tag{2.19}$$

In the case of homogeneous Dirichlet boundary conditions, let $(G_d)_m^n(t,x,y)$ denote the corresponding approximation of the Green function G_d defined by

$$(G_d)_m^n(t,x,y) = \sum_{\underline{\mathbf{k}} \in \{1,\cdots,(n-1)^d\}} \big(1 + Tm^{-1} \lambda_{\underline{\mathbf{k}}} \big)^{[mtT^{-1}]} \; \varphi_{\underline{\mathbf{k}}}^n(x) \; \varphi_{\underline{\mathbf{k}}}(\kappa_n(y)). \tag{2.20}$$

Again for $d = 1$, let $G_m^n = (G_1)_m^n$. Then for $t = t_i = iTm^{-1}$, when completing the solution $u_m^n(t_i, .)$ from the space lattice $\mathcal{L}$ to Q, we obtain the solution to the following equation

$$u_m^n(t,x) = \int_Q (G_d)_m^n(t,x,y) u_0(\kappa_n(y)) dy + \int_0^t \int_Q (G_d)_m^n \big(t - s + T/m, x, y \big)$$

$$\times\Big[\sigma(\Lambda_m(s),\kappa_n(y),u^n_m(\Lambda_m(s),\kappa_n(y)))F(ds,dy)+b(\Lambda_m(s),\kappa_n(y),u^n_m(\Lambda_m(s),\kappa_n(y)))dyds\Big]. \tag{2.21}$$

We then complete the process $u^n_m(.,x)$ by time linear interpolation and obvious changes yield the explicit scheme for homogeneous Neumann boundary conditions.

3 Convergence results for the discretization schemes

In this section, we study the speed of convergence for the d-dimensional space scheme and then of the d-dimensional implicit and explicit space-time schemes. For the sake of simplicity, we only write the proofs in the case of homogeneous Dirichlet boundary conditions. The following result states that the solutions u, u^n, $u^{n,m}$ and u^n_m exist and have bounded moments uniformly in n,m. The proofs for u can be found in [14]; see also [4] and [3]. The arguments for the approximations are similar using (A.9), (A.10), (A.13) and (A.14) and the version of Gronwall's lemma in stated in [8] Lemma 3.4.

Proposition 3.1 *Let $u_0\in\mathcal{C}(Q)$ satisfy the homogeneous Neumann or Dirichlet boundary conditions, and suppose that the coefficients σ and b satisfy the conditions (2.4) and (2.5); then the equation (2.10) (resp. (2.14), (2.18) and (2.21)) has a unique solution u (resp. u^n, $u^{n,m}$ and u^n_m) such that for every $p\in[1,+\infty[$ and $T>0$:*

$$\sup_{n,m}\sup_{0\le t\le T}\sup_{x\in Q}\mathbb{E}\big(|u(t,x)|^{2p}+|u^n(t,x)|^{2p}+|u^{n,m}(t,x)|^{2p})+|u^n_m(t,x)|^{2p}\big)<\infty\,. \tag{3.1}$$

We now prove Hölder regularity properties of the trajectories of u and u^n. Note that for u, a similar result has been proved in [17] for the heat equation with free boundary with a perturbation driven by a Gaussian process with a more general space covariance; see also [3] for a related result in the case of a more general even order differential operator.

Proposition 3.2 *Suppose that the coefficients b and σ satisfy the Lipschitz property (2.5), that the initial condition u_0 satisfies the homogeneous Dirichlet or Neumann boundary condition.*

(i) Suppose furthermore that $u_0\in\mathcal{C}^{1-\frac{\alpha}{2}}(Q)$ and fix $T>0$. Then, for every $p\in[1,+\infty[$, there exists a constant C such that for $x,x'\in Q$ and $0\le t<t'\le T$,

$$\sup_{0\le t\le T}\mathbb{E}\left(|u(t,x)-u(t,x')|^{2p}\right)\le C|x'-x|^{p(2-\alpha)}\,, \tag{3.2}$$

$$\sup_{x\in Q}\mathbb{E}\left(|u(t',x)-u(t,x)|^{2p}\right)\le C|t'-t|^{p(1-\frac{\alpha}{2})}\,. \tag{3.3}$$

(ii) Suppose furthermore that $u_0\in\mathcal{C}^2(Q)$; then for every $p\in[1,+\infty[$, there exists a constant C such that for $x,x'\in Q$ and $0\le t<t'\le T$,

$$\sup_{n\ge1}\sup_{x\in Q}\mathbb{E}\left(|u^n(t',x)-u^n(t,x)|^{2p}\right)\le C|t'-t|^{p(1-\frac{\alpha}{2})} \tag{3.4}$$

$$\sup_{n\ge1}\sup_{0\le t\le T}\mathbb{E}\left(|u^n(t,x')-u^n(t,x)|^{2p}\right)\le C|x'-x|^{p(2-\alpha)} \tag{3.5}$$

Proof: The proofs of (3.2) and (3.3) can be adapted from Sanz-Sarrà [17] (see also [3]), and are therefore omitted. For the sake of completeness, we sketch the proof of (3.4). For every

$t>0$, let $v^n(t,x)=\int_Q (G_d)^n(t,x,y)\,u_0(\kappa_n(y)\,dy$ and $w^n(t,x)=u^n(t,x)-v^n(t,x)$, where $(G_d)^n$ is the fundamental solution of $\frac{\partial}{\partial t}-\Delta_n=0$,

$$\Delta_n U(y) = n^2\sum_{i=1}^d \Big[U\Big(\sum_{j\neq i}\frac{[ny_j]}{n}e_j+\frac{[ny_i]+1}{n}e_i\Big)-2U\Big(\frac{[ny]}{n}\Big)+U\Big(\sum_{j\neq i}\frac{[ny_j]}{n}e_j+\frac{[ny_i]-1}{n}e_i\Big)\Big], \tag{3.6}$$

and $(e_i, 1\le i\le d)$ denotes the canonical basis of $\mathbb{R}^d$. Then if $u_0\in\mathcal{C}^2(Q)$,

$$v^n(t,x)=u_0(x)+\int_0^t\int_Q (G_d)^n(s,x,y)\,\Delta_n u_0(y)\,dy\,ds\,. \tag{3.7}$$

Using the fact that $\Delta_n u_0$ is bounded if $u_0\in\mathcal{C}^2(Q)$, and (A.22), we deduce that for any $\lambda>0$

$$\sup_{n\ge 1}\sup_{x\in Q}|v_n(t,x)-v_n(t',x)|\le C\,|t'-t|^{1-\lambda}\,. \tag{3.8}$$

Computations similar to those used in [17], using Burkholder's and Hölder's inequalities with respect to suitable measures, (A.22)-(A.23) and (3.1), show the existence of $C_p>0$ such that for any $0\le t<t'\le T$

$$\sup_{n\ge 1}\sup_{x\in Q}\mathbb{E}(|w^n(t,x)-w^n(t',x)|^{2p})\le C_p\,|t'-t|^{p(1-\frac{\alpha}{2})}\,. \tag{3.9}$$

The inequalities (3.8) and (3.9) conclude the proof of (3.4). □

The first convergence result of this section is that of u^n to u.

Theorem 3.3 *Let σ and b satisfy the conditions (2.4) and (2.6), u and u^n be the solutions to (2.10) and (2.14) respectively, where the Green functions G_d and $(G_d)^n$ are defined with the homogeneous Neumann or Dirichlet boundary conditions on Q.*
(i) If the initial condition u_0 belongs to $\mathcal{C}^3(Q)$, then for every $T>0$ and $p\in[1+\infty[$, there exists a constant $C_p(T)>0$ such that:

$$\sup_{(t,x)\in[0,T]\times Q}\mathbb{E}\left(|u(t,x)-u^n(t,x)|^{2p}|\right)\le C_p(T)\,n^{-(2-\alpha)\,p}\,. \tag{3.10}$$

(ii) If the initial condition u_0 belongs to $\mathcal{C}^{1-\frac{\alpha}{2}}(Q)$, then there exists $\nu>0$ such that given any $p\in[1,+\infty[$, there exists a constant $C_p>0$ such that, for every $t>0$:

$$\sup_{x\in Q}\mathbb{E}\left(|u(t,x)-u^n(t,x)|^{2p}|\right)\le C_p\,t^{-\nu}\,n^{-(2-\alpha)\,p}\,. \tag{3.11}$$

(iii) Finally, if u_0 belongs to $\mathcal{C}_0(Q)$, then for all $p\in[1,+\infty[$, as $n\to+\infty$, $\sup_{(t,x)\in[0,T]\times Q}\mathbb{E}\big(|u(t,x)-u^n(t,x)|^{2p}|\big)$ converges to 0, and the sequence $u^n(t,x)$ converges a.s. to $u(t,x)$ uniformly on $[0,T]\times Q$.

Proof: As in [7], set $v(t,x)=\int_Q G_d(t,x,y)\,u_0(y)\,dy$, $v^n(t,x)=\int_Q (G_d)^n(t,x,y)\,u_0(\kappa_n(y))\,dy$, and $u(t,x)=v(t,x)+w(t,x)$ $u^n(t,x)=v^n(t,x)+w^n(t,x)$. If $u_0\in\mathcal{C}^{1-\frac{\alpha}{2}}(Q)$ (and hence is bounded), using (4.1) and (A.9), we deduce that for any $\lambda\in]0,1[$, there exists $\mu>0$, $C>0$ such that for $t>0$, $\nu=\lambda\vee\mu$,

$$\sup_{x\in Q}|v(t,x)-v^n(t,x)|\le\int_Q\Big[|G_d(t,x,y)-(G_d)^n(t,x,y)|\,|u_0(y)|$$

$$+ |(G_d)^n t, x, y)|\,|u_0(y) - u_0(\kappa_n(y))|\Big]\,dy \le C\,(1+t^{-\nu})\,e^{-ct}\,n^{-(1-\frac{\alpha}{2})}\,. \tag{3.12}$$

If $u_0 \in \mathcal{C}^3(Q)$, then since G_d (resp. $(G_d)^n$) is the fundamental solution of $\frac{\partial}{\partial t} - \Delta = 0$ (resp $\frac{\partial}{\partial t} - \Delta_n = 0$), where Δ_n is defined by (3.6), integrating by parts we deduce that $v(t,x) = u_0(x) + \int_0^t \int_Q G_d(s,x,y)\,\Delta u_0(y)\,dy$ and $v^n(t,x) = u_0(x) + \int_0^t \int_Q (G_d)^n(s,x,y)\,\Delta_n u_0(y)\,dy$. Hence $|v(t,x) - v^n(t,x)| \le \sum_{i=1}^3 A_i(t,x)$, where

$$A_1(t,x) = |u_0(t,x) - u_0(\kappa_n(x))|\,, \quad A_2(t,x) = \Big|\int_0^t \int_Q [G_d(s,x,y) - (G_d)^n(s,x,y)]\Delta u_0(y)dyds\Big|,$$

$$A_3(t,x) = \Big|\int_0^t \int_Q (G_d)^n(s,x,y)[\Delta u_0(y) - \Delta_n u_0(\kappa_n(y))]dyds\Big|.$$

Since Δu_0 is bounded and $\|u_0(.) - u_0(\kappa_n(.))\|_\infty + \|\Delta u_0(.) - \Delta_n u_0(\kappa_n(.))\|_\infty \le C\,n^{-1}$, the inequalities (A.9) and (4.2) imply

$$\sup_{(t,x)\in[0,+\infty[\times Q} |v(t,x) - v^n(t,x)| \le C\,n^{-1}\,. \tag{3.13}$$

Furthermore, for every $t \in]0,T]$, $\sup_{x\in Q} \mathbb{E}(|w(t,x) - w^n(t,x)|^{2p}) \le C \sum_{i=1}^6 B_i(t)$, where

$$\begin{aligned}
B_1(t) =& \sup_{x\in Q} \mathbb{E}\Big(\Big|\int_0^t\int_Q G_d(t-s,x,y)\Big(\sigma(s,y,u(s,y)) - \sigma(s,\kappa_n(y),u(s,\kappa_n(y))\Big)F(ds,dy)\Big|^{2p}\Big),\\
B_2(t) =& \sup_{x\in Q} \mathbb{E}\Big(\Big|\int_0^t\int_Q G_d(t-s,x,y)\Big(\sigma(s,\kappa_n(y),u(s,\kappa_n(y)))\\
&\qquad - \sigma(s,\kappa_n(y),u^n(s,\kappa_n(y))\Big)F(ds,dy)\Big|^{2p}\Big),\\
B_3(t) =& \sup_{x\in Q} \mathbb{E}\Big(\Big|\int_0^t\int_Q G_d(t-s,x,y) - (G_d)^n(t-s,x,y)\Big)\sigma(s,\kappa_n(y),u^n(s,\kappa_n(y))F(ds,dy)\Big|^{2p}\Big),\\
B_4(t) =& \sup_{x\in Q} \mathbb{E}\Big(\Big|\int_0^t\int_Q G_d(t-s,x,y)\Big(b(s,y,u(s,y)) - b(s,\kappa_n(y),u(s,\kappa_n(y))\Big)dyds\Big|^{2p}\Big),\\
B_5(t) =& \sup_{x\in Q} \mathbb{E}\Big(\Big|\int_0^t\int_Q G_d(t-s,x,y)\Big(b(s,\kappa_n(y),u(s,\kappa_n(y))) - b(s,\kappa_n(y),u^n(s,\kappa_n(y))\Big)dyds\Big|^{2p}\Big),\\
B_6(t) =& \sup_{x\in Q} \mathbb{E}\Big(\Big|\int_0^t\int_Q \Big(G_d(t-s,x,y) - (G_d)^n(t-s,x,y)\Big)b(s,\kappa_n(y),u^n(s,\kappa_n(y))dyds\Big|^{2p}\Big).
\end{aligned} \tag{3.14}$$

Burkholder's inequality, (A.1), Hölder's inequality with respect to $\|G_d(t-s,x,.)\|^2_{(\alpha)}\,ds$, Fubini's theorem, (2.6), Schwarz's inequalities and (3.2) imply that

$$\begin{aligned}
B_1(t) &\le C_p \sup_{x\in Q} \mathbb{E}\Big|\int_0^t \Big\||G_d(t-s,x,.)|\big(n^{-\frac{2-\alpha}{2}} + |u(s,.) - u(s,\kappa_n(.)|\big)\Big\|^2_{(\alpha)}\,ds\Big|^p\\
&\le C_p \int_0^t (t-s)^{-\frac{\alpha}{2}}\Big[n^{-p\,(2-\alpha)} + \sup_{(s,\xi)\in[0,t]\times Q} \mathbb{E}(|u(s,\xi) - u(s,\kappa_n(\xi))|^{2p})\Big]ds \le C_p\,n^{-p(2-\alpha)}.
\end{aligned} \tag{3.15}$$

Similar arguments based on (A.1) and (2.6) (resp. (4.3), (2.4) and (3.1)) imply that

$$B_2(t) \le C_p \int_0^t (t-s)^{-\frac{\alpha}{2}}\Big(\sup_{x\in Q}|v(s,x) - v^n(s,x)|^{2p} + \sup_{x\in Q}\mathbb{E}(|w(s,x) - w^n(s,x)|^{2p})\Big)ds, \tag{3.16}$$

$$B_3(t) \le C_p\, n^{-(2-\alpha)\,p}\,. \tag{3.17}$$

The deterministic integrals are easier to deal with. Using Hölder's inequality with respect to the measure $|G(t-s,x,y)|\,dy\,ds$, (2.9), (2.6) and (3.2) we deduce that

$$\begin{aligned} B_4(t) \le & C_p \sup_{x\in Q} \mathbb{E}\Big(\Big| \int_0^t \int_Q |G_d(t-s,x,y)| dyds\Big)^{2p-1} \int_0^t \int_Q |G_d(t-s,x,y)| \\ & \times \Big(n^{-(2-\alpha)\,p} + \mathbb{E}(|u(s,y) - u(s,\kappa_n(y))|^{2p})\Big) dydzds \le C n^{-p(2-\alpha)}. \end{aligned} \tag{3.18}$$

Similarly, Hölder's inequality, (2.6) and (2.9) (resp. (2.4), (3.1) and (4.2)) imply

$$B_5(t) \le C \int_0^t \Big[\sup_{x\in Q} |v(s,x) - v^n(s,x)|^{2p} + \sup_{x\in Q} \mathbb{E}(|w(s,x) - w^n(s,x)|^{2p}\Big] ds, \tag{3.19}$$

$$B_6(t) \le C\, n^{-2p}\,. \tag{3.20}$$

The inequalities (3.15)-(3.20) imply that for any $T > 0$ and $p \in [1, +\infty[$, there exists a constant $C > 0$ such that for $0 \le t \le T$,

$$\begin{aligned} \sup_{x\in Q} \mathbb{E}(|w(t,x) - w^n(t,x)|^{2p}) \le & C\Big(n^{-p(2-\alpha)} + \int_0^t (t-s)^{-\frac{\alpha}{2}} \\ & \times \Big[\sup_{x\in Q} |v(s,x) - v^n(s,x)|^{2p} + \sup_{x\in Q} \mathbb{E}(|w(s,x) - w^n(s,x)|^{2p})\Big] ds\Big). \end{aligned} \tag{3.21}$$

Thus, (3.13) and Gronwall's lemma (see e.g. [8], lemma 3.4) imply that if $u_0 \in \mathcal{C}^3(Q)$,

$$\begin{aligned} \sup_{x\in Q} \mathbb{E}(|w(t,x) - w^n(t,x)|^{2p}) \le & C_p \Big[n^{-p(2-\alpha)} + \int_0^t (t-s)^{-\frac{\alpha}{2}} \\ & \times \sup_{x\in Q} \mathbb{E}(|w(s,x) - w^n(s,x)|^{2p})\, ds\Big] \le C_p\, n^{-p(2-\alpha)}. \end{aligned}$$

This inequality together with (3.13) yield (3.10). If $u \in \mathcal{C}^{1-\frac{\alpha}{2}}(Q)$, using again Gronwall's lemma and (3.12), we deduce that for some $\lambda \in]0,1[$, one has

$$\begin{aligned} \sup_{x\in Q} \mathbb{E}(|w(t,x) - w^n(t,x)|^{2p}) \le & C_p\Big(n^{-p(2-\alpha)} + \int_0^t (t-s)^{-\frac{\alpha}{2}} \Big[s^{-\lambda} n^{-p(2-\alpha)} \\ & + \sup_{x\in Q} \mathbb{E}(|w(s,x) - w^n(s,x)|^{2p})\Big] ds\Big) \le C_p n^{-p(2-\alpha)}. \end{aligned}$$

This inequality and (3.12) imply (3.11).

Finally, let $u_0 \in \mathcal{C}^0(Q)$ and for any $\varepsilon > 0$, let $u_{0,\varepsilon}$ denote a function in $\mathcal{C}^3(Q)$ such that $\|u_0 - u_{0,\varepsilon}\|_\infty \le \varepsilon$. Let $u_\varepsilon = v_\varepsilon + w_\varepsilon$ and $u^n_\varepsilon = v^n_\varepsilon + w^n_\varepsilon$ denote the previous decompositions of the solution u_ε and its space discretization u^n_ε with the initial condition $u_{0,\varepsilon}$. Then

$$\begin{aligned} \sup_{(t,x)\in[0,T]\times Q} |v(t,x) - v^n(t,x)| \le & \sup_{(t,x)\in[0,T]\times Q} |v_\varepsilon(t,x) - v^n_\varepsilon(t,x)| \\ & + \Big| \int_Q G_d(t,x,y)[u_0(y) - u_{0,\varepsilon}(y)] dy\Big| + \Big| \int_Q (G_d)^n(t,x,y)[u_0(\kappa_n(y)) - u_{0,\varepsilon}(\kappa_n(y))] dy\Big| \\ \le & C\varepsilon + \sup_{(t,x)\in[0,T]\times Q} |v_\varepsilon(t,x) - v^n_\varepsilon(t,x)|. \end{aligned} \tag{3.22}$$

Hence (3.21) and (3.22) imply that

$$\sup_{x\in Q}\mathbb{E}(|w(t,x)-w^n(t,x)|^{2p})\leq C\Big[\varepsilon+n^{-p(2-\alpha)}+\int_0^t (t-s)^{-\frac{\alpha}{2}}\sup_{x\in Q}\mathbb{E}(|w(s,x)-w^n(t,x)|^{2p})ds\Big].$$

Gronwall's lemma concludes the proof of the theorem. □

We now prove the convergence of $u^{n,m}$ and of u^n_m to u^n as $m\to+\infty$.

Theorem 3.4 *Let σ and b satisfy the conditions (2.4) and (2.7). Then*
(i) If $u_0\in\mathcal{C}^2(Q)$, then for every $T>0$ and $p\in[1,+\infty[$, there exists a constant $C_p(T)>0$ such that

$$\sup_{n\geq 1}\sup_{t\in[0,T]}\sup_{x\in Q}\mathbb{E}(|u^n(t,x)-u^{n,m}(t,x)|^{2p})\leq C_p(T)\,m^{-p(1-\frac{\alpha}{2})}\,. \tag{3.23}$$

(ii) If $u_0\in\mathcal{C}(Q)$, then $\sup_{n\geq 1}\sup_{t\in[0,T]}\sup_{x\in Q}|u^n(t,x)-u^{n,m}(t,x)|$ converges to 0 as $m\to+\infty$ and for every $t>0$ and $p\in[1,+\infty[$ there exists a constant $C_p(t)$ such that

$$\sup_{n\geq 1}\sup_{x\in Q}\mathbb{E}(|u^n(t,x)-u^{n,m}(t,x)|^{2p})\leq C_p(t)\,m^{-p(1-\frac{\alpha}{2})}\,.$$

(iii) The results of (i) and (ii) hold with u^n_m instead of $u^{n,m}$ if one requires that $\frac{n^2T}{m}\leq q<\frac{1}{2}$.

Proof: Let $v^n(t,x)=\int_Q (G_d)^n(t,x,y)u_0(\kappa_n(y))dy$, $v^{n,m}(t,x)=\int_Q (G_d)^{n,m}(t,x,y)u_0(\kappa_n(y))dy$. Suppose at first that $u_0\in\mathcal{C}^2(Q)$ and as in the proof of (3.23) in [8], for $d=1$ set $I=\sup_{t\in[0,T]}\sup_{x\in[0,1]}|v^{n,m}(t,x)-v^n(t,x)|\leq\sum_{i=1}^3 I_i$, where

$$\begin{aligned}
I_1 &= \sup_{t\in[0,T]}\sup_{x\in[0,1]}\big|v^{n,m}\big(\big[mtT^{-1}\big]\,Tm^{-1},x\big)-v^n\big(\big[mtT^{-1}\big]\,Tm^{-1},x\big)\big|,\\
I_2 &= \sup_{t\in[0,T]}\sup_{x\in[0,1]}\big|v^n\big(\big[mtT^{-1}\big]\,Tm^{-1},x\big)-v^n(t,x)\big|,\\
I_3 &= \sup_{t\in[0,T]}\sup_{x\in[0,1]}\big|v^n\big(\big[\big(mtT^{-1}+1\big)\big]\,Tm^{-1},x\big)-v^n(t,x)\big|\,.
\end{aligned}$$

The inequalities (3.27) and (3.28) in [8] imply that $I_2+I_3\leq C\,m^{-\frac{1}{2}}$. Furthermore, using an estimate of [8], we deduce that

$$I_1\leq C\sup_{n\geq 1}\sup_{t\in[0,T]}\sup_{x\in[0,1]}\sum_{j=1}^{n-1} j^{-2}\exp\Big(\lambda^n_j\,[\frac{mt}{T}]\,\frac{T}{m}\Big)\Big|1-\exp\Big[[\frac{mt}{T}]\,\Big(\lambda^n_j\,\frac{T}{m}+\ln\Big(1-\lambda^n_j\,\frac{T}{m}\Big)\Big]\Big|.$$

For $t\leq T\,m^{-1}$, $[\frac{mt}{T}]=0$ and the right hand-side of the previous inequality is 0. If $t\geq T\,m^{-1}$, then there exists a constant $c>0$ such that $\frac{T}{m}\,[\frac{mt}{T}]\geq c\,t$ and using (A.7) we deduce that

$$\begin{aligned}
I_1 \leq& C\sup_{n\geq 1}\sup_{t\in[\frac{T}{m},T]}\sum_{j=1}^{n-1} j^{-2}\,e^{-ctj^2}\,\big|1-\exp(-j^4\,t\,m^{-1})\big|\\
\leq& \sup_{n\geq 1}\sup_{t\in[\frac{T}{m},T]} m^{-1}\sum_{j=1}^{n-1} j^2\,t\,e^{-ctj^2}\leq C\sup_{n\geq 1}\sup_{t\in[\frac{T}{m},T]} m^{-1}\sum_{j=1}^{n-1} e^{-ctj^2}\leq C\,m^{-\frac{1}{2}}\,.
\end{aligned}$$

Hence for $d=1$, $\sup_{n\geq 1}\sup_{t\in[0,T]}\sup_{x\in Q}|v^n(t,x)-v^{n,m}(t,x)|\leq C\,m^{-\frac{1}{2}}$, and an easy argument shows that this inequality can be extended to any $d\geq 1$. Furthermore, for any $m\geq 1$ and $t\in[0,T]$, $\sup_{n\geq 1}\sup_{x\in Q}\mathbb{E}(|w^n(t,x)-w^{n,m}(t,x)|^{2p})\leq C\,\sum_{i=1}^6\tilde{B}_i(t)$, where $\tilde{B}_1(t)$

and $\tilde{B}_4(t)$ are similar to $B_1(t)$ and $B_4(t)$ in the proof of (3.14) replacing $\varphi(s,y,u(s,y)) - \varphi(s,\kappa_n(y),u(s,\kappa_n(y)))$ by $\varphi(s,\kappa_n(y),u^n(s,\kappa_n(y))) - \varphi(\Lambda_m(s),\kappa_n(y),u^n(s,\kappa_n(y)))$, $\tilde{B}_2(t)$ and $\tilde{B}_5(t)$ are similar to $B_2(t)$ and $B_5(t)$ replacing $\varphi(s,\kappa_n(y),u(s,\kappa_n(y))) - \varphi(s,\kappa_n(y),u^n(s,\kappa_n(y)))$ by $\varphi(\Lambda_m(s),\kappa_n(y),u^n(\Lambda_m(s),\kappa_n(y))) - \varphi(\Lambda_m(s),\kappa_n(y),u^{n,m}(\Lambda_m(s),\kappa_n(y)))$ with $\varphi=\sigma$ or b respectively, and finally $\tilde{B}_3(t)$ and $\tilde{B}_6(t)$ are similar to $B_3(t)$ and $B_6(t)$ replacing $G_d-(G_d)^n$ by $(G_d)^n-(G_d)^{n,m}$. The argument is then similar to that used in the proof of Theorem 3.3. The inequalities (2.7), (A.13), (3.1) and (3.4) provide an upper estimate of $\tilde{B}_1$, (4.37) and (3.1) give an upper estimate of $\tilde{B}_3$ so that $\tilde{B}_1(t)+\tilde{B}_3(t)\le C\,m^{-(1-\frac{\alpha}{2})\,p}$. On the other hand, (A.14) and (2.7) show that for some $\lambda\in]0,1[$,

$$\tilde{B}_2(t)\le\int_0^t (t-s)^{-\lambda}\sup_{n\ge1}\sup_{y\in Q}\mathbb{E}\big(|u^n(\Lambda_m(s),\kappa_n(y))-u^{n,m}(\Lambda_m(s),\kappa_n(y))|^{2p}\big)\,ds\,.$$

A similar argument based on (A.13), (4.36), (3.1) (3.4) proves that $\tilde{B}_4(t)+\tilde{B}_6(t)\le C\,m^{-\mu}$ for any $\mu\in]0,1[$ and shows that for some $\lambda\in]0,1[$,

$$\tilde{B}_5(t)\le\int_0^t (t-s)^{-\lambda}\sup_{n\ge1}\sup_{y\in Q}\mathbb{E}\big(|u^n(\Lambda_m(s),\kappa_n(y))-u^{n,m}(\Lambda_m(s),\kappa_n(y))|^{2p}\big)\,ds\,.$$

Thus, Gronwall's lemma concludes the proof of (3.23). The rest of the proof of the theorem, which is similar to that of Theorem 3.3 is omitted. □

4 Refined estimates of differences of Green functions

This section is devoted to prove some crucial evaluations for the norms of the difference between G_d and its space discretizations $(G_d)^n$, $(G_d)^{n,m}$ or $(G_d)^n_m$; indeed, as shown in the previous section, they provide the speed of convergence of the scheme. We suppose again that these kernels are defined in terms of the homogeneous Dirichlet boundary conditions. Simple modifications of the proof yield similar estimates for the homogeneous Neumann ones.

The main ingredient in the proofs will be the so-called *Abel's summation method*, which is a discrete "integration-by-parts" formula and is classically used in analysis to evaluate non absolutely convergent series. More precisely :

Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be sequences of real numbers, $A_{-1}=0$ and $A_n=\sum_{k=0}^n a_k$ if $n\ge0$. Then, for any $0\le N_0<N$, one has

$$\sum_{k=N_0}^N a_k\,b_k=\sum_{k=N_0}^N (A_k-A_{k-1})b_k=A_Nb_N-A_{N_0-1}b_{N_0}-\sum_{k=N_0}^{N-1}A_k\,(b_k-b_{k+1})\,.$$

In particular, this technique will be employed repeatedly throughout the proofs with $x\in]0,2[$ and $a_k=\cos(k\pi x)$, for which the corresponding sequence A_k satisfies the property $|A_k|\le\frac{C}{|\sin(\frac{\pi x}{2})|}$, or $a_k=\sin(k\pi x)$, for which A_k satisfies a similar inequality, and various monotonous sequences (b_k); see [16] pages 17 - 18 for a more detailed account on the subject.

Lemma 4.1 *There exists positive constants c, C, μ such that for $t>0$, $n\ge2$:*

$$\sup_{x\in Q}\|G_d(t,x,.)-(G_d)^n(t,x,.)\|_1\le C\,n^{-1}\,(1+t^{-\mu})\,e^{-ct}\,,\qquad(4.1)$$

$$\int_0^{+\infty} \sup_{x\in Q} \int_Q |G_d(t,x,y) - (G_d)^n(t,x,y)|\, dydt \le Cn^{-1}\,, \tag{4.2}$$
$$\int_0^{+\infty} \sup_{x\in Q} \| G_d(t,x,\cdot) - (G_d)^n(t,x,\cdot)\|^2_{(\alpha)}\, dt \le Cn^{-(2-\alpha)}\,. \tag{4.3}$$

Proof : Let $\gamma > 0$ to be fixed later on; the inequalities (A.9), (A.1) and (A.11) imply that for $0<\lambda<1$,

$$\int_0^{\gamma n^{-2}} \sup_{x\in Q} \|G_d(t,x,.) - (G_d)^n(t,x,.)\|_1\, dt \le C\, n^{-2+\lambda}\,, \tag{4.4}$$
$$\int_0^{\gamma n^{-2}} \sup_{x\in Q} \| G_d(t,x,.) - (G_d)^n(t,x,.)\|^2_{(\alpha)}\, dt \le C\, n^{-2+\alpha}\,. \tag{4.5}$$

To estimate $\int_{\gamma n^{-2}}^{+\infty} \sup_x \|G_d(t,x,.) - (G_d)^n(t,x,.)\|\, dt$, where $\|\;\|$ denotes either the $\|\;\|_1$ or $\|\;\|_{(\alpha)}$ norm, we first deal with the case $d=1$ and $\alpha<1$.
Case $d=1$ and $\alpha<1$. As in Gyöngy [7], write $|G(t,x,y) - G^n(t,x,y)| \le \sum_{i=1}^4 T_i(t,x,y)$, where

$$\begin{aligned}
T_1(t,x,y) =& \Big| \sum_{j\ge n} e^{-j^2\pi^2 t}\, \varphi_j(x)\, \varphi_j(y)\Big|\,,\\
T_2(t,x,y) =& \Big| \sum_{1\le j\le n-1} \Big[e^{\lambda_j^n t} - e^{-j^2\pi^2 t}\Big]\, \varphi_j(x)\, \varphi_j(y)\Big|\,,\\
T_3(t,x,y) =& \Big| \sum_{1\le j\le n-1} e^{\lambda_j^n t}\, [\varphi_j(x) - \varphi_j^n(x)]\, \varphi_j(y)\Big|\,,\\
T_4(t,x,y) =& \Big| \sum_{1\le j\le n-1} e^{\lambda_j^n t}\, \varphi_j^n(x)\, [\varphi_j(y) - \varphi_j(k_n(y))]\Big|\,.
\end{aligned} \tag{4.6}$$

To study the $\|\|_{(\alpha)}$ norm of a non-negative function $R(t,x,.)$, for $x\in[0,1]$ and $i\in\{1,2,3\}$, let

$$A_n^i(x) = \{y\in[0,1] : |y-x| \le i\, n^{-1} \text{ or } \; y+x \le i\, n^{-1} \text{ or } \; 2-x-y \le i\, n^{-1}\}\,. \tag{4.7}$$

Then $dy(A_n^i(x)) \le Cn^{-1}$ and for $x\in[0,1]$, $y,z\in A_n^i(x)$, $|y-z|\le 2i\, n^{-1}$; furthermore,

$$\|R(t,x,.)\|^2_{(\alpha)} \le 2\;\big[\, \|R(t,x,.)\, 1_{A_n^2(x)}(.)\|^2_{(\alpha)} + \|R(t,x,.)\, 1_{A_n^2(x)^c}(.)\|^2_{(\alpha)}\,\big]\,.$$

Set $\mathcal{A}_n^{(1)}(x) = \{(y,z)\in Q^2 : |y-x|\vee|z-x| \le 2\, n^{-1}\}$, $\mathcal{A}_n^{(2)}(x) = \{(y,z)\in Q^2 : |y-x|\vee(x+z) \le 2\, n^{-1}\}$, and $\mathcal{A}_n^{(3)}(x) = \{(y,z)\in Q^2 : |y-x|\vee(2-x-z) \le 2\, n^{-1}\}$ and for $i=1,2,3$, let

$$R^{(i)}(t,x) = \int_{\mathcal{A}_n^{(1)}(x)} R(t,x,y)|y-z|^{-\alpha} R(t,x,z)\, dydz\,. \tag{4.8}$$

Then $\|R(t,x,.)1_{A_n^2(x)}(.)\|^2_{(\alpha)} \le C\sum_{i=1}^3 R^{(i)}(t,x)$. Let $\mathcal{B}_n^{(1)}(x) = \{(y,z)\in Q^2 \,:\, 2\, n^{-1} \le |y-x| \wedge |z-x|\,,\, |y-z| \le 2\, n^{-1}\}$ and $\mathcal{B}_n^{(2)}(x) = \{(y,z)\in Q^2 \,:\, 2\, n^{-1} \le |y-z|\wedge|y-x|\wedge|z-x|\}$ and for $i=1,2$ set

$$\bar{R}^{(i)}(t,x) = \int_{\mathcal{B}_n^{(i)}(x)} R(t,x,y)|y-z|^{-\alpha} R(t,x,z)\, dydz\,. \tag{4.9}$$

Then $\|R(t,x,.)1_{A_n^2(x)^c}(.)\|^2_{(\alpha)} \le C\sum_{i=1}^2 \bar{R}^{(i)}(t,x)$. These notations will be used repeatedly throughout the proof for various functions R.

Estimate of T_2 This term is the most delicate to handle. Set $\Delta_j^n(t) := e^{\lambda_j^n t} - e^{-j^2\pi^2 t}$. Then for any $A \in [0,2]$ we have

$$0 \le \Delta_j^n(t) \le C\,(j/n)^2\, j^2\, t\, e^{-cj^2 t} \le C\, n^{-A}\, j^A\, e^{-cj^2 t}\,, \tag{4.10}$$

so that (A.7) with $K = A$ yields

$$\sup_{x\in[0,1]} T_2(t,x,y) \le C\, n^{-A}\, t^{-\frac{A+1}{2}}\, e^{-ct}\,. \tag{4.11}$$

Furthermore, $T_2(t,x,y) \le |\sum_{j=1}^{n-1} e^{\lambda_j^n t}\, \varphi_j(x)\, \varphi_j(y)| + |\sum_{j=1}^{n-1} e^{-j^2\pi^2 t}\, \varphi_j(x)\, \varphi_j(y)|$, and Abel's summation method yields that for $1 \le N_1(n) < N_2(n) \le n-1$,

$$\Big| \sum_{j=N_1(n)}^{N_2(n)} \Delta_j^n(t)\, \varphi_j(x)\, \varphi_j(y) \Big| \le e^{-c\, N_1(n)^2\, t} \Big[\frac{1}{|\sin(\frac{\pi\,(x-y)}{2})|} + \frac{1}{|\sin(\frac{\pi\,(x+y)}{2})|} \Big]\,. \tag{4.12}$$

Hence for $A \in]0,2]$ and $\lambda \in]0, \frac{2}{A+1} \wedge 1[$, we have for any $t > 0$

$$\sup_{x\in[0,1]} \int_0^1 T_2(t,x,y)\, dy \le C\, e^{-ct}\, n^{-A\lambda}\, t^{-\frac{A+1}{2}\lambda}\,. \tag{4.13}$$

In order to bound the $\|\ \|_{(\alpha)}$ norm of $T_2(t,x,.)$ for $t \ge \gamma\, n^{-2}$, let $N_1(n) = [\sqrt{n}]$, $N_2(n) = [n/2]$ and $N_3(n) = n-1$. Then $T_2(t,x,y) \le \sum_{i=1}^3 T_{2,i}(t,x,y)$ where $T_{2,1}(t,x,y) = \sum_{j=1}^{[\sqrt{n}]} |\Delta_j^n(t)|$ and for $i = 2,3$, $T_{2,i}(t,x,y) = \Big| \sum_{j=N_{i-1}(n)+1}^{N_i(n)} \Delta_j^n(t) \varphi_j(x) \varphi_j(y) \Big|$. The inequalities (4.10) with $A = 2$ and (A.7) with $\beta = 0$ yield $\sup_{x,y\in[0,1]} T_{2,1}(t,x,y) \le C\, \sum_{j=1}^{[\sqrt{n}]} n^{-1} e^{-cj^2 t} \le C\, n^{-1}\, [1 + t^{-\frac{1}{2}}]\, e^{-ct}$. Furthermore, $\sup\{T_{2,1}(t,x,y)\, ;\, (t,x,y) \in]0,+\infty[\times[0,1]^2\} \le C\sqrt{n}$. Hence both estimates yield

$$\sup_{x\in[0,1]} \|T_{2,1}(t,x,.)\|_{(\alpha)}^2 \le C\, e^{-ct}\, (1 + t^{-1+\frac{\alpha}{3}})\, n^{-2+\alpha}. \tag{4.14}$$

For $\lambda \in]\alpha, 1[$ and $\mu \in]0, 1-\lambda[$, using (4.12) and (4.11) with $A = 0$, we have for $t \ge \gamma\, n^{-2}$, with the notations defined in (4.8) and (4.9): $\sup_{x\in[0,1]} T_{2,3}^{(1)}(t,x) \le C n^\alpha e^{-ctn^2} [1 + (nt^{\frac{1}{2}})^{-(\lambda+\mu)}]$. Similar computations for integrals over the sets $\mathcal{A}_n^{(i)}(x)$ for $i = 2,3$ yield $\sup_{x\in[0,1]} \|T_{2,3}(t,x,.) \times 1_{A_n^2(x)}(.)\|_{(\alpha)}^2 \le C n^\alpha e^{-ctn^2}$. Furthermore, (4.12) implies that $\bar{T}_{2,3}^{(1)}(t,x) \le C e^{-cn^2 t} \Big(\int_0^{2n^{-1}} u^{-2} du \Big) \times \Big(\int_0^{2n^{-1}} v^{-\alpha} dv \Big) \le C e^{-cn^2 t} n^\alpha$. For $(y,z) \in \mathcal{B}_n^{(2)}(x)$, let $I(y,z) \le M(y,x) \le S(y,z)$ denote the ordered values of $|x-y|$, $|y-z|$ and $|x-z|$. Then $\bar{T}_{2,3}^{(2)}(t,x) \le C e^{-cn^2 t} \int_{2n^{-1} \le I(y,z) \le S(y,z) \le 2} \Big(I(y,z) \times M(y,z) \Big)^{-1-\frac{\alpha}{2}} dydz \le C e^{-cn^2 t} n^\alpha$. The previous inequalities on $T_{2,3}$ yield that for $t \ge \gamma n^{-2}$,

$$\sup_{x\in[0,1]} \|T_{2,3}(t,x,.)\|_{(\alpha)}^2 \le C\, n^\alpha\, e^{-ctn^2}\,. \tag{4.15}$$

We now estimate $T_{2,2}$. Let $C_0 > 0$ be a "large" constant to be chosen later on, and suppose that $t \ge C_0\, n^{-2}$. For fixed n, t and $j \in \big[[\sqrt{n}] + 1, [n/2]\big]$, set $\phi(j) := \Delta_j^n(t)$. Then $\phi'(j) = 2j\pi^2 t \exp[-j^2\pi^2 t]\, \big(1 - \psi(\frac{j\pi}{2n})\big)$, where for $\frac{\pi}{2\sqrt{n}} \le u \le \frac{\pi}{4}$ one sets $\psi(u) := \frac{\sin(2u)}{2u}\, \exp\big[4n^2 t\, (u^2 - \sin^2 u) \big]$. Hence, to apply Abel's summation method, we have to compare $\psi(u)$ and 1.

Using Taylor's expansion of the functions sine and exponential, we deduce that there exists a positive constant C_1 such that if $\tilde{C}_1 = (\frac{2\,C_1}{\pi})^2$, for C_0 large enough, the map $j \mapsto \phi(j)$ decreases on $\big[[\sqrt{n}],[n/2]\big]$ for $t \geq \frac{\tilde{C}_1}{n}$. Let $t \in [\frac{C_0}{n^2}, \frac{\tilde{C}_1}{n}]$. Then there exists a constant $C_2 \in]0, C_1[$ such that $j \mapsto \phi(j)$ increases on $\big[[\sqrt{n}], [\frac{2\,C_2}{\pi\,\sqrt{t}}]\big]$ and decreases on $\big[[\frac{2\,C_1}{\pi\,\sqrt{t}}]+1, [\frac{n}{2}]\big]$. For $t \in [\frac{C_0}{n^2}, \frac{\tilde{C}_1}{n}]$ $T_{2,2}(t,x,y) \leq \sum_{i=1}^2 T_{2,2,i}(t,x,y)$, where one set $B_i = \frac{2C_i}{\pi}$ and

$$T_{2,2,1}(t,x,y) = \Big|\sum_{j=[\sqrt{n}]}^{[B_2/\sqrt{t}]} \Delta_j^n(t)\,\varphi_j(x)\,\varphi_j(y)\Big| + \Big|\sum_{j=[B_1/\sqrt{t}]}^{[n/2]} \Delta_j^n(t)\,\varphi_j(x)\,\varphi_j(y)\Big|,$$

$$T_{2,2,2}(t,x,y) = \Big|\sum_{j=[B_2/\sqrt{t}]}^{[B_1/\sqrt{t}]} \Delta_j^n(t)\,\varphi_j(x)\,\varphi_j(y)\Big|.$$

There exists a constant $C>0$ such that if $\gamma > C_0$, for every $t \geq \frac{\gamma}{n^2}$

$$\sup_{(x,y)\in[0,1]^2} T_{2,2,1}(t,x,y) \leq C\,n\,e^{-ct}. \tag{4.16}$$

For $t \geq \frac{\tilde{C}_1}{n}$, let $T_{2,2,1}(t,x,y) = T_{2,2}(t,x,y)$. Using (4.12) we deduce that for $t \geq \frac{C_0}{n^2}$ and $\beta \in [0,1]$

$$\begin{aligned} T_{2,2,1}(t,x,y) \leq & C\Big[\frac{1}{|\sin(\frac{\pi(x-y)}{2})|} + \frac{1}{|\sin(\frac{\pi(x+y)}{2})|}\Big]\Big[\Delta^n_{[\sqrt{n}]+1}(t) + \sum_{i=1}^2 \Delta^n_{[\frac{B_i}{\sqrt{t}}]}(t) 1_{\{C_0 n^{-2}\leq t\leq \tilde{C}_1 n^{-1}\}}\Big] \\ \leq & C\Big[\frac{1}{|x-y|} + \frac{1}{x+y} + \frac{1}{2-x-y}\Big]\Big[\frac{e^{-ctn}}{n} + n^{-2\beta} t^{-\beta} 1_{C_0 n^{-2}\leq t\leq \tilde{C}_1 n-1\}}\Big]. \end{aligned} \tag{4.17}$$

For $t \in [\frac{C_0}{n^2}, \frac{\tilde{C}_1}{n}]$, it remains to bound directly the sum $T_{2,2,2}(t,x,y)$. The inequality (4.10) implies that for $B_2\,t^{-\frac{1}{2}} \leq j \leq B_1\,t^{-\frac{1}{2}}$, $\Delta_j^n(t) \leq Cn^{-2A}\,t^{-A}\,e^{-c\,t\,j^2}$ for any $A \in [0,1]$. Therefore, the inequality (A.7) implies that for $A \in [0,1]$,

$$\sup_{x\in[0,1]} T_{2,2,2}(t,x,y) \leq C\,n^{-2A}\,t^{-(A+\frac{1}{2})}. \tag{4.18}$$

Finally, if C_0 is large enough, for $t \in [\frac{C_0}{n^2}, \frac{\tilde{C}_1}{n}]$ the function ψ is increasing on the interval $[\frac{C_2}{n\sqrt{t}}, \frac{C_1}{n\sqrt{t}}]$ and $\sup\{\phi(u) : u \in [\frac{C_2}{n\sqrt{t}}, \frac{C_1}{n\sqrt{t}}]\} \leq C\,n^{-2}\,t^{-1}$. Hence Abel's summation method implies that for $t \in [\frac{C_0}{n^2}, \frac{\tilde{C}_1}{n}]$,

$$\sup_{x\in[0,1]} T_{2,2,2}(t,x,y) \leq C\,n^{-2}\,t^{-1}\Big[\frac{1}{|\sin(\pi\,\frac{x-y}{2})|} + \frac{1}{|\sin(\pi\,\frac{x+y}{2})|}\Big]. \tag{4.19}$$

The inequalities (4.17) applied with $\beta = \frac{1}{2}$ and $\beta = 1$ respectively and (4.16) imply that for $\lambda \in]0,\alpha[$ and $\mu \in]0, 1-\lambda[$, $\nu = \lambda + \mu$, there exists a constant $C>0$ such that for every $t \geq \gamma n^{-2}$: $\sup_{x\in[0,1]} T^{(1)}_{2,2,1}(t,x) \leq C\Big[n^{-1+\alpha} e^{-ctn} + 1_{\{C_0 n^{-2}\leq t\leq \tilde{C}_1 n^{-1}\}} n^{-4+\alpha+2\nu} t^{-\nu}\Big]$. Similar computations for the integrals over the sets $\mathcal{A}_n^{(i)}(x)$, $i=2,3$ imply that the same upper estimates hold for $T^{(i)}_{2,2,1}(t,x)$. Furthermore, (4.17) with $\beta \in]\frac{1}{2},1[$ yields $\sum_{j=1}^2 \sup_{x\in[0,1]} \bar{T}^{(j)}_{2,2,1}(t,x) \leq C\,\big(n^{-2+\alpha} \times e^{-ctn} + n^{-4\beta+\alpha}\,t^{-2\beta}\,1_{\{C_0\,n^{-2}\leq t\leq \tilde{C}_1\,n^{-1}\}}\big)$. The inequalities on $T^{(i)}_{2,2,1}$ and $\bar{T}^{(j)}_{2,2,1}$ yield that for $\nu \in]\alpha,1[$ and $\beta \in]\frac{1}{2},1[$,

$$\sup_{x\in[0,1]} \|T_{2,2,1}(t,x,.)\|^2_{(\alpha)} \leq C\big[n^{-1+\alpha} e^{-ctn} + 1_{\{\frac{C_0}{n^2}\leq t\leq \frac{\tilde{C}_1}{n}\}}\big(t^{-\nu} n^{-4+\alpha+2\nu} + t^{-2\beta} n^{-4\beta+\alpha}\big)\big]. \tag{4.20}$$

For $t \in [\frac{C_0}{n^2}, \frac{\tilde{C}_1}{n}]$ (4.18) and (4.12) yield that for $A \in [0,1]$, $\lambda \in]0,\alpha[$, $\mu \in]0,1[$ and $\nu = \lambda + \mu$, $\sup_{x\in[0,1]} \sum_{i=1}^3 T^{(i)}_{2,2,2}(t,x) \leq C n^{-(2A+1)\nu+\alpha} t^{-(A+\frac{1}{2})\nu}$. Proceeding as for the estimates of $\bar{T}^{(i)}_{2,3}$ we deduce that for $A=1$ and $\lambda \in]\frac{1}{3},\frac{1}{2}[$ (resp. for $0<\tilde{\lambda}<\frac{\alpha}{2}$), $\sup_{x\in[0,1]} \bar{T}^{(1)}_{2,2,2}(t,x) \leq C\, n^{-6\lambda+\alpha}\, t^{-3\lambda}$ and $\sup_{x\in[0,1]} \bar{T}^{(2)}_{2,2,2}(t,x) \leq C\, n^{-4}\, t^{-2-\tilde{\lambda}} \left(\int_{2\,n^{-1}}^2 u^{-1+\tilde{\lambda}-\frac{\alpha}{2}}\, du\right)^2 \leq C\, n^{-4+\alpha}\, t^{-2}$.
The upper estimates of $T^{(i)}_{2,2,2}(t,x)$, $\bar{T}^{(j)}_{2,2,2}(t,x)$ and (4.20) imply that for γ large enough, $\mu \in]\frac{1}{3},\frac{1}{2}[$ there exists $C>0$ such that for every $t \geq \gamma n^{-2}$:

$$\sup_{x\in[0,1]} \|T_{2,2}\|^2_{(\alpha)} \leq C n^{\alpha} \Big[e^{-ctn} n^{-1} + 1_{\{\frac{C_0}{n^2} \leq t \leq \frac{\tilde{C}_1}{n}\}} \Big(n^{-4+2\nu} t^{-\nu} + t^{-2} n^{-4} + t^{-\lambda} n^{-2\lambda} \Big)\Big]. \qquad (4.21)$$

Thus (4.14), (4.15) and (4.21) yield for $\lambda \in]1,\frac{3}{2}[$, $\nu \in]\alpha,1[$ and $t \geq \gamma n^{-2}$ with γ large enough,

$$\sup_{x\in[0,1]} \|T_2(t,x,.)\|^2_{(\alpha)} \leq C n^{\alpha} \Big[e^{-ctn^2} + n^{-2} e^{-ct}(1+t^{-1+\frac{\alpha}{3}}) + n^{-1} e^{-ctn}$$
$$+ 1_{\{\frac{C_0}{n^2} \leq t \leq \frac{\tilde{C}_1}{n}\}} \Big(n^{-4+2\nu} t^{-\nu} + n^{-4} t^{-2} + n^{-2\lambda} t^{-\lambda} \Big)\Big]. \qquad (4.22)$$

Estimates of T_3 Using (A.7) we deduce that $T_3(t,x,y) \leq C\, n^{-1} \left[1+t^{-1}\right] e^{-ct}$. Furthermore, set $A(l) := [0,1] \cap \big(\big[\frac{l-1}{n}, \frac{l+2}{n}\big] \cup \big[0, \frac{(2-l)^+}{n}\big] \cup \big[(2-\frac{l+2}{n}) \wedge 1, 1\big]\big)$. Then $dx(A(l)) \leq C\, n^{-1}$. The study of the monotonicity of the function H defined by $H(z) = z \exp\left[-4n^2 t \sin^2\left(\frac{z\pi}{2n}\right)\right]$ and Abel's summation method yield for large enough γ, $t \geq \gamma\, n^{-2}$, $0<\lambda<1$ and $y \in A(l)$, $\sup_{x\in[\frac{l}{n},\frac{l+1}{n}]} T_3(t,x,y) \leq C\,(1+t^{-\frac{1+\lambda}{2}})\, n^{-\lambda}\, e^{-ct}$, while for $y \notin A(l)$, $\sup_{x\in[\frac{l}{n},\frac{l+1}{n}]} T_3(t,x,y) \leq \frac{C}{n}\,(1+t^{-\frac{1}{2}})\, e^{-ct} \left[\, \left|y-\frac{2l+1}{2n}\right|^{-1} + \left|y+\frac{2l+1}{2n}\right|^{-1} + \left|2n-y-\frac{2l+1}{2n}\right|^{-1} \right]$. Then, using the partition $\{A(l), A(l)^c\}$ and the three previous inequalities we deduce that for $\bar{\lambda} \in]0,1[$ and $t \geq \gamma\, n^{-2}$ one has

$$\sup_{x\in[0,1]} T_3(t,x,y) \leq C e^{-ct} (1+t^{-\frac{1+\bar{\lambda}}{2}}) n^{-1}. \qquad (4.23)$$

Similarly, for $t \geq \gamma\, n^{-2}$ and $\lambda \in]0,1[$ there exists a constant $C>0$ such that for every $l \in \{0,\cdots,n-1\}$ and $x \in [\frac{l}{n},\frac{l+1}{n}]$, $\|T_3(t,x,.)1_{A(l)}(.)\|^2_{(\alpha)} \leq C e^{-ct}(1+t^{-1-\lambda}) n^{-2+\alpha-2\lambda}$. Furthermore, when $t \geq \gamma\, n^{-2}$ separate estimates in the cases $y,z \notin A(l)$ and either $|y-z| \leq n^{-1}$ or $|y-z| \geq n^{-1}$ yield that given $\nu \in]0,\frac{\alpha}{2}[$, there exists a constant $C>0$ such that for every $l \in \{0,\cdots,n-1\}$ and $x \in [\frac{l}{n},\frac{l+1}{n}]$, one has $\|T_3(t,x,.)1_{A(l)^c}(.)\|^2_{(\alpha)} \leq C n^{-2-\nu+\alpha} e^{-ct}(1+t^{-\frac{2+\nu}{2}})$. These inequalities imply that for $t \geq \gamma\, n^{-2}$ and $\lambda \in]0,\frac{\alpha}{4}[$,

$$\sup_{x\in[0,1]} \|T_3(t,x,.)\|^2_{(\alpha)} \leq C\, e^{-ct}\, n^{-2+\alpha-2\lambda}\, (1+t^{-1-\lambda})\,. \qquad (4.24)$$

Estimates of T_4 We suppose that $x=\frac{l}{n}$, $1 \leq l \leq n-1$. The general case is easily deduced by linear interpolation. For $\frac{k}{n} \leq y \leq \frac{k+1}{n}$, $0 \leq k \leq n-1$, one has $k_n(y) = \frac{k}{n}$ and using (A.7), we deduce $T_4(t,x,y) \leq C\, n^{-1} [\, t^{-1} + 1\,]\, e^{-ct}$. Let $B(l) := \{u \in [0,1];\, \left|\frac{l}{n}-u\right| \leq \frac{1}{n}$ or $\frac{l}{n}-u \leq \frac{1}{n}$ or $2-\frac{l}{n}-u \leq \frac{1}{n}\}$; as usual, $dx(B(l)) \leq cn^{-1}$. Let then $C^1(l) := \{y \in [0,1]; \exists u \in B(l) \cap [k_n(y), y]\}$ and for $i=1,2$, let $\tilde{C}^i(l) := \{z \in [0,1]; \exists y \in C^1(l), |y-z| \leq \frac{i}{n}\}$. Then $dx(\tilde{C}^i(l)) \leq C\, n^{-1}$ and for $y \notin \tilde{C}^1(l)$, one has $|y-x| \wedge (y+x) \wedge (2-x-y) \geq n^{-1}$. Computations similar to that made to estimate T_3 yield for $\lambda \in]0,1[$ the existence of a constant $C>0$ such that for every $l \in \{0,\cdots,n\}$, and $y \in \tilde{C}^2(l)$, $T_4(t,l/n,y) \leq C n^{-\lambda}\big(1+t^{-\frac{1+\lambda}{2}}\big) e^{-ct}$ and for $y \in \tilde{C}^2(l)^c$, $T_4(t,l/n,y) \leq C n^{-1}(1+t^{-\frac{1}{2}}) e^{-ct} \big[\left|y-\frac{l}{n}\right|^{-1} + \big(y+\frac{l}{n}\big)^{-1} + \left|2-y-\frac{l}{n}\right|^{-1}\big]$. An argument similar to that proving (4.23) and (4.24) implies that for $\lambda \in]0,1[$ and $\nu \in]0,\frac{\alpha}{4}[$, there exists a constant $C>0$ such that for any $t \geq \gamma\, n^{-2}$

$$\sup_{x\in[0,1]} T_4(t,x,y) \leq C e^{-ct}(1+t^{-\frac{1+\lambda}{2}})n^{-1}, \tag{4.25}$$

$$\sup_{x\in[0,1]} \|T_4(t,x,\cdot)\|_{(\alpha)}^2 \leq n^{-2+\alpha} e^{-ct}(1+t^{-(1+\nu)})n^{-2\nu}. \tag{4.26}$$

Estimate of $T_1(t,x,.)$. Using (A.7) with $\beta=0$ and $J_0=n$, we have

$$\sup_{x,y\in[0,1]} T_1(t,x,y) \leq C \sum_{j\geq n} e^{-ctj^2} \leq C\, e^{-ctn^2}\,[1+t^{-\frac{1}{2}}]. \tag{4.27}$$

On the other hand, since $j \to e^{-j^2\pi^2 t}$ decreases, Abel's summation method yields $T_1(t,x,y) \leq C\, e^{-cn^2t}\big[\frac{1}{|\sin(\pi\frac{x-y}{2})|} + \frac{1}{|\sin(\pi\frac{x+y}{2})|}\big]$. Thus, for $\lambda \in [0,1[$ and $t>0$ we have

$$\sup_{x\in[0,1]} \int_0^1 T_1(t,x,y)\, dy \leq C\, e^{-cn^2t}\,[1+t^{-\frac{\lambda}{2}}]\,. \tag{4.28}$$

Given $\lambda \in]\alpha,1[$, $\mu \in]0,1[$ and $t \geq \gamma n^{-2}$, computations similar to the previous ones using the partition $\{A_n^2(x), A_n^2(x)^c\}$ yield that for $t \geq \gamma\, n^{-2}$, we have

$$\sup_{x\in[0,1]} \|T_1(t,x,.)\|_{(\alpha)}^2 \leq C\, n^{\alpha}\, e^{-cn^2t}\,. \tag{4.29}$$

The inequalities (4.28), (4.13) with $A=2$, $\lambda=\frac{1}{2}$, and $(T_3+T_4)(t,x,y) \leq Cn^{-1}(1+t^{-1})e^{-ct}$ imply the existence of $c, C>0$, $\lambda \in]0,1[$ such that for $t>0$,

$$\sup_{x\in[0,1]} \|G(t,x,.) - G^n(t,x,.)\|_1 \leq C\, n^{-1}\left[\left(1+t^{-1}\right)e^{-ct} + t^{-\frac{\lambda}{2}}\, e^{-ctn^2}\right], \tag{4.30}$$

which implies (4.1) for $d=1$ with $\mu=1$. The inequalities (4.28), (4.13) with $A=2$ and $\lambda=\frac{1}{2}$, (4.23) and (4.25) with $\bar\lambda=\frac{1}{2}$ imply that for some $\mu \in]0,\frac{1}{2}[$ there exists a constant $C>0$ such that for every $t \geq \gamma\, n^{-2}$ with $\gamma>0$ large enough, one has

$$\sup_{x\in[0,1]} \|G(t,x,.) - G^n(t,x,.)\|_1 \leq C\left[n^{-1}\left(1+t^{-\frac{3}{4}}\right) + e^{-c\,t\,n^2}\left(1+t^{-\mu}\right)\right]. \tag{4.31}$$

On the other hand, the inequalities (4.29), (4.22), (4.24) and (4.26) imply that for $\nu \in]0,\frac{\alpha}{4}[$, $\lambda \in]1,\frac{3}{2}[$, $\mu \in]\alpha,1[$, there exist positive constants C and $\tilde C_1$ such that for $t \geq \gamma\, n^{-2}$ with $\gamma>0$ large enough, one has

$$\begin{aligned}\sup_{x\in[0,1]} \|G(t,x,.) - G^n(t,x,.)\|_{(\alpha)}^2 \leq Cn^{\alpha}\Big[& e^{-ctn^2} + \frac{e^{-ctn}}{n} + \frac{e^{-ct}}{n^2}\Big(1+\frac{t^{-(1+\nu)}}{n^{2\nu}} + t^{-1+\frac{\alpha}{3}}\Big)\\ &+ \Big(n^{-4+2\mu}t^{-\mu} + n^{-4}t^{-2} + n^{-2\lambda}t^{-\lambda}\Big)1_{\{\gamma n^{-2}\leq t\leq \tilde C_1 n^{-1}\}}\Big],\end{aligned} \tag{4.32}$$

which proves (4.1)-(4.3) for $d=1$.

The general case We extend the inequalities (4.30), (4.31) and (4.32) to any dimension d. We use the fact that for any $d \geq 2$, we have $|G_d(t,x,y) - (G_d)^n(t,x,y)| \leq \sum_{i=1}^d \Pi_i$, where

$$\Pi_i = \Big(\prod_{j=1}^{i-1} |G(t,x_j,y_j)|\Big)|G(t,x_i,y_i) - G^n(t,x_i,y_i)|\Big(\prod_{j=i+1}^{d} |G^n(t,x_j,y_j)|\Big). \tag{4.33}$$

Hence, the inequalities (2.9), (A.9), (4.30), (4.31) and (4.33) imply (4.1) with some $\mu > 0$, and that for any $\lambda \in]0,1[$ and any $\nu \in]0,1/4[$, there exists $C > 0$ such that for $t \geq \gamma\, n^{-2}$,

$$\sup_{x\in Q} \|G_d(t,x,.) - (G_d)^n(t,x,.)\|_1 \leq Ct^{-\nu}\big[\big(1+t^{-\frac{3}{4}}\big)n^{-1} + e^{-ctn^2}(1+t^{-\nu})\big]. \tag{4.34}$$

Using (4.4) and integrating (4.34) with respect to t on $[\gamma\, n^{-2}, +\infty[$ we obtain (4.2). Finally, for $1 \leq k \leq d-1$ set $\alpha_k = \alpha\, 2^{-k}$ and set $\alpha_d = \alpha_{d-1}$; then using (4.33), (2.2), (A.1), (A.11) and (4.32), we deduce that for $\alpha \in]0,2[$, $C_1 > 0$, $\lambda \in]\alpha, 1[$, $\mu \in]1, \frac{3}{2}[$, $\nu \in]0, \frac{\alpha_d}{4}[$ and for $t \geq \gamma\, n^{-2}$ for $\gamma > 0$ large enough

$$\begin{aligned}\sup_{x\in Q} \|G_d(t,x,.-(G_d)^n(t,x,.)\|_{(\alpha)} &\leq C \sum_{i=1}^{d} t^{-\frac{1}{2}\sum_{j=1}^{i-1}\alpha_j} \\ &\quad\times \|G(t,x_i,.) - G^n(t,x_i,.)\|_{(\alpha_i)}\, n^{\sum_{j=i+1}^{d}\alpha_j} \\ &\leq C\, n^{\alpha}\Big[e^{-ctn^2} + n^{-1} e^{-ctn} + n^{-2} e^{-ct}\big(t^{-(1+\nu)} n^{-2\nu} + t^{-(1+\lambda)} n^{-2\lambda} + 1\big) \\ &\quad + \big(n^{-4} t^{-2} + n^{-4+2\nu} t^{-2+\nu} + n^{-2\mu} t^{-\mu}\big) 1_{\{\gamma n^{-2} \leq t \leq \tilde{C}_1 n^{-1}\}}\Big].\end{aligned} \tag{4.35}$$

Integrating on $[\frac{\gamma}{n^2}, +\infty[$ and using (4.5), we deduce (4.3). □

We now estimate the norm of the difference $(G_d)^n$ and $(G_d)^{n,m}$.

Lemma 4.2 *Given any $T > 0$ and $\nu > 0$ there exists $C > 0$ such that*

$$\begin{aligned}\sup_{x\in Q}\int_0^T \int_Q \big[\, &|(G_d)^n(t,x,y) - (G_d)^{n,m}(t+T\,m^{-1},x,y)| \\ &+ |(G_d)^n(t,x,y) - (G_d)^n_m(t+T\,m^{-1},x,y)|\,\big]\, dydt \leq C\, m^{-1+\nu}, \end{aligned}\tag{4.36}$$

$$\begin{aligned}\sup_{x\in Q}\int_0^T \big[\, &\|(G_d)^n(t,x,\cdot) - (G_d)^{n,m}(t+T\,m^{-1},x,\cdot)\|^2_{(\alpha)} \\ &+ \|(G_d)^n(t,x,.) - (G_d)^n_m(t,x,.)\|^2_{(\alpha)}\,\big]\, dt \leq C\, m^{-1+\frac{\alpha}{2}}. \end{aligned}\tag{4.37}$$

Proof : We only prove these inequalities for $G^n - G^{n,m}$ and we at first suppose that $d = 1$. Let $\bar{\mathcal{G}}^{n,m} = G^n - \tilde{\mathcal{G}}^{n,m}$ and $\bar{G}^{n,m} = G^{n,m} - \tilde{G}^{n,m}$ where $\tilde{G}^{n,m}$ is defined by (A.15), and $\tilde{\mathcal{G}}^{n,m}$ is defined by

$$\tilde{\mathcal{G}}^{n,m}(t,x,y) = \sum_{j=1}^{(n\wedge\sqrt{m})-1} e^{\lambda_j^n t}\, \varphi_j^n(x)\varphi_j(\kappa_n(y)). \tag{4.38}$$

Then (A.20) and (A.21) provide upper estimates of the norms of $\bar{G}^{n,m}$. Similar computations prove that the same upper estimates hold for the norms of $\bar{\mathcal{G}}^{n,m}$, i.e., for $\lambda \in]0, \frac{1}{2}[$ and $\beta \in]\alpha, 1[$,

$$\sup_{x\in[0,1]} \|\bar{\mathcal{G}}^{n,m}(t,x,.)\|_1 \leq C(1+t^{-\lambda})e^{-ctm}, \quad \sup_{x\in[0,1]} \|\bar{\mathcal{G}}^{n,m}(t,x,.)\|^2_{(\alpha)} \leq Cm^{\frac{\alpha-\beta}{2}} t^{-\frac{\beta}{2}} e^{-ctm}. \tag{4.39}$$

Let $\tilde{c}$ be a positive constant to be fixed later on; for $t \leq \tilde{c}\, T\, m^{-1}$ we estimate separately the norms of $\tilde{G}^{n,m}(t,x,.)$ and $\tilde{\mathcal{G}}^{n,m}(t,x,.)$. The inequalities (A.16) and (A.17) provide the estimates of $\tilde{G}^{n,m}$. For $\tilde{\mathcal{G}}^{n,m}$, we proceed in a similar way. Indeed, $j \to \exp(\lambda_j^n t)$ is decreasing, $\exp(\lambda_j^n t) \leq e^{-ctj^2}$ for $c > 0$, and $|\tilde{\mathcal{G}}^{n,m}(t,x,y)| \leq C\,(n \wedge \sqrt{m})$. Hence the arguments used in the proof of Lemma A.5 yield that for any $\tilde{c} > 0$ there exists a constant $C > 0$ such that for $t \in]0, \frac{\tilde{c}T}{m}]$,

$$\sup_{x\in[0,1]} \|\tilde{\mathcal{G}}^{n,m}(t,x,.) - \tilde{G}^{n,m}(t,x,.)\|_1 \le C\left(1 + t^{-\lambda}\right), \tag{4.40}$$

$$\sup_{x\in[0,1]} \| \tilde{\mathcal{G}}^{n,m}(t,x,.) - \tilde{G}^{n,m}(t,x,.)\|_{(\alpha)}^2 \le C\,(n\wedge\sqrt{m})^{\alpha}\,. \tag{4.41}$$

Furthermore, if $t \in [\tilde{c}\,T\,m^{-1}, T]$, $|\tilde{\mathcal{G}}^{n,m}(t,x,y) - \tilde{G}^{n,m}(t,x,y)| \le \tilde{T}(t,x,y) = \tilde{T}_1(t,x,y) + \tilde{T}_2(t,x,y)$, where $\tilde{T}_i(t,x,y) = \big|\sum_{j=1}^{(n\wedge\sqrt{m})-1} A^i_{n,m}(t)\varphi^n_j(x)\varphi_j(\kappa_n(y))\big|$, $A^1_{n,m}(t) = \exp\left(\frac{([\frac{mt}{T}]+1)\,\lambda^n_j\,T}{m}\right) - \left(1-\lambda^n_j\,\frac{T}{m}\right)^{-([\frac{mt}{T}]+1)}$ and $A^2_{n,m}(t,x) = \exp\left(\lambda^n_j t\right) - \exp\left(\frac{([\frac{mt}{T}]+1)\,\lambda^n_j\,T}{m}\right)$. Using Abel's summation method, we have for $i = 1,2$, $t \ge \frac{\tilde{c}}{m}$ for $\tilde{c}$ large enough, $x = \frac{l}{n}$ and $\kappa_n(y) = \frac{k}{n}$, $|\tilde{T}_i(t,x,y)| \le C\frac{T}{mt}\Big[\frac{1}{|x-\kappa_n(y)|} + \frac{1}{x+\kappa_n(y)} + \frac{1}{2-x-\kappa_n(y)}\Big]$. Furthermore, $|\tilde{T}(t,x,y)| \le C(n\wedge\sqrt{m})$. These inequalities yield that for $t \ge \frac{\tilde{c}T}{m}$ for large enough $\tilde{c}$ and $\lambda \in]0,1[$,

$$\|\tilde{T}(t,x,.)\|_1 \le C t^{-1+\lambda} m^{-1+\frac{3\lambda}{2}}. \tag{4.42}$$

For $\mu \in]\alpha,1[$, $\nu \in]0, 1-\mu[$ and $\beta = \mu+\nu \in]\alpha,1[$, using the sets $\mathcal{A}^{(i)}_{n\wedge\sqrt{m}}(x)$ for $i \le 3$ and $\mathcal{B}^{(j)}_{n\wedge\sqrt{m}}(x)$ for $j = 1,2$ and the fact that $\frac{1}{n} \le \frac{1}{n\wedge\sqrt{m}}$, we deduce that given $\tilde{c}$ large enough, there exists constants $c, C > 0$ such that for every $t \in [\frac{\tilde{c}T}{m}, T]$:

$$\|\tilde{T}(t,x,.)\|_{(\alpha)}^2 \le C\,(n\wedge\sqrt{m})^{\alpha}\,m^{-2}\,t^{-2} \le C\,m^{-2+\frac{\alpha}{2}}\,t^{-2}\,. \tag{4.43}$$

For $d = 1$, the inequalities (A.20), (4.39), (4.40) and (4.42) imply the existence of $\lambda \in]0, \frac{1}{2}[$ and positive constants c, C such that for any $t \in]0,T]$:

$$\sup_{x\in Q} \|\big((G_d)^n - (G_d)^{n,m}\big)(t,x,.)\|_1 \le C\left[\left(1+t^{-\lambda}\right)e^{-ctm} + t^{-1+\lambda}\,m^{-1+\frac{3\lambda}{2}}\right], \tag{4.44}$$

while the inequalities (A.21), (4.39), (4.41) and (4.43) yield the existence of $\beta \in]0, \alpha\wedge d[$ and positive constants $\tilde{c}, c$ and C such that for every $t \in]0,T]$:

$$\sup_{x\in Q} \|\big((G_d)^n - (G_d)^{n,m}\big)(t,x,.)\|_{\alpha}^2 \le C m^{\frac{\alpha}{2}}\Big[\big(1+(tm)^{-\frac{\beta}{2}}\big)e^{-ctm} + 1_{[\tilde{c}Tm^{-1},T]}(t)m^{-2}t^{-2}\Big]. \tag{4.45}$$

Let $\alpha_k = \alpha\,2^k$ for $1 \le k \le d-1$ and $\alpha_d = \alpha_{d-1}$. The inequalities (4.44) for $d = 1$, (A.9) and (A.13) yield (4.44) for any d, while (4.45) for $d = 1$, (A.11) and (A.14) yield (4.45) for any d. Integrating with respect to t we deduce the inequalities (4.36) and (4.37). □

5 Some numerical results

In order to study the influence of the correlation coefficient α of the Gaussian noise on the speed of convergence, we have implemented in C the implicit discretization scheme $u^{n,m}$ in the case of homogeneous boundary conditions in dimension $d = 1$ for the equation (2.16).

To check the influence of the time mesh, we have fixed the space mesh n^{-1} with $n = 500$ and taken the smallest time mesh m_0^{-1} with $m_0 = 20736$. Using one trajectory of the noise F, we have approximated by the Monte-Carlo method $e(m_i) = \mathbb{E}(|u^{n,m_0}(1,.5) - u^{n,m_i}(1,.5)|^2)$ and $\hat{e}(m_i) = \sup_{x\in[0,1]} \mathbb{E}(|u^{n,m_0}(1,x) - u^{n,m_i}(1,x)|^2)$ for 13 divisors m_i of m_0, ranging from $m_1 = 854$ to $m_{13} = 144$. These simulations have been done for various values of α, including the case of the space-time white noise. Assuming that u^{n,m_0} is close to u, according to (3.10) and (3.23), these errors should behave like $C\,[m_i^{-(1-\frac{\alpha}{2})} + n^{-(2-\alpha)}] \sim m_i^{-(1-\frac{\alpha}{2})}$ for this choice of n and m_i. Thus, we have computed the linear regression coefficients $c(t)$ and $d(t)$ (resp. $\hat{c}(t)$ and $\hat{d}(t)$) of $\ln(e(m_i))$ (resp. of $\ln(\hat{e}(m_i))$), i.e., of the approximation of $\ln(e(m_i))$ by $c(t)\ln(m_i) + d(t)$ as well as the corresponding standard deviation sd (resp. $\hat{sd}$) for $K = 3200$ Monte-Carlo iterations in the case $\sigma(x) = 0.2\,x + 1$ and $b(x) = x + 2$.

The study of the influence of the space mesh is done in a similar way; we fix the time mesh m^{-1} with $m = 32000$ and let the smallest space mesh $n_0 = 432$. Again for various divisors of n_0, using one trajectory of the noise F we have approximated $\varepsilon(n_i) = \mathbb{E}(|u^{n_0,m}(1,.5) - u^{n_i,m}(1,.5)|^2)$ and $\tilde{\varepsilon}(n_i) = \mathbb{E}(|u^{n_0,m}(1,.5) - u^{n_i,m}(1,.5)|^2)$ for the 7 divisors n_i of n_0 ranging from 72 to 12. Assuming that $u^{n_0,m}$ is close to u, according to (3.10) and (3.23), these errors should behave like $C\,[m^{-(1-\frac{\alpha}{2})} + n_i^{-(2-\alpha)}] \sim n_i^{-(2-\alpha)}$ for this choice of n_i and m. Thus, we have computed the linear regression coefficients $\gamma(x)$ and $\delta(x)$ (resp. $\hat{\gamma}(t)$ and $\hat{\delta}(t)$) of $\ln(\varepsilon(n_i))$ (resp. of $\ln(\hat{\varepsilon}(n_i))$), i.e., of the approximation of $\ln(\varepsilon(n_i))$ by $\gamma(x)\ln(n_i) + \delta(x)$ as well as the corresponding standard deviation SD (resp. $\hat{SD}$) for $K = 3200$ iterations in the case $\sigma(x) = 1$ and $b(x) = 2x+3$. Both sets of results are summarized as follows.

α	Theoretical exponent	$c(t)$ $x=\frac{1}{2}$	sd $x=\frac{1}{2}$	$\hat{c}(t)$ $\sup_x$	$\hat{sd}$ $\sup_x$
White noise	0.5	0.6665	0.0063	0.6330	0.0108
0.9	0.55	0.6954	0.0121	0.6853	0.0130
0.8	0.6	0.7548	0.0098	0.7203	0.0134
0.7	0.65	0.7512	0.0089	0.7508	0.0186
0.6	0.7	0.8158	0.0143	0.8007	0.0090
0.5	0.75	0.8826	0.0144	0.8512	0.0089
0.4	0.8	0.8987	0.0100	0.9112	0.0113
0.3	0.85	0.9592	0.0117	0.9135	0.0117
0.2	0.9	0.9891	0.0116	0.9563	0.0147
0.1	0.95	1.1797	0.0114	1.0219	0.0120

α	Theoretical exponent	$\gamma(x)$ $x=\frac{1}{2}$	SD $x=\frac{1}{2}$	$\hat{\gamma}(x)$ $\sup_x$	$\hat{SD}$ $\sup_x$
White noise	1.0	1.2513	0.0346	1.2504	0.0268
0.9	1.1	1.3467	0.0340	1.3361	0.0201
0.8	1.2	1.4347	0.0336	1.4251	0.0211
0.7	1.3	1.5460	0.0305	1.5050	0.0298
0.6	1.4	1.5869	0.0210	1.5859	0.0274
0.5	1.5	1.6714	0.0280	1.6671	0.0272
0.4	1.6	1.7704	0.0283	1.7259	0.0259
0.3	1.7	1.8381	0.0280	1.7911	0.0232
0.2	1.8	1.8978	0.0274	1.8503	0.0208
0.1	1.9	1.9236	0.0208	1.9054	0.0229

Finally, since our method applies in the case of non-linear coefficients, we have performed similar computations for $e(m_i)$, $\hat{e}(m_i)$, $\varepsilon(n_j)$ and $\hat{\varepsilon}(n_j)$ for $1 \le i \le 13$ and $1 \le j \le 7$ with $K = 3000$ iterations in the case $\sigma(x) = b(x) = 1 + 0.2\,\cos(x)$. The corresponding results are summarized as follows

α	Theoretical exponent	$c(t)$ $x=\frac{1}{2}$	sd $x=\frac{1}{2}$	$\hat{c}(t)$ $\sup_x$	$\hat{sd}$ $\sup_x$
White noise	0.5	0.4915	0.0602	0.5200	0.0431
0.8	0.6	0.5550	0.0449	0.6070	0.0496
0.5	0.75	0.7244	0.0176	0.7947	0.0431
0.2	0.9	0.8607	0.0225	0.8571	0.0429
α	Theoretical exponent	$\gamma(x)$ $x=\frac{1}{2}$	SD $x=\frac{1}{2}$	$\hat{\gamma}(x)$ $\sup_x$	$\hat{SD}$ $\sup_x$
White noise	1.0	1.0278	0.0790	0.8263	0.1056
0.8	1.2	1.3628	0.0830	1.1276	0.0684
0.5	1.5	1.5626	0.0710	1.5507	0.0686
0.2	1.8	1.7351	0.0708	1.4875	0.0768

In this semi-linear case, the speed of convergence is worse and the precision is less than in the previous linear case.

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A Appendix.

We start this section with some results concerning the Green kernel G_d in arbitrary dimension $d \geq 1$. As in the previous sections, we will suppose that G_d and its discretized versions are defined with the homogeneous Dirichlet conditions on δQ; all the results stated remain true for the Neumann ones.

Lemma A.1 *Let $d \geq 1$ and $\alpha \in]0, 2 \wedge d[$. There exists some constant $C > 0$ depending only on α, such that for all x, x' in $Q = [0,1]^d$ and $0 < t \leq t' \leq T$:*

$$\sup_{y \in Q} \|G_d(t,y,\cdot)\|^2_{(\alpha)} \leq C\, t^{-\frac{\alpha}{2}}\,, \tag{A.1}$$

$$\int_0^{+\infty} \|G_d(t,x,\cdot) - G_d(t,x',\cdot)\|^2_{(\alpha)}\, dt \leq C\, |x - x'|^{2-\alpha}\,, \tag{A.2}$$

$$\sup_{x \in Q} \int_0^t \|G_d(t'-s,x,\cdot) - G_d(t-s,x,\cdot)\|^2_{(\alpha)}\, ds \leq C\, |t'-t|^{1-\frac{\alpha}{2}}\,, \tag{A.3}$$

$$\sup_{x \in Q} \int_t^{t'} \|G_d(t'-s,x,\cdot)\|^2_{(\alpha)}\, ds \leq C\, |t'-t|^{1-\frac{\alpha}{2}}\,. \tag{A.4}$$

Proof: To prove (A.1), recall the upper estimate of $|G_d|$ given in (2.9). We remark that $\exp\big(-c|x-y|^2 t^{-1}\big) \leq \exp\big(-c\,|y-z|^2\, t^{-1}\big)$ if $|x-y| \geq |y-z|$, while $|y-z|^{-\alpha} \geq |x-y|^{-\alpha}$ if $|x-y| \leq |y-z|$. Hence (A.1) follows from

$$\sup_{x \in Q} \|G_d(t,x,\cdot)\|^2_{(\alpha)} \leq C t^{-d} \left(\int_0^{+\infty} e^{-c\frac{u^2}{t}} u^{-\alpha+d-1}\, du \right) \left(\int_0^{+\infty} e^{-c\frac{v^2}{t}} v^{d-1}\, dv \right).$$

We now prove (A.2) and set $x' = x + v$. Then, for $0 < t \leq |v|^2$, we have

$$\|G_d(t,x,\cdot) - G_d(t,x',\cdot)\|^2_{(\alpha)} \leq 2\,[\,\|\,G_d(t,x,\cdot)\,\|^2_{\alpha} + \|\,G_d(t,x',\cdot)\,\|^2_{\alpha}\,]\,.$$

The change of variables defined by $x-y=|v|\,\eta$, $x-z=|v|\,\xi$ and $t=|v|^2\,s$ in the first integral (and a similar one with x' instead of x in the second one), combined with (A.1), yields

$$\int_0^{|v|^2}\|G_d(t,x,\cdot)-G_d(t,x',\cdot)\|_{(\alpha)}^2\,dt \le C|v|^{2-\alpha}\int_0^1 s^{-d}\,ds\Big\{\iint_{\{|\xi-\eta|\le|\eta|\}} e^{-c\frac{|\xi-\eta|^2}{s}} \times|\xi-\eta|^{-\alpha}e^{-c\frac{|\xi|^2}{s}}d\xi d\eta+\iint_{\{|\xi-\eta|\ge|\eta|\}} e^{-c\frac{|\eta|^2+|\xi|^2}{s}}|\eta|^{-\alpha}d\xi d\eta\Big\}\le C|v|^{2-\alpha}.$$

On the other hand, if $t\ge|v|^2$, for every $j\in\{1,\cdots,d\}$, we use the following well-known estimate (see e.g. [6]): $\left|\frac{\partial}{\partial x_j}G_d(t,x,y)\right|\le Ct^{-\frac{d+1}{2}}\exp\left(-c\frac{|x-y|^2}{t}\right)$ and the fact that $|G_d(t,x,y)-G_d(t,x',y)|\le\sum_{i=1}^d\left(\prod_{j=1}^{i-1}|G(t,x_j,y_j)|\right)|G(t,x_i,y_i)-G(t,x'_i,y_i)|\left(\prod_{j=i+1}^d|G(t,x'_j,y_j)|\right)$ for $G:=G_1$. Thus, Taylor's formula and for every $i\in\{1\cdots,d\}$ such that $v_i\ne0$, the change of variables $x_j-y_j=v_i\,\eta_j$, $x_j-z_j=v_i\,\xi_j$ for $j\le i$, $x'_j-y_j=v_i\,\eta_j$, $x'_j-z_j=v_i\,\xi_j$ for $j\ge i+1$, and $t=v_i^2\,s$ yield

$$\int_{|v|^2}^{+\infty}\|G_d(t,x,\cdot)-G_d(t,x',\cdot)\|_{(\alpha)}^2\,dt\le C\sum_{i=1}^d 1_{\{v_i\ne0\}}|v_i|^{2-\alpha}\int_1^{+\infty}s^{-(d+1)}ds\int_{-1}^1 d\lambda \times\Big(\prod_{j\ne i}\int_{\mathbb{R}^2}e^{-c\frac{|\eta_j|^2}{s}}|\eta_j-\xi_j|^{-\alpha_j}e^{-c\frac{|\xi_j|^2}{s}}d\xi_j d\eta_j\Big)\int_{\mathbb{R}^2}e^{-c\frac{|\eta_i+\lambda|^2}{s}}|\eta_i-\xi_i|^{-\alpha_i}e^{-c\frac{|\xi_i+\lambda|^2}{s}}d\xi_i d\eta_j.$$

Splitting again the integrals between $\{|\xi_j-\eta_j|\le|\eta_j|\}$ and $\{|\xi_j-\eta_j|\ge|\eta_j|\}$ for $j\ne i$ and $\{|\xi_i-\eta_i|\le|\eta_i+\lambda|\}$ and $\{|\xi_i-\eta_i|\ge|\eta_i+\lambda|\}$ yields

$$\int_{|v|^2}^{+\infty}\|G_d(t,x,\cdot)-G_d(t,x',\cdot)\|_{(\alpha)}^2\le C\,|v|^{2-\alpha}\int_1^{+\infty}s^{-(1+\alpha)}ds\le C\,|v|^{2-\alpha}.$$

This completes the proof of (A.2). On the other hand, (A.3) is obtained using similar arguments and the change of variables defined by $t-s=h\,r$, $y-x=\sqrt{h}\,\eta$ and $z-x=\sqrt{h}\,\xi$ (where $h=t'-t>0$), Taylor's formula and the estimate $\left|\frac{\partial}{\partial t}G_d(t,x,y)\right|\le Ct^{-\frac{d+2}{2}}e^{-c\frac{|x-y|^2}{t}}$. □

We recall the following well-known set of estimates. The proofs can be found in [18] for $d=1$ and are easily deduced for any $d\ge2$. By convention set $G_d(t,x,y)=0$ if $t\le0$.

Lemma A.2 *For $x,x'\in Q$, $0\le t<t'\le T$ and $\mu\in]0,1[$,*

$$\int_0^{+\infty}\int_Q|G_d(t,x,y)-G_d(t,x',y)|\,dy\,dt \le C\,|x-x'|\,, \tag{A.5}$$

$$\int_0^{t'}\int_Q|G_d(t-s,x,y)-G_d(t'-s,x,y)|\,dy\,ds \le C\,|t'-t|^{\mu}\,. \tag{A.6}$$

The following technical results are needed to obtain refined estimates for the discretized kernels G^n and $G^{n,m}$. The proofs, based on simple comparison between series and integrals for piecewise monotone functions, are omitted.

Lemma A.3 *For any $c\ge0$ there exists a constant $C>0$ such that, for $K\ge0$, $\beta\in[0,1[$, $t>0$, $a>1$ and $J_0\ge1$,*

$$\sum_{j=J_0}^{\infty}j^{-\beta}e^{-ctj^2}\le Ce^{-ctJ_0^2}\left[1+t^{-\frac{1-\beta}{2}}\right],\ \sum_{j=1}^{\infty}j^Ke^{-ctj^2}\le C\,\left[1+t^{-\frac{K+1}{2}}\right]\,e^{-ct}\,, \tag{A.7}$$

$$\sum_{j=J_0}^{\infty}\Big(1+\frac{cTj^2}{m}\Big)^{-1}\leq Cm^{\frac12}T^{-\frac12},\ \sum_{j=J_0}^{\infty}\Big(1+\frac{cTj^2}{m}\Big)^{-a}\leq\frac{Cm}{J_0T(a-1)}\Big(1+\frac{cTJ_0^2}{m}\Big)^{-a+1}. \quad (A.8)$$

The following lemma bounds the $\|\ \|_1$ and $\|\ \|_\alpha$ norms of $(G_d)^n(t,x,.)$.

Lemma A.4 *There exists a constant $C>0$ such that for every $t>0$, $d\geq 1$, $\lambda>0$ and $0<\alpha<\beta<d\wedge 2$,*

$$\sup_n\sup_{x\in Q}\|(G_d)^n(t,x,.)\|_1\leq C\left(1+t^{-\lambda}\right)e^{-ct}, \quad (A.9)$$

$$\sup_n\sup_{x\in Q}\|(G_d)^n(t,x,.)\|_{(\alpha)}^2\leq C\,t^{-\frac{\beta}{2}}\,e^{-ct}, \quad (A.10)$$

$$\sup_{x\in Q}\|(G_d)^n(t,x,.)\|_{(\alpha)}^2\leq C\,n^\alpha\,e^{-ct}. \quad (A.11)$$

Proof: It suffices to check these inequalities for $x=l/n$, $0\leq l\leq n$. We at first prove them for $d=1$. Using Abel's summation method, we have, since $j\longmapsto\lambda_j^n=-4n^2\sin^2\left(\frac{j\pi}{2n}\right)$ is decreasing, for $\kappa=\kappa_n(y)$,

$$|G^n(t,x,y)|\leq Ce^{\lambda_1^n t}\Big\{\Big|\sin\big(\pi\frac{x-\kappa_n(y)}{2}\big)\Big|^{-1}+\Big|\sin\big(\pi\frac{x+\kappa_n(y)}{2}\big)\Big|^{-1}\Big\}. \quad (A.12)$$

Fix $0<\lambda<\frac12$ and $t>0$; (A.9) and (A.7) with $\beta=0$, $J_0=1$ yield (A.9) for $d=1$. Let $A_n^i(x)$ be the sets defined by (4.7) and for $0\leq l\leq n$, set $D_n^{(i)}(l)=A_n^i(l/n)$. Then $dx\left(D_n^{(i)}(l)\right)\leq\frac{C}{n}$ and, if $y\notin D_n^{(3)}(l)$, one has $|x-k_n(y)|\geq\dfrac23|x-y|$, $|x+k_n(y)|\geq\dfrac23|x+y|$ and similarly, if $y\notin D_n^{(2)}(l)$, $|x-k_n(y)|\geq\dfrac12|x-y|$, $|x+k_n(y)|\geq\dfrac12|x+y|$. Thus, for every $n\geq 1$ and $0\leq l\leq n$, $\|G^n(t,l/n,.)\|_{(\alpha)}^2\leq C(T_1+T_2+T_3)$, where T_i, $1\leq i\leq 3$ is the integral of $|G^n(t,l/n,y)|\,|y-z|^{-\alpha}\,|G(t,l/n,z)|$ respectively on the set $A_1=\{(y,z):y\in D_n^{(2)}(l),z\in D_n^{(2)}(l)\}$, $A_2=\{(y,z):y\in D_n^{(2)}(l)^c,\,z\in D_n^{(2)}(l)^c,\,|y-z|\leq n^{-1}\}$ and $A_3=\{(y,z):y\in D_n^{(2)}(l)^c,\,z\in D_n^{(2)}(l)^c,\,|y-z|\geq n^{-1}\}$. Thus, using (A.7) and (A.12), we deduce upper estimates of T_i for $1\leq i\leq 3$ which imply (A.10) when $d=1$.
Again, to prove (A.11), it suffices to show that $\sup_{0\leq l\leq n}\|G^n(t,l/n,.)\|_{(\alpha)}^2\leq C\,e^{-ct}\,n^\alpha$. Using the sets A_i, $1\leq i\leq 3$, the inequality (A.12), the crude estimate $|G^n(t,x,y)|\leq C\,n\,e^{-ct}$, and replacing in products involving two of the terms $|x-y|^{-1}$, $|x-z|^{-1}$ and $|y-z|^{-\frac{\alpha}{2}}$ the largest norm by the smallest one, a similar computation yields for $\alpha<\lambda<1$ and $0<\mu<1$,

$$\|G^n(t,x,.)\|_{(\alpha)}^2\leq C\left[n^{\lambda+\mu}\,n^{-\lambda-\mu+\alpha}+n^\alpha\right]e^{-ct}\leq C\,n^\alpha\,e^{-ct}.$$

Since $(G_d)^n(t,x,y)=\prod_{i=1}^d G^n(t,x_i,y_i)$, (A.9) for $d=1$ immediately yields (A.9) for any d. For $d\geq 2$, and $1\leq i\leq d-1$, set $\alpha_i=\alpha\,2^{-i}$ and set $\alpha_d=\alpha_{d-1}$. Then using (2.2), the inequality (A.10) (resp. (A.11)) for $d=1$, we deduce (A.10) (resp. (A.11)) for every d. □
We now prove a similar result for the norms of $(G_d)^{n,m}(t,x,.)$.

Lemma A.5 *For every $\lambda\in]0,\frac12[$ and $\beta\in]\alpha,d\wedge 2[$, there exist positive constants c and C such that for every $t\in]0,T]$,*

$$\sup_{x\in Q}\|(G_d)^{n,m}(t,x,.)\|_1\leq Ce^{-ct}\big(1+t^{-\lambda}\big) \quad (A.13)$$

$$\sup_{x\in Q}\|(G_d)^{n,m}(t,x,.)\|_{(\alpha)}^2\leq Ce^{-ct}\Big[(n\wedge\sqrt{m})^\alpha\wedge\big(1+t^{-\frac{\beta}{2}}\big)\Big]+Ct^{-\frac{\beta}{2}}e^{-ctm}. \quad (A.14)$$

Proof : For $m \geq 1$, set $(\bar{G}_d)^{n,m}(t,x,y) := (G_d)^{n,m}(t,x,y) - (\tilde{G}_d)^{n,m}(t,x,y)$, where

$$(\tilde{G}_d)^{n,m}(t,x,y) = \sum_{\underline{\mathbf{k}} \in \{1,\cdots,[(n\wedge\sqrt{m})-1)]\}^d} \prod_{i=1}^{d} \big(1 - Tm^{-1}\lambda^n_{k_i}\big)^{-[\frac{mt}{T}]} \varphi^n_{\underline{\mathbf{k}}}(x)\varphi_{\underline{\mathbf{k}}}(\kappa_n(y)). \tag{A.15}$$

Let $(\tilde{G}_1)^{n,m} = \tilde{G}^{n,m}$ and $(\bar{G}_1)^{n,m} = \bar{G}^{n,m}$. Since $j \to \big(1 - \frac{T}{m}\lambda^n_j\big)^{-[\frac{mt}{T}]}$ is decreasing, $|\tilde{G}^{n,m}(t + Tm^{-1}, x, y)| \leq C(n \wedge \sqrt{m})e^{-ct}$ and Abel's summation method yields that for $x = \frac{l}{n}$ and $\kappa_n(y) = \frac{k}{n}$: $|\tilde{G}^{m,n}(t + T\,m^{-1}, x, y)| \leq Ce^{-ct}\Big[\frac{1}{|x-\kappa_n(y)|} + \frac{1}{x+\kappa_n(y)} + \frac{1}{2-x-\kappa_n(y)}\Big]$. Finally, since for $j \leq \sqrt{m}$, $\ln\Big(1+\frac{T}{m}\,4n^2\sin\big(\frac{j\pi}{2n}\big)\Big) \geq C\,j^2\,T\,m^{-1}$. Using (A.7) we deduce $|\tilde{G}^{n,m}(t+Tm^{-1}, x, y)| \leq C\sum_{j=1}^{(n\wedge\sqrt{m})-1} e^{-ctj^2} \leq Ce^{-ct}\big(1 + t^{-\frac{1}{2}}\big)$. Thus, repeating the arguments used to prove (A.9) - (A.11) we deduce that for $\lambda > 0$, $0 < \alpha < \beta < d \wedge 2$,

$$\sup_{x\in Q} \|\tilde{G}^{n,m}(t,x,.)\|_1 \leq C\,e^{-ct}\left(1 + t^{-\lambda}\right), \tag{A.16}$$

$$\sup_{x\in Q} \|\tilde{G}^{n,m}(t,x,.)\|^2_{(\alpha)} \leq C\,e^{-ct}\left[\left(1 + t^{-\frac{\beta}{2}}\right) \wedge (n \wedge \sqrt{m})^{\alpha}\right]. \tag{A.17}$$

We finally give an upper estimate of the norms of $(\bar{G}_d)^{n,m}(t,x,.)$ and thus we suppose that $\sqrt{m} < n$. Using (A.8) we deduce the existence of positive constants c, C such that for $t \leq \frac{2T}{m}$, $\sup_{x,y\in Q} |\bar{G}^{n,m}(t,x,y)| \leq C\int_{\sqrt{cT2^{-1}}}^{+\infty}(mT)^{-\frac{1}{2}}(1+y^2)^{-([\frac{mt}{T}]+1)}dy$. Hence for $t \leq 2Tm^{-1}$, since $[\frac{mt}{T}] = 1$ or 2, for $x, y \in Q$, $|\bar{G}^{n,m}(t,x,y)| \leq C \leq \sqrt{m} \leq t^{-\frac{1}{2}}$ while for $t \geq 2Tm^{-1}$, $|\bar{G}^{n,m}(t,x,y)| \leq C(mT)^{-\frac{1}{2}}\int_{\sqrt{cT}}^{+\infty} y(1+y^2)^{-([\frac{mt}{T}]+1)}dy \leq Ct^{-1}m^{-\frac{1}{2}} \leq Ct^{-\frac{1}{2}}$. This implies that

$$\sup_{x,y\in Q} |\bar{G}^{n,m}(t,x,y)| \leq C\left(1 + t^{-\frac{1}{2}}\right). \tag{A.18}$$

Furthermore, $j \to (1 - T\lambda^n_j/m)^{-[\frac{mt}{T}]}$ decreases and $\big(1 - T\lambda^n_{\sqrt{m}}/m\big)^{-([\frac{mt}{T}]+1)} \leq C\,e^{-cTm}$. Hence for $x = l/n$ and $\kappa_n(y) = k/n$ by Abel's summation method

$$|\bar{G}^{n,m}(t + T\,m^{-1}, x, y)| \leq C\,e^{-ctm}\left[\frac{1}{|x-\kappa_n(y)|} + \frac{1}{x+\kappa_n(y)} + \frac{1}{2-x-\kappa_n(y)}\right]. \tag{A.19}$$

An argument similar to that used to prove (A.9) implies that for $\lambda \in]0, \frac{1}{2}[$, there exists $C > 0$ such that for $t \in]0, T[$,

$$\sup_{x\in Q} \|\bar{G}^{n,m}(t,x,.)\|_1 \leq C\left(1 + t^{-\lambda}\right). \tag{A.20}$$

Finally, for $x = l/n$ let $\bar{D}^i_m(l) = \{z \in [0,1] : |x-z| \leq i\sqrt{m}, \text{or } x+z \leq i\sqrt{m}, \text{or } 2-x-z \leq i\sqrt{m}\}$. Then since $n \geq \sqrt{m}$, for $y \notin \bar{D}^3_m(l)$ we deduce that $|x - \kappa_n(y) \geq \frac{1}{2}|x-y|$, $|x + \kappa_n(y) \geq \frac{1}{2}|x-y|$ and $|2-x-\kappa_n(y) \geq \frac{1}{2}|x-y|$. Hence, the arguments used to prove (A.10) with $d = 1$ and (A.11) show that $\|\bar{G}^{n,m}(t+T\,m^{-1}, x, .)1_{\bar{D}^3_m(l)^c}\|^2_{(\alpha)} \leq Ce^{-ctm}m^{\frac{\alpha}{2}}$. Furthermore, (A.18) and (A.19) imply that the same upper estimate holds for $\|\bar{G}^{n,m}(t+T\,m^{-1}, x, .)1_{\bar{D}^3_m(l)}\|^2_{(\alpha)}$. These inequalities imply that for $\beta \in]\alpha, d \wedge 2[$,

$$\sup_{x\in[0,1]} \|\bar{G}^{n,m}(t,x,.)\|^2_{(\alpha)} \leq C\,e^{-ctm}\,m^{\frac{\alpha}{2}}\left[1 + (tm)^{-\frac{\beta}{2}}\right]. \tag{A.21}$$

Hence (A.21) and (A.17) imply that (A.14) holds for $d = 1$. Finally, as in the proof of Lemma A.4, (A.13) and (A.14) hold for any $d \geq 1$. □

We finally prove upper estimates for the norms of time increments of $(G_d)^n$ and set by convention $(G_d)^n(t,x,.) = 0$ if $t \leq 0$.

Lemma A.6 *For any $T > 0$, there exists $C > 0$ such that for any $h > 0$*

$$\sup_{n\geq 1}\sup_{x\in Q}\sup_{t\in[0,T]}\int_0^{t+h}\|(G_d)^n(t-s,x,.)-(G_d)^n(t+h-s,x,.)\|_1\,ds\leq C\,h^{\frac{1}{2}}\,, \tag{A.22}$$

$$\sup_{n\geq 1}\sup_{x\in Q}\sup_{t\in[0,T]}\int_0^{t+h}\|(G_d)^n(t-s,x,.)-(G_d)^n(t+h-s,x,.)\|_{(\alpha)}^2\,ds\leq C\,h^{1-\frac{\alpha}{2}}\,. \tag{A.23}$$

Proof: Inequality (A.9) implies that for $x\in Q$, $\lambda>0$ and $n\geq 1$, $\int_t^{t+h}\int_Q|(G_d)^n(t+h-s,x,y)|dyds\leq C\int_0^h s^{-\lambda}ds\leq Ch^{1-\lambda}$. Using (A.11) we deduce that for any $\tilde c>0$ and $h\leq n^{-2}$,

$$\int_0^{\tilde c h}\|(G_d)^n(s,x,.)\|_{(\alpha)}^2\,ds\leq\int_0^{\tilde c h}n^\alpha\,ds\leq C\,h^{1-\frac{\alpha}{2}}. \tag{A.24}$$

Suppose that $s\geq n^{-2}$ and that $d=1$. Then by (A.7), $|G^n(s,x,y)|\leq C\,(1+s^{-\frac{1}{2}})$. Then using (A.12) and proceeding as in the proof of (A.11), replacing the sets $\mathcal{A}_n^i(x)$ defined by (4.7) by the sets $\mathcal{A}_h^i(x)=\{y\in[0,1]:|y-x|\leq i\sqrt{h}$ or $y+x\leq i\sqrt{h}$ or $2-x-y\leq i\sqrt{h}\}$, since we have assumed that $n^{-1}\leq\sqrt{s}$, we deduce that $\|G^n(s,x,.)\|_{(\alpha)}^2\leq C\,s^{-\frac{\alpha}{2}}\,e^{-ct}$. Let $\alpha_k=\alpha\,2^{-k}$ for $1\leq k\leq d-1$ and $\alpha_d=\alpha_{d-1}$; using the inequality (2.2), we deduce that for $s\geq n^{-2}$, then $\|(G_d)^n(s,x,.)\|_{(\alpha)}^2\leq Cs^{-\frac{\alpha}{2}}e^{-ct}$. Hence this inequality and (A.24) imply $\int_t^{t+h}\|(G_d)^n(t+h-s,x,.)\|_{(\alpha)}^2ds\leq C\big[\int_0^{h\wedge n^{-2}}n^\alpha ds+\int_{h\wedge n^{-2}}^h s^{-\frac{\alpha}{2}}ds\big]\leq Ch^{1-\frac{\alpha}{2}}$, which yields (A.23). To complete the proof of (A.22) and (A.23), set $t'=t+h$ and consider the integrals on the interval $[0,t[$. Then for $d=1$ and $x=\frac{l}{n}$ one has $|G^n(t,x,y)-G^n(t',x,y)|=\Big|\sum_{j=1}^{n-1}e^{-4tn^2\sin^2\left(\frac{j\pi}{2n}\right)}\Big[1-e^{-4n^2h\sin^2\left(\frac{j\pi}{2n}\right)}\Big]\varphi_j(x)\varphi_j(\kappa_n(y))\Big|$. Thus (A.7) implies the existence of $C>0$ such that for any $n\geq 1$ and $x=l\,n^{-1}$, $\int_0^t\int_Q|G^n(t-s,x,y)-G^n(t'-s,x,y)|dyds\leq C\sum_{j=1}^{n-1}j^{-2}\big[(j^2h)\wedge 1\big]\leq Ch^{\frac{1}{2}}$, which proves (A.22). The inequality (A.24) proves that given any $\tilde c>0$,

$$\sup_{n\geq 1}\sup_{x\in[0,1]}\int_0^{\tilde c h}\Big(\|G^n(s,x,.)\|_{(\alpha)}^2+|G^n(s+h,x,.)\|_{(\alpha)}^2\Big)\,ds\leq C\,h^{1-\frac{\alpha}{2}}\,. \tag{A.25}$$

Fix $\tilde c>0$ large enough, let $t\geq\tilde c\,h$ and set $\Phi(j)=\exp(-4n^2t\sin^2(\frac{j\pi}{2n}))-\exp(-4n^2t'\sin^2(\frac{j\pi}{2n}))\geq 0$. Then $\Phi'(j)=2n\sin\left(\frac{j\pi}{n}\right)\big[t'\exp\big(-4n^2t'\sin^2\left(\frac{j\pi}{2n}\right)\big)-t\exp\big(-4n^2t'\sin^2\left(\frac{j\pi}{2n}\right)\big)\big]$. Then the arguments used to estimate $\Phi_2(x)$ and then $|T_2(t,x,y)|$ in the proof of Lemma 4.2 show that there exists $C>0$ such that for any $s\in[\tilde ch,T]$, $x=l\,n^{-1}$ and $y\in[0,1]$, $\big|\sum_{j=[h^{-\frac{1}{2}}]}^{n-1}\big(e^{\lambda_j^n s}-e^{\lambda_j^n(s+h)}\big)\varphi_j(x)\,\varphi_j(\kappa_n(y))\big|\leq C\,\exp\big(\lambda^n_{[h^{-\frac{1}{2}}]}s\big)\big[\frac{1}{|x-\kappa_n(y)|}+\frac{1}{x+\kappa_n(y)}+\frac{1}{2-x-\kappa_n(y)}\big]$, while (A.7) implies that $\big|\sum_{j=[h^{-\frac{1}{2}}]}^{n-1}\big(e^{\lambda_j^n s}-e^{\lambda_j^n(s+h)}\big)\varphi_j(x)\varphi_j(\kappa_n(y)\big|\leq Cs^{-\frac{1}{2}}\exp\big(\lambda^n_{[h^{-\frac{1}{2}}]}s\big)$. Using again the sets $\mathcal{A}_h^i(x)$, we deduce that for any $s\in[\tilde ch,T]$, $\sup_{n\geq 1}\sup_{x\in[0,1]}\|G^n(s,x,.)-G^n(s+h,x,.)\|_{(\alpha)}^2\leq Ch^{1-\frac{\alpha}{2}}$. Thus, $\sup_{n\geq 1}\sup_{x\in[0,1]}\int_{\tilde ch}^T\|G^n(s,x,.)-G^n(s+h,x,.)\|_{(\alpha)}^2ds\leq Ch^{1-\frac{\alpha}{2}}$. This inequality and (A.25) yield (A.23) for $d=1$. We extend the lemma in any dimension $d\geq 1$ as in the proof of Lemma A.4. □

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