STOCHASTIC 2D HYDRODYNAMICAL SYSTEMS: WONG-ZAKAI APPROXIMATION AND SUPPORT THEOREM

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ABSTRACT. We deal with a class of abstract nonlinear stochastic models with multiplicative noise, which covers many 2D hydrodynamical models including the 2D Navier-Stokes equations, 2D MHD models and 2D magnetic Bénard problems as well as some shell models of turbulence. Our main result describes the support of the distribution of solutions. Both inclusions are proved by means of a general Wong–Zakai type result of convergence in probability for nonlinear stochastic PDEs driven by a Hilbert-valued Brownian motion and some adapted finite dimensional approximation of this process.

1. INTRODUCTION

Our goal in this paper is to continue the unified investigation of statistical properties of some stochastic 2D hydrodynamical models which started in our previous paper [9]. The model introduced there covers a wide class of mathematical coupled models from 2D fluid dynamics. This class includes the 2D Navier-Stokes equations and also some other classes of two dimensional hydrodynamical models such as the magneto-hydrodynamic equation, the Boussinesq model for the Bénard convection and the 2D magnetic Bénard problem. We also cover the case of regular higher dimensional problems such as the 3D Leray α-model for the Navier-Stokes equations and some shell models of turbulence. For details we refer to [9, Sect.2.1].

Our unified approach is based on an abstract stochastic evolution equation in some Hilbert space of the form

$$\partial_t u + Au + B(u,u) + R(u) = \Xi(u)\,\dot{W}, \quad u|_{t=0} = \xi, \tag{1.1}$$

where $\dot{W}$ is a multiplicative noise white in time with spatial correlation. The hypotheses concerning the linear operator A, the bilinear mapping B and the operators R and Ξ are stated below. These hypotheses guarantee unique solvability of problem (1.1).

For general abstract stochastic evolution equations in infinite dimensional spaces we refer to [11]. However the hypotheses in [11] do not cover our hydrodynamical type model. We also note that stochastic Navier-Stokes equations were studied by many authors (see, e.g., [6, 14, 21, 32] and the references therein). In [9] we prove existence, uniqueness and provide a priori estimates for a weak (variational) solution to the abstract problem of the form (1.1), where the forcing term may also include a stochastic control term with a multiplicative coefficient. In all the concrete hydrodynamical examples mentioned above, the diffusion coefficient may contain a small multiple of the gradient of the solution. This result contains the corresponding existence and uniqueness theorems and a priori bounds for 2D Navier-Stokes equations, the Boussinesq model of the Bénard convection, and also for the GOY and Sabra shell models of turbulence. Theorem 2.4 [9] generalizes the existence result for Boussinesq or MHD equations given in [13] or [3] to the case of multiplicative noise (see also [12]) and also covers new situations such as the 2D magnetic

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Bénard problem or the 3D Leray α-model. Our main result in [9] is a Wentzell-Freidlin type large deviation principle (LDP) in an appropriate Polish space X for stochastic equations of the form (1.1) with $\Xi^\varepsilon := \sqrt{\varepsilon}\sigma$ as $\varepsilon \to 0$, which describes the exponential rate of convergence of the solution $u := u^\varepsilon$ to the deterministic solution u^0. One of the key arguments is a time increment control which provides the weak convergence needed in order to prove the large deviations principle. We refer to [9] for detailed discussion and references.

Another classical problem is that of approximation of solutions in terms of a simpler model, where the stochastic integral is changed into a "deterministic" one, replacing the noise by a random element of its reproducing kernel Hilbert space, such as a finite dimensional space approximation of its piecewise linear interpolation on a time grid. This is the celebrated Wong-Zakai approximation of the solution and once more the lack of continuity of the solution as a function of the noise has to be dealt with. This requires to make a drift correction coming from the fact that the Itô integral is replaced by a Stratonovich one. For finite-dimensional diffusion processes, this kind of approximation is well-known (see, e.g., [19], [25], [26], [33] and also the survey [29] and the references therein). There is a substantial number of publications devoted to Wong–Zakai approximations of infinite-dimensional stochastic equations. For instance, in [16], and [17] I. Gyöngy established Wong-Zakai approximations of linear parabolic evolution equations satisfying a coercivity and stochastic parabolicity condition and subject to a random finite-dimensional perturbation driven by a continuous martingale; some applications to filtering and some stochastic dynamo models are given. Z. Brzezniak, M. Capińsky and F. Flandoli [4] studied a similar problem for a linear parabolic equation subject to an perturbation driven by an infinite dimensional Gaussian noise. In [5] Z. Brzezniak and F. Flandoli and in [18] I. Gyöngy and A. Shmatkov obtained some more refined convergence, either a.s. or with some rate of convergence. Let us also mention the reference [8] by I. Chueshov and P. Vuillermot which deals with semilinear non-autonomous parabolic PDE systems perturbed by multiplicative noise and considers some applications to invariance of deterministic sets with respect to the corresponding evolutions and the reference [27] by G. Tessitore and J. Zabczyk which studies Wong-Zakai type approximations of mild solutions to abstract semilinear parabolic type equations. Similar Wong Zakai approximations were proven in Hölder spaces by A. Millet and M. Sanz-Solé in [22] for a semi-linear stochastic hyperbolic equation and by V. Bally, A. Millet and M. Sanz-Solé in [2] for the one-dimensional heat equation with a multiplicative stochastic perturbation driven by a space-time white noise. Similarly, in [7] C. Cardon-Weber and A. Millet proved similar Wong-Zakai approximation results in various topologies for the stochastic one-dimensional Burgers equation with a multiplicative perturbation driven by space-time white noise. In references [16], [17], [22], [2] and [7], these approximations were the main step to characterize the support of the distribution of these stochastic evolution equations.

Except for [7] which studies a toy model of turbulence and has a truly non-linear feature, all the above papers require linear or Lipschitz assumptions on the coefficients which do not cover non-linear models such as the Navier-Stokes equations or general hydrodynamical models. Some result on the Wong-Zakai approximation for the 2D Navier–Stokes system is proved by W. Grecksch and B. Schmalfuss in [15], but for a linear finite dimensional noise, which is a particular case of the framework used in this paper. We also mention the paper of K. Twardowska [30] which claims the convergence of Wong-Zakai approximations for 2D Navier–Stokes system with a rather general diffusion part. However, the argument used in [30] is incomplete and we were not able to fill the gaps.

In the same spirit, a slightly more general approximation result, using adapted linear interpolation of the Gaussian noise, provides a description of the support for the abstract system (1.1) and thus covers a wide class of hydrodynamical models. This is the well-known Stroock-Varadhan characterization of the support of the distribution of the solution in the Polish space X where it lives. Note that the approach used in the present paper is different from the original one of D.W. Stroock and S.R.S. Varadhan [26], and is similar to that introduced in [22] and [23]. Indeed, only one result of convergence in probability provides both inclusions needed to describe the support of the distribution. See also V. Mackevičius [20] as well as S. Aida, S. Kusuoka and D. Stroock [1], where related results were obtained for diffusion processes using a non-adapted linear time interpolation of the noise. The technique proposed in [20] was used in [16] and [17] to characterize the support of the solution of stochastic quasi-linear parabolic evolution equations in (weighted) Sobolev spaces. The references [2], [7] and the paper by T. Nakayama [24] establish a support theorem for the one-dimensional heat or Burgers equation, and for general mild solutions to semi-linear abstract parabolic equations along the same line. However, unlike these references in a parabolic setting, the argument used in this paper does not rely on the Green function associated with the second order differential operator and deals with a nonlinear physical model. As in [9], the control equation is needed with some control defined in terms of both an element of the Reproducing Kernel Hilbert Space of the driving Brownian motion and an adapted linear finite-dimensional approximation of this Brownian. For this class of control equations we first establish a Wong-Zakai type approximation theorem (see Theorem 3.1), which is the main step of our proof and, as we believe, has an independent interest. Note that all previous works were using intensively a time Hölder regularity of the solution of either the diffusion or the evolution equation. Such a time regularity is out of reach for the Navier-Stokes equations and the general hydrodynamical models we cover.

A key ingredient of the proof of the main convergence theorem is some "time integrated" time increment which can be obtained with a better speed of convergence to zero than that needed in [9]. To our best knowledge, there is only one publication related to the support characterization of solutions to 2D hydrodynamical models. The short note [31], which states a characterization of the support for 2D Navier-Stokes equations with the Dirichlet boundary conditions, does not provide a detailed proof and refers to [30] where the argument is incomplete.

The paper is organized as follows. In Section 2 we recall the mathematical model introduced in [9]. In this section we also formulate our abstract hypotheses. Our main results are stated in Section 3 under some additional integrability property on the solution. We first formulate the Wong-Zakai type approximation Theorem 3.1, which is the main tool to characterize the support of the distribution of the solution to the stochastic hydrodynamical equations. This characterization is given in Theorem 3.2 and we show how the support characterization can be deduced. In Section 4 we provide some preliminary step where the noise is truncated. Section 6 contains the proof of Theorem 3.1. It heavily depends on the time increment speed of convergence, which is proved in Section 5. In the appendix (see Section 7) we discuss with details the way our result can be applied to different classes of hydrodynamical models and give conditions which ensure that the solution fulfills the extra integrability assumption we have imposed (see (3.1)).

2. Description of the model

2.1. **Deterministic analog.** Let $(H, |.|)$ denote a separable Hilbert space, A be an (unbounded) self-adjoint positive linear operator on H. Set $V = Dom(A^{\frac{1}{2}})$. For $v \in V$ set $\|v\| = |A^{\frac{1}{2}} v|$. Let V' denote the dual of V (with respect to the inner product $(.,.)$ of H). Thus we have the Gelfand triple $V \subset H \subset V'$. Let $\langle u, v\rangle$ denote the duality between

$u \in V$ and $v \in V'$ such that $\langle u, v\rangle = (u,v)$ for $u\in V$, $v\in H$, and let $B : V\times V \to V'$ be a mapping satisfying the condition (**B**) given below.

The goal of this paper is to study stochastic perturbations of the following abstract model in H

$$\partial_t u(t) + Au(t) + B\big(u(t), u(t)\big) + Ru(t) = f, \tag{2.1}$$

where R is a continuous operator in H. We assume that the mapping $B : V \times V \to V'$ satisfies the following antisymmetry and bound conditions:

Condition (B):

(1) *$B : V \times V \to V'$ is a bilinear continuous mapping.*

(2) *For $u_i \in V$, $i = 1, 2, 3$,*

$$\langle B(u_1, u_2)\,,\, u_3\rangle = -\,\langle B(u_1, u_3)\,,\, u_2\rangle. \tag{2.2}$$

(3) *There exists a Banach (interpolation) space $\mathcal{H}$ possessing the properties*

(i) $V \subset \mathcal{H} \subset H$;

(ii) there exists a constant $a_0 > 0$ such that

$$\|v\|^2_{\mathcal{H}} \le a_0 |v|\,\|v\| \quad \textit{for any } v \in V; \tag{2.3}$$

(iii) for every $\eta > 0$ there exists $C_\eta > 0$ such that

$$|\langle B(u_1, u_2)\,,\, u_3\rangle| \le \eta\,\|u_3\|^2 + C_\eta\,\|u_1\|^2_{\mathcal{H}}\,\|u_2\|^2_{\mathcal{H}}, \quad \textit{for } u_i \in V,\ i = 1, 2, 3. \tag{2.4}$$

Note (see [9, Remark 2.1]) that the upper estimate in (2.4) can also be written in the following two equivalent forms:

(iii-a) there exist positive constants C_1 and C_2 such that

$$|\langle B(u_1, u_2)\,,\, u_3\rangle| \le C_1\|u_3\|^2 + C_2\,\|u_1\|^2_{\mathcal{H}}\,\|u_2\|^2_{\mathcal{H}}, \quad \text{for } u_i \in V,\ i = 1, 2, 3; \tag{2.5}$$

(iii-b) there exists a constant $C > 0$ such that for $u_i \in V$, $i = 1, 2, 3$ we have:

$$|\langle B(u_1, u_2)\,,\, u_3\rangle| = |\langle B(u_1, u_3)\,,\, u_2\rangle| \le C\,\|u_1\|_{\mathcal{H}}\,\|u_2\|\,\|u_3\|_{\mathcal{H}}. \tag{2.6}$$

For $u \in V$ set $B(u) := B(u, u)$; with this notation, relations (2.2), (2.3) and (2.6) yield for every $\eta > 0$ the existence of $C_\eta > 0$ such that for $u_1, u_2 \in V$,

$$|\langle B(u_1)\,,\, u_2\rangle| \le \eta\,\|u_1\|^2 + C_\eta\,|u_1|^2\,\|u_2\|^4_{\mathcal{H}}. \tag{2.7}$$

Relations (2.2) and (2.7) imply that for any $\eta > 0$ there exists $C_\eta > 0$ such that

$$|\langle B(u_1) - B(u_2)\,,\, u_1 - u_2\rangle| = |\langle B(u_1 - u_2), u_2\rangle| \le \eta\|u_1 - u_2\|^2 + C_\eta\,|u_1 - u_2|^2\,\|u_2\|^4_{\mathcal{H}} \tag{2.8}$$

for all $u_1, u_2 \in V$. As it was explained in [9] the main motivation for condition (**B**) is that it covers a wide class of 2D hydrodynamical models including Navier-Stokes equations, magneto-hydrodynamic equations, Boussinesq model for the Bénard convection, magnetic Bénard problem, 3D Leray α-model for Navier-Stokes equations, Shell models of turbulence (GOY, Sabra, and dyadic models).

2.2. **Noise.** We will consider a stochastic external random force f in equation (2.1), driven by a Wiener process W and whose intensity may depend on the solution u. More precisely, let Q be a linear positive operator in the Hilbert space H which belongs to the trace class, and hence is compact. Let $H_0 = Q^{\frac12} H$. Then H_0 is a Hilbert space with the scalar product

$$(\phi, \psi)_0 = (Q^{-\frac12}\phi, Q^{-\frac12}\psi),\ \forall \phi, \psi \in H_0,$$

together with the induced norm $|\cdot|_0 = \sqrt{(\cdot,\cdot)_0}$. The embedding $i : H_0 \to H$ is Hilbert-Schmidt and hence compact, and moreover, $i\ i^* = Q$. Let $L_Q \equiv L_Q(H_0, H)$ denote the space of linear operators $S : H_0 \mapsto H$ such that $SQ^{\frac12}$ is a Hilbert-Schmidt operator from

H to H. The norm on the space L_Q is defined by $|S|^2_{L_Q} = tr(SQS^*)$, where S^* is the adjoint operator of S. The L_Q-norm can be also written in the form:

$$|S|^2_{L_Q} = tr([SQ^{1/2}][SQ^{1/2}]^*) = \sum_{k\geq 1} |SQ^{1/2}\psi_k|^2 = \sum_{k\geq 1} |[SQ^{1/2}]^*\psi_k|^2 \tag{2.9}$$

for any orthonormal basis $\{\psi_k\}$ in H.

Let $W(t)$ be a Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, taking values in H and with covariance operator Q. This means that W is Gaussian, has independent time increments and that for $s,t\geq 0$, $f,g\in H$,

$$\mathbb{E}(W(s),f) = 0 \quad \text{and} \quad \mathbb{E}(W(s),f)(W(t),g) = \big(s\wedge t\big)\,(Qf,g).$$

We also have the representation

$$W(t) = \lim_{n\to\infty} W_n(t) \ \text{ in } \ L^2(\Omega; H) \ \text{ with } W_n(t) = \sum_{1\leq j\leq n} q_j^{1/2}\beta_j(t)e_j, \tag{2.10}$$

where β_j are standard (scalar) mutually independent Wiener processes, $\{e_j\}$ is an orthonormal basis in H consisting of eigen-elements of Q, with $Qe_j = q_je_j$. For details concerning this Wiener process we refer to [11], for instance. Let $(\mathcal{F}_t, t\geq 0)$ denote the Brownian filtration, that is the smallest right-continuous complete filtration with respect to which $(W(t), t\geq 0)$ is adapted.

We now define some adapted approximations of the processes β_j and W. For all integers $n\geq 1$ and $k=0,1,\ldots,2^n$, set $t_k = kT2^{-n}$ and define step functions $\underline{s}_n, s_n, \bar{s}_n : [0,T]\to[0,T]$ by the formulas

$$\underline{s}_n = t_k,\ s_n = t_{k-1}\vee 0,\ \bar{s}_n = t_{k+1} \quad \text{for} \quad s\in[t_k, t_{k+1}[. \tag{2.11}$$

It is clear that $s_n < \underline{s}_n < \bar{s}_n$. Now we set $\dot{\tilde{\beta}}^n_j(s) = T^{-1}2^n\left(\beta_j(\underline{s}_n) - \beta_j(s_n)\right)$, for every $s\in[0,T]$, and thus we obtain an adapted approximation for $\dot{\beta}_j(s)$ given by the formula

$$\dot{\tilde{\beta}}^n_j(s) = T^{-1}2^n\left[\beta_j(t_k) - \beta_j(t_{k-1}\vee 0)\right], \quad \text{for} \quad s\in[t_k,t_{k+1}[. \tag{2.12}$$

Clearly $\dot{\tilde{\beta}}^n_j(s)\equiv 0$ for $s\in[0,t_1[$ and $\dot{\tilde{\beta}}^n_j(s) = T^{-1}2^n\beta_j(t_1)$ for $s\in[t_1,t_2[$. We also let

$$\dot{\widetilde{W}}^n(s) = \sum_{1\leq j\leq n} \dot{\tilde{\beta}}^n_j(s) q_j^{1/2} e_j \tag{2.13}$$

denote the corresponding finite-dimensional adapted approximation of $\dot{W}$.

Lemma 2.1. *There exists an absolute constant $\alpha_0>0$ such that for every $\alpha > \alpha_0/\sqrt{T}$ and $t\in[0,T]$ we have as $n\to\infty$*

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{1\leq j\leq n}\sup_{s\leq t}\left|\dot{\tilde{\beta}}^n_j(s)\right| \geq \alpha n^{1/2}2^{n/2}\right) = \lim_{n\to\infty}\mathbb{P}\left(\sup_{s\leq t}\left|\dot{\widetilde{W}}^n(s)\right|_{H_0} \geq \alpha n 2^{\frac{n}{2}}\right) = 0.$$

Proof. One can see that

$$\tilde{\Omega}_n(t) = \Big\{\sup_{1\leq j\leq n}\sup_{s\leq t}\left|\dot{\tilde{\beta}}^n_j(s)\right| \geq \alpha n^{1/2}2^{n/2}\Big\} \subset \bigcup_{1\leq j\leq n}\ \bigcup_{0\leq k<2^n}\Big\{|\gamma^k_j| \geq \alpha T^{1/2}n^{1/2}\Big\},$$

where $\gamma^k_j = T^{-1/2}2^{n/2}[\beta_j(t_{k+1}) - \beta_j(t_k)]$ are independent standard normal Gaussian random variables. Therefore, if $\alpha > \sqrt{T^{-1}2\ln 2}$ we have

$$\mathbb{P}\big(\tilde{\Omega}_n(t)\big) \ \leq \ n2^nP(|\gamma^0_1|\geq\alpha\sqrt{Tn}) = \frac{n2^{n+1}}{\sqrt{2\pi}}\int_{\alpha\sqrt{Tn}}^{\infty} e^{-z^2/2}dz$$

$$\leq \frac{n^{1/2}2^{n+1}}{\alpha\sqrt{2\pi T}}\int_{\alpha\sqrt{Tn}}^{\infty} ze^{-z^2/2}dz = \frac{2n^{1/2}}{\alpha\sqrt{2\pi T}}\exp\Big[n\Big(-\alpha^2 T/2+\ln 2\Big)\Big].$$

This proves the first convergence result. The second one follows immediately from the estimate

$$\sup_{s\leq t}\Big|\dot{\widetilde{W}}^n(s)\Big|^2_{H_0} = \sup_{s\leq t}\sum_{1\leq j\leq n}\Big|\dot{\tilde{\beta}}_j^n(s)\Big|^2 \leq n\sup_{1\leq j\leq n}\sup_{s\leq t}\Big|\dot{\tilde{\beta}}_j^n(s)\Big|^2.$$

□

In the sequel, we will localize the processes using the following set:

$$\Omega_n(t) = \left\{\sup_{j\leq n}\sup_{s\leq t}\Big|\dot{\tilde{\beta}}_j^n(s)\Big| \leq \alpha n^{1/2}2^{n/2}\right\}\cap\left\{\sup_{s\leq t}\Big|\dot{\widetilde{W}}^n(s)\Big|_{H_0}\leq \alpha n 2^{n/2}\right\}. \tag{2.14}$$

It is clear that $\Omega_n(t)\subset\Omega_n(s)$ for $t>s$ and $\Omega_n(t)\in\mathcal{F}_t$. Furthermore, Lemma 2.1 implies that $\mathbb{P}\left(\Omega_n(T)^c\right)\to 0$ as $n\to\infty$.

For any $n\geq 1$, we introduce the following localized processes $\dot{\tilde{\beta}}_j^n$ and $\dot{\tilde{W}}^n$:

$$\dot{\beta}_j^n(t) = \dot{\tilde{\beta}}_j^n(t)1_{\Omega_n(t)},\ j\leq n, \quad \dot{W}^n(t) = \dot{\widetilde{W}}^n(t)1_{\Omega_n(t)}. \tag{2.15}$$

For all integers n and $j=1,\cdots,n$, $(\dot{\beta}_j^n(t), 0\leq t\leq T)$ (resp. $(\dot{W}^n(t), 0\leq t\leq T)$) are $(\mathcal{F}_t)$-adapted $\mathbb{R}$ (resp. H_0) valued processes.

2.3. **Diffusion coefficients.** We need below two diffusion coefficients σ and $\tilde{\sigma}$ which map H into $L_Q(H_0,H)$. They are assumed to satisfy the following growth and Lipschitz conditions:

Condition (S): *The maps $\sigma,\tilde{\sigma}$ belong to $\mathcal{C}\big(H;L_Q(H_0,H)\big)$ and satisfy:*

(1) *There exist non-negative constants K_i and L such that for every $u,v\in H$:*

$$|\sigma(u)|^2_{L_Q}+|\tilde{\sigma}(u)|^2_{L_Q} \leq K_0+K_1|u|^2, \tag{2.16}$$

$$|\sigma(u)-\sigma(v)|^2_{L_Q}+|\tilde{\sigma}(u)-\tilde{\sigma}(v)|^2_{L_Q} \leq L|u-v|^2. \tag{2.17}$$

(2) *Moreover, for every $N>0$,*

$$\lim_{n\to\infty}\sup_{|u|\leq N}|\tilde{\sigma}(u)-\tilde{\sigma}(u)\circ\Pi_n|_{L_Q}=0, \tag{2.18}$$

where $\Pi_n : H_0\to H_0$ denote the projector defined by $\Pi_n u=\sum_{k=1}^n\big(u\,,\,e_k\big)\,e_k$, where $\{e_k, k\geq 1\}$ is the orthonormal basis of H made by eigen-elements of the covariance operator Q and used in (2.10).

Condition (DS): *For every integer $j\geq 1$ let $\sigma_j,\tilde{\sigma}_j : H\mapsto H$ be defined by*

$$\sigma_j(u)=q_j^{1/2}\sigma(u)e_j\,,\quad \tilde{\sigma}_j(u)=q_j^{1/2}\tilde{\sigma}(u)e_j\,,\quad \forall u\in H. \tag{2.19}$$

We assume that the maps $\tilde{\sigma}_j$ are twice Fréchet differentiable and satisfy

(1) *For every integer $N\geq 1$ there exist positive constants $C_i(N)$, $i=1,2,3$ such that:*

$$C_1(N):=\sup_j\sup_{|u|\leq N}|D\tilde{\sigma}_j(u)|_{L(H,H)}<\infty, \tag{2.20}$$

$$C_2(N):=\sup_j\sup_{|u|\leq N}|D^2\tilde{\sigma}_j(u)|_{L(H\times H,H)}<\infty, \tag{2.21}$$

$$\sup_j\sup_{|u|\leq N}\|[D\tilde{\sigma}_j(u)]^*v\|\leq C_3(N)\|v\| \quad \textit{for every} \quad v\in V. \tag{2.22}$$

(2) *For every integer $n \geq 1$, let the functions ϱ_n, $\tilde{\varrho}_n : H \mapsto H$ be defined by*

$$\varrho_n(u) = \sum_{1\leq j\leq n} D\tilde{\sigma}_j(u)\,\sigma_j(u)\,, \quad \tilde{\varrho}_n(u) = \sum_{1\leq j\leq n} D\tilde{\sigma}_j(u)\tilde{\sigma}_j(u), \quad \forall u \in H, \tag{2.23}$$

where σ_j and $\tilde{\sigma}_j$ are given by (2.19). *For every integer $N \geq 1$ there exist positive constants $\bar{K}_N, \bar{C}_N$ such that:*

$$\sup_{|u|\leq N} \sup_n \{|\varrho_n(u)| + |\tilde{\varrho}_n(u)|\} \leq \bar{K}_N, \tag{2.24}$$

$$\sup_{|u|,|v|\leq N} \sup_n \{|\varrho_n(u) - \varrho_n(v)| + |\tilde{\varrho}_n(u) - \tilde{\varrho}_n(v)|\} \leq \bar{C}_N|u - v|. \tag{2.25}$$

(3) *Furthermore, there exist mappings ϱ, $\tilde{\varrho} : H \mapsto H$ such that every integer $N \geq 1$*

$$\lim_{n\to\infty} \sup_{|u|\leq N} \{|\varrho_n(u) - \varrho(u)| + |\tilde{\varrho}_n(u) - \tilde{\varrho}(u)|\} = 0. \tag{2.26}$$

Remark 2.2. As a simple (non-trivial) example of diffusion coefficient σ and $\tilde{\sigma}$ satisfying Conditions **(S)** and **(DS)**, we can consider the case when $\tilde{\sigma}(u)$ is proportional to $\sigma(u)$, i.e. $\tilde{\sigma}(u) = c_0\sigma(u)$ for some constant c_0 and $\sigma(u)$ is an affine function of u of the form:

$$\sigma(u)f = \sum_{j\geq 1} f_j\sigma_j(u) \quad \text{for} \quad f = \sum_{j\geq 1} f_j\sqrt{q_j}e_j \in H_0,$$

where $\sigma_j(u) = g_j + S_j u$, $j = 1, 2, \ldots$. Here $g_j \in H$ satisfy $\sum_{j\geq 1} |g_j|^2 < \infty$ and $S_j : H \mapsto H$ are linear operators such that $S_j^* : V \mapsto V$ and $\sum_{j\geq 1} |S_j|^2_{L(H,H)} + \sup_{j\geq 1} |S_j^*|^2_{L(V,V)} < \infty$. For instance, in the case $H = L_2(D)$ and $V = H^1(D)$, where D is a bounded domain in $\mathbb{R}^d$, our framework includes diffusion terms of the form

$$\sigma(u)dW(t) = \sum_{1\leq j\leq N} (g_j(x) + \phi_j(x)u(x))d\beta_j(t),$$

where $g_j \in L_2(D)$ and $\phi_j \in C^1(\bar{D})$, $j = 1, 2, \ldots, N$ are arbitrary functions. In this situation, another possibility to satisfy Conditions **(S)** and **(DS)** is

$$\sigma(u)dW(t) = \sum_{1\leq j\leq N} s_j([\mathcal{R}_j u](x))d\beta_j(t),$$

where $s_j : \mathbb{R} \mapsto \mathbb{R}$ are $\mathcal{C}^2$-functions such that s_j' and s_j'' are bounded, and $[\mathcal{R}_j u](x) = \int_D r_j(x, y)u(y)dy$ with sufficiently smooth kernels $r_j(x, y)$.

In order to define the sequence of processes u^n converging to u in the Wong-Zakai approximation, we need a control term, that is a coefficient G of the process acting on an element of the RKHS of W. We impose that G and R satisfy the following:

Condition (GR): *Let $G : H \mapsto L(H_0, H)$ and $R : H \mapsto H$ satisfy the following growth and Lipschitz conditions:*

$$|G(u)|^2_{L(H_0,H)} \leq K_0 + K_1|u|^2, \quad |G(u) - G(v)|^2_{L(H_0,H)} \leq L|u - v|^2\,, \tag{2.27}$$

$$|R(u)| \leq R_0(1 + |u|), \quad |R(u) - R(v)| \leq R_1|u - v|, \tag{2.28}$$

for some nonnegative constants K_i, R_i, $i = 0, 1$, L and for every $u, v \in H$ (we can assume that K_i and L are the same constants as in (2.16) *and* (2.17)*).*

2.4. **Basic problem.** Let $X := \mathcal{C}\big([0,T];H\big)\cap L^2\big(0,T;V\big)$ denote the Banach space endowed with the norm defined by

$$\|u\|_X = \Big\{ \sup_{0\le s\le T} |u(s)|^2 + \int_0^T \|u(s)\|^2 ds \Big\}^{\frac{1}{2}}. \tag{2.29}$$

The class $\mathcal{A}$ of admissible random shifts is the set of H_0-valued $(\mathcal{F}_t)-$predictable stochastic processes h such that $\int_0^T |h(s)|_0^2 ds < \infty$, a.s. For any $M>0$, let

$$S_M = \Big\{ h\in L^2(0,T;H_0) : \int_0^T |h(s)|_0^2 ds \le M \Big\},\ \mathcal{A}_M = \{h\in\mathcal{A} : h(\omega)\in S_M,\ a.s.\}. \tag{2.30}$$

Assume that $h\in\mathcal{A}_M$ and $\xi\in H$ is $\mathcal{F}_0$-measurable random element such that $\mathbb{E}|\xi|^4<\infty$. Then under the conditions **(B)**, **(GR)**, (2.16) and (2.17) in **(S)**, Theorem 2.4 [9] implies that there exists a unique $(\mathcal{F}_t)$-predictable solution $u\in X$ to the stochastic problem:

$$\begin{aligned} u(t) \;&=\; \xi - \int_0^t [Au(s)+B(u(s))+R(u(s))]\,ds \\ &\quad + \int_0^t \big(\sigma+\tilde\sigma\big)(u(s))dW(s) + \int_0^t G(u(s))h(s)ds, \quad \text{a.s. for all } t\in[0,T]. \end{aligned} \tag{2.31}$$

This solution is weak in the PDE sense and strong in the probabilistic meaning. Moreover, for this solution there exists a constant $C := C(K_i, L, R_i, T, M)$ such that for $h\in\mathcal{A}_M$,

$$\mathbb{E}\Big(\sup_{0\le t\le T} |u(t)|^4 + \int_0^T \|u(t)\|^2\,dt + \int_0^T \|u(t)\|_{\mathcal{H}}^4\,dt\Big) \le C\,\big(1+\mathbb{E}|\xi|^4\big). \tag{2.32}$$

Remark 2.3. In the case of 2D Navier–Stokes equations in a domain $D\subset\mathbb{R}^2$, we can choose $\mathcal{H}$ as the space of divergent free 2D vector fields from $[L^4(D)]^2$ (see [9]). Therefore, the finiteness of the integral $\int_0^T \|u(t)\|_{\mathcal{H}}^4 dt$ stated in (2.32) is a Serrin's type condition. In the case of deterministic Navier–Stokes equations this condition implies additional regularity of weak solutions (see, e.g., [10]). For instance, they become strong solutions for an appropriate choice of the initial data. However we do not know whether similar regularity properties can be established for our abstract model without additional requirements concerning the diffusion part of the equation.

2.5. **Approximate problem.** We also consider the evolution equation on the time interval $[0,T]$:

$$\begin{aligned} u^n(t) \;&=\; \xi - \int_0^t [Au^n(s)+B(u^n(s))+R(u^n(s))]\,ds + \int_0^t \sigma((u^n(s))dW(s) \\ &\quad + \int_0^t \tilde\sigma(u^n(s))\dot{\widetilde{W}}^n(s)ds - \int_0^t \big(\varrho+\frac{1}{2}\tilde\varrho\big)(u^n(s))ds + \int_0^t G(u^n(s))h(s)ds, \quad \text{a.s.}, \end{aligned} \tag{2.33}$$

where $\dot{\widetilde{W}}^n(t)$ is defined in (2.13). Let again $h\in\mathcal{A}_M$, ξ be $\mathcal{F}_0$ measurable such that $\mathbb{E}|\xi|^4<+\infty$.

First, since $(\dot{\widetilde{W}}^n_t, t\in[0,T])$ is H_0-valued and $(\mathcal{F}_t)$-adapted, we check that the following infinite dimensional version of the Benes criterion holds: for some $\delta>0$ we have that $\sup_{0\le s\le T}\mathbb{E}\big(\big(\exp(\delta|\dot{\widetilde{W}}^n(s)|_0^2)\big)<+\infty$. This is a straightforward consequence of the inequality for some standard Gaussian random variable Z:

$$\begin{aligned} \sup_{0\le s\le T} \mathbb{E}\big(\exp\big(\delta|\dot{\widetilde{W}}^n(s)|_0^2\big)\big) &\le \prod_{1\le j\le n} \sup_{0\le s\le T} \mathbb{E}\Big(\exp(\delta T^{-2}2^{2n}|\beta_j(\underline{s}_n)-\beta_j(s)|^2)\Big) \\ &\le \Big(\mathbb{E}\big(\exp(\delta T^{-1}\,2^n\,|Z|^2)\big)\Big)^n < +\infty \end{aligned}$$

for $\delta > 0$ small enough. Therefore, for any constant γ, the measure with density $L_T^\gamma = \exp\Big(\gamma\int_0^T \dot{\widetilde{W}}^n(s)dW(s) - \frac{\gamma^2}{2}\int_0^T |\dot{\widetilde{W}}^n(s)|_0^2\,ds\Big)$ with respect to $\mathbb{P}$ is a probability $\mathbb{Q}_\gamma << \mathbb{P}$, such that the process

$$\big(W_\gamma(s) := W(s) - \gamma\widetilde{W}^n(s), 0 \le s \le T\big) \text{ is a } \mathbb{Q}_\gamma \text{ Brownian motion} \tag{2.34}$$

with values in H, and the same covariance operator Q. Then Theorem 3.1 [9] shows that

$$\begin{aligned} u^n(t) &= \xi - \int_0^t [Au^n(s) + B(u^n(s)) + R(u^n(s))]\,ds + \int_0^t \sigma((u^n(s))dW_\gamma(s) \\ &\quad - \int_0^t \big(\varrho + \frac{1}{2}\tilde{\varrho}\big)(u^n(s))ds + \int_0^t G(u^n(s))h(s)ds, \quad \text{a.s.}, \end{aligned}$$

has a unique $(\mathcal{F}_t)$-predictable solution $u^n \in X$ which satisfies (2.32) where $\mathbb{P}$ is replaced by $\mathbb{Q}_\gamma$ Therefore, $u^n \in X$ is the unique solution to problem (2.33) when $\tilde{\sigma} = -\gamma\sigma$, and u^n also satisfies (2.32) with expected values under the given probability $\mathbb{P}$, but the constant C in the right hand side depends on the constants K_i, L, R_i, T, M, and in addition, it may also depend on n.

To keep the convergence result as general as possible, in the sequel we only suppose that $\tilde{\sigma}$ satisfies Conditions **(DS)** and that problem (2.33) is well-posed in X.

3. Main results

In this section we first state Wong–Zakai approximation results (see Theorem 3.1) and then show in Theorem 3.2 how the description support can be derived from Theorem 3.1.

More precisely, let u and u^n be solutions to (2.31) and (2.33) respectively. Our first main result proves that the X norm of the difference $u^n - u$ converges to 0 in probability. This Wong–Zakai type result is the key point of the support characterization stated in Theorem 3.2 below.

Theorem 3.1. *Let conditions* **(B)**, **(S)**, **(DS)** *and* **(GR)** *hold,* $h \in \mathcal{A}_M$ *for some* $M > 0$ *and* ξ *be* $\mathcal{F}_0$ *measurable such that* $\mathbb{E}|\xi|^4 < +\infty$. *Let* u *be the solution to* (2.31) *such that:*
(i) $t \mapsto \|u(t)\|_{\mathcal{H}}$ *is continuous on* $[0,T]$ *almost surely,*
(ii) there exists $q > 0$ *such that for any constant* $C > 0$ *we have*

$$\mathbb{E}\Big(\sup_{[0,\tau_C]} \|u(t)\|_{\mathcal{H}}^q\Big) < \infty, \tag{3.1}$$

where $\tau_C := \inf\{t : \sup_{s\le t}|u(s)|^2 + \int_0^t \|u(s)\|^2 ds \ge C\} \wedge T$ *is a stopping time.*

Suppose that for every $n \ge 1$ *problem* (2.33) *is well posed and let* u^n *denote its solution. Then for every* $\lambda > 0$ *we have*

$$\lim_{n\to\infty} \mathbb{P}\Big(\sup_{t\in[0,T]} |u(t) - u^n(t)|^2 + \int_0^T \|u(s) - u^n(s)\|^2 ds \ge \lambda\Big) = 0. \tag{3.2}$$

The convergence in (3.2) allows to deduce both inclusions characterizing the support of the distribution of the solution U to the following stochastic perturbation of the evolution equation (2.1):

$$dU(t) + \big[AU(t) + B\big(U(t)\big) + R(U(t))\big]\,dt = \Xi(U(t))\,dW(t), \quad U(0) = \xi \in H, \tag{3.3}$$

where $\Xi \in \mathcal{C}\big(H; L_Q(H_0,H)\big)$ is such that conditions **(S)** and **(DS)** hold with $\tilde{\sigma} \equiv \sigma \equiv \Xi$. Thus problem (3.3) is a special case of problem (2.31).

Let $\phi \in L^2(0,T;H_0)$ and $\varrho_\Xi \equiv \varrho$ be defined by (2.23) and (2.26) with $\tilde\sigma = \sigma = \Xi$. We also consider the following (deterministic) nonlinear PDE

$$\partial_t v_\phi(t)+Av_\phi(t)+B\big(v_\phi(t)\big)+R(v_\phi(t))+\frac{1}{2}\varrho_\Xi(v_\phi(t)) = \Xi(v_\phi(t))\phi(t), \quad v_\phi(0) = \xi \in H. \quad (3.4)$$

If $B(u)$ satisfies condition **(B)** and $R : H \mapsto H$ possesses property (2.28) we can use Theorem 3.1 in [9] to obtain the existence (and uniqueness) of the solution v_ϕ to (3.4) in the space $X = \mathcal{C}\big([0,T];H\big) \cap L^2\big(0,T;V\big)$. Let

$$\mathcal{L} = \big\{v_\phi \;:\; \phi \in L^2(0,T;H_0)\big\} \subset X.$$

Our second main result is the following consequence of the approximation Theorem 3.1.

Theorem 3.2. *Let conditions* **(B)** *and* **(S)**, **(DS)** *with $\tilde\sigma \equiv \sigma \equiv \Xi$ be in force. Assume that $R : H \mapsto H$ satisfies* (2.28). *Let $(U(t), t \in [0,T])$ denote the solution to the stochastic evolution equation* (3.3) *with deterministic initial data $\xi \in H$. Suppose that conditions (i) and (ii) of Theorem 3.1 hold for this solution U. Then $\operatorname{supp} U(\cdot) = \bar{\mathcal{L}}$, where $\bar{\mathcal{L}}$ is the closure of $\mathcal{L}$ in X and $\operatorname{supp} U(\cdot)$ denotes the support of the distribution $\mathbb{P}\circ U^{-1}$, i.e., the support of the Borel measure on X defined by $\mu(\mathcal{B}) = \mathbb{P}\{\omega \;:\; U(\cdot) \in \mathcal{B}\}$ for any Borel subset $\mathcal{B}$ of X.*

Remark 3.3. Both Theorems 3.1 and 3.2 are *conditional* in the sense that they provide an approximation of the solution u or the description of its support when u satisfy some additional conditions concerning its properties in the space $\mathcal{H}$ (see the requirements (i) and (ii) in Theorem 3.1). We do not know whether these conditions can be derived from the basic requirements which we already have imposed on the model. However they can be established under additional conditions concerning operators in (2.31). For instance, we can assume that the bilinear operator B possesses the property

$$(B(u,u), Au) = 0 \;\text{ for }\; u \in Dom(A) \quad (3.5)$$

(this is the case of 2D Navier-Stokes equations in a periodic domain, see, e.g., [10]) and the diffusion coefficient $\sigma + \tilde\sigma$ satisfies the estimate

$$|A^{\frac12}(\sigma(u)+\tilde\sigma(u))|^2_{L_Q(H_0,H)} \le K(1+\|A^{1-\delta}u\|^2) \;\text{ for all }\; u \in Dom(A)$$

for some $K > 0$ and $\delta > 0$. Under these conditions we can apply Itô's formula to the norm $\|u(t)\|^2 = |A^{1/2}u(t)|^2$ and obtain that

$$\|u(t\wedge\tau_N)\|^2 + 2\int_0^{t\wedge\tau_N}|Au(s)|^2 ds = \|\xi\|^2 + 2\int_0^{t\wedge\tau_N}\big((\sigma+\tilde\sigma)(u(s))dW(s), Au(s)\big)$$
$$- 2\int_0^{t\wedge\tau_N}\big(R(u(s)) + G(u(s))h(s), Au(s)\big)\, ds + \int_0^{t\wedge\tau_N}|A^{1/2}(\sigma+\tilde\sigma)(u(s))|^2_{L_Q}\, ds$$

for an appropriate sequence of stopping times $\{\tau_N\}$ (see [9] for similar calculations). In the standard way (see [11]) this implies that $u(t) \in C(0,T;V)$ a.s. and

$$\mathbb{E}\left\{\sup_{t\in[0,T]}\|u(t)\|^2 + 2\int_0^T|Au(s)|^2 ds\right\} \le C(1+\mathbb{E}\|\xi\|^2)$$

for some constant $C > 0$. Since $V \subset \mathcal{H}$, this implies the requirements (i) and (ii) in Theorem 3.1. Unfortunately the assumption in (3.5) is rather restrictive. To our best knowledge, in our 2D hydrodynamical framework it is only valid for 2D Navier-Stokes equations with the periodic boundary conditions.

Other simple examples where we can apply Theorem 3.1 and 3.2 are the shell models of turbulence. We can consider either the GOY model or the Sabra model, or else the so-called dyadic model. Indeed, (see [9, Sect.2.1.6]) in all these models we have that

$|\langle B(u,v),w\rangle| \le C|u||A^{1/2}v||w|$, $\forall u,w\in H, \forall v \in Dom(A^{1/2})$. Thus condition (**B**) holds with $\mathcal{H} = Dom(A^s)$ for any choice of $s\in[0,1/4]$. In particular we can choose $\mathcal{H}=H$. In this case conditions (i) and (ii) in Theorems 3.1 and 3.2 trivially hold.

In the Appendix (see Section 7) we show that conditions (i) and (ii) in the statement of Theorem 3.1 can be also established under another set of hypotheses which hold for several important cases of hydrodynamical models such as (non-periodic) 2D Navier-Stokes equations and 2D MHD equations.

Proof of Theorem 3.2. The argument is similar to that introduced in [23].

In the definition of the evolution equation (2.33) let $\tilde\sigma = \Xi$, $\sigma=0$, $G=0$. Then if v_h is the solution to (3.4), where $\phi = \dot{\widetilde{W}}_n$ is the (random) element of $L^2(0,T;H_0)$ defined by (2.13), then we have that $u^n = v_{\dot{\widetilde{W}}_n}$ for u^n defined by (2.33). Note that is this case, well posedeness of (2.33) in X is easy to prove on ω by ω. Under this choice of $\tilde\sigma$, σ and G for the solution u to (2.31), we obviously have that $U(t)=u(t)$, where U solves (3.3). Therefore the Wong-Zakai approximation stated in Theorem 3.1 implies that

$$\lim_n \mathbb{P}(\|v_{\dot{\widetilde{W}}_n} - U\|_X \ge \lambda) = 0 \quad \text{for any } \lambda>0,$$

where $\|\cdot\|_X$ is the norm defined by (2.29). Therefore Support$(\mathbb{P}\circ U^{-1}) \subset \overline{\mathcal{L}}$.

Conversely, fix $h\in L^2(0,T;H_0)$, let $n\ge 1$ be an integer, $\tilde\sigma = -\sigma$ and $G=\sigma := \Xi$. Let $T_n^h : \Omega\to\Omega$ be defined by

$$T_n^h(\omega) = W(\omega) - \widetilde{W}^n(\omega) + \int_0^{\cdot} h(s)ds. \tag{3.6}$$

Then for every fixed integer $n\ge 1$, by Girsanov's theorem there exists a probability $\mathbb{Q}_n^h << \mathbb{P}$ such that T_n^h is a $\mathbb{Q}_n^h$-Brownian motion with values in H and the same covariance operator Q. Indeed, the proof is easily decomposed in two steps, using Theorem 10.14 and Proposition 10.17 in [11].

First, since $(\dot{\widetilde{W}}^n_t, t\in[0,T])$ is H_0-valued and $(\mathcal{F}_t)$-adapted, the argument used at the end of section 2 with $\gamma=1$ proves that the measure with density $L_T^1 = \exp\Big(\int_0^T \dot{\widetilde{W}}^n(s)dW(s) - \frac12\int_0^T |\dot{\widetilde{W}}^n(s)|_0^2\,ds\Big)$ with respect to $\mathbb{P}$ is a probability $\mathbb{Q}_1 << \mathbb{P}$, such that the process $\big(W_1(s) := W(s) - \widetilde{W}^n(s), 0\le s\le t\big)$ is a $\mathbb{Q}_1$ Brownian motion with values in H, and the same covariance operator Q.

Then using once more these two results, since $h\in L^2([0,T],H_0)$, the measure with density $L_T^2 = \exp\Big(-\int_0^T h(s)dW_1(s) - \frac12\int_0^T |h(s)|^2 ds\Big)$ with respect to $\mathbb{Q}_1$ is a probability $\mathbb{Q}_2 << \mathbb{P}$, such that the process

$$W_2(t) = W_1(t) + \int_0^t h(s)ds = W(t) - \widetilde{W}^n(t) + \int_0^t h(s)ds$$

is a H-valued Brownian motion under $\mathbb{Q}_2$, with covariance operator Q. Clearly $\mathbb{Q}_n^h = \mathbb{Q}_2$.

Let U denote the solution to (3.3); then, since T_n^h can be seen as a transformation of the standard Wiener space with Brownian motion $W(t)$, we deduce that $U(\cdot)(T_n^h(\omega)) = u^n(\cdot)(\omega)$ in distribution on $[0,T]$. Thus Theorem 3.1 implies that for every $\varepsilon>0$,

$$\limsup_n \mathbb{P}(\{\omega : \|U(T_h^n(\omega)) - v_h\|_X < \varepsilon\}) > 0.$$

Let $n_0\ge 1$ be an integer such that

$$\mathbb{Q}_{n_0}^h(\{\omega : \|U(\omega) - v_h\|_X < \varepsilon\}) \equiv \mathbb{P}(\{\omega : \|U(T_h^{n_0}(\omega)) - v_h\|_X < \varepsilon\}) > 0.$$

Since $\mathbb{Q}^h_{n_0} << \mathbb{P}$, this implies $\mathbb{P}(\{\omega : \|U(\omega) - v_h\|_X < \varepsilon) > 0$ which yields:

$$\overline{\mathcal{L}} \subset \text{Support}(\mathbb{P} \circ U^{-1}).$$

This completes the proof of Theorem 3.2.

4. Preliminary step in the proof of Theorem 3.1

Let $M > 0$ be such that $h \in \mathcal{A}_M$. Without loss of generality we may and do assume in the sequel that $0 < \lambda \le 1$. Fix $N \ge 1$, $m \ge 1$ and $\lambda \in]0, 1]$. Let us introduce the following stopping times which will enable us to bound several norms for u and u^n:

$$\tau_N^{(1)} = \inf\Big\{t > 0 : \sup_{s\in[0,t]} |u(s)|^2 + \int_0^t \|u(s)\|^2 ds \ge N\Big\} \wedge T,$$

$$\tau_n^{(2)} = \inf\Big\{t > 0 : \sup_{s\in[0,t]} |u(s) - u^n(s)|^2 + \int_0^t \|u(s) - u^n(s)\|^2 ds \ge \lambda\Big\} \wedge T,$$

$$\tau_n^{(3)} = \inf\Big\{t > 0 : \Big[\sup_{j\le n} \sup_{s\in[0,t]} \Big|\dot{\tilde{\beta}}_j^n(s)\Big|\Big] \vee \Big[n^{-\frac{1}{2}} \sup_{s\in[0,t]} \Big|\dot{\widetilde{W}}^n(s)\Big|_{H_0}\Big] \ge \alpha n^{1/2} 2^{n/2}\Big\} \wedge T,$$

and

$$\tau_m^{(4)} = \inf\Big\{t > 0 : \sup_{s\in[0,t]} \|u(s)\|_{\mathcal{H}} \ge m\Big\} \wedge T.$$

In the sequel, the constants N and m will be chosen to make sure that, except on small sets, $\tau_N^{(1)}$ and $\tau_m^{(4)}$ are equal to T; once this is done, only the dependence in n will be relevant. Thus once N and m have been chosen in terms of the limit process u, we let

$$\tau_n = \tau_N^{(1)} \wedge \tau_n^{(2)} \wedge \tau_n^{(3)} \wedge \tau_m^{(4)}. \tag{4.1}$$

One can see from the definition of $\tau_N^{(1)}$ and $\tau_n^{(2)}$ that

$$\sup_{s\in[0,\tau_n]} \big(|u(s)|^2 \vee |u^n(s)|^2\big) + \int_0^{\tau_n} \big(\|u(s)\|^2 \vee \|u^n(s)\|^2\big)\, ds \le 2(N+1); \tag{4.2}$$

the definition of $\tau_n^{(3)}$ yields

$$\sup_{s\le\tau_n} \Big(\Big[\sup_{j\le n} \big|\dot{\tilde{\beta}}_j^n(s)\big|\Big] \vee \Big[n^{-\frac{1}{2}} \sup_{s\le t} \big|\dot{\widetilde{W}}^n(s)\big|_{H_0}\Big]\Big) \le \alpha n^{1/2} 2^{n/2}. \tag{4.3}$$

Furthermore, the definition of $\tau_m^{(4)}$ implies

$$\sup_{s\in[0,\tau_n]} \|u(s)\|_{\mathcal{H}} \le m. \tag{4.4}$$

We use the following obvious properties; their standard proof is omitted.

Lemma 4.1. *Let $\Psi(t) \equiv \Psi(\omega, t)$ be a random, a.s. continuous, nondecreasing process on the interval $[0, T]$. Let $\tau_\lambda = \inf\{t > 0 : \Psi(t) \ge \lambda\} \wedge T$. Then*

$$\mathbb{P}(\Psi(T) \ge \lambda) = \mathbb{P}(\Psi(\tau_\lambda) \ge \lambda).$$

Let τ_* be a stopping time such that $0 \le \tau_* \le T$ and $\mathbb{P}\big(\tau_* < T\big) \le \varepsilon$. Then

$$\mathbb{P}\big(\Psi(T) \ge \lambda\big) \le \mathbb{P}\big(\Psi(\tau_\lambda \wedge \tau_*) \ge \lambda\big) + \varepsilon.$$

Apply Lemma 4.1 with $\tau_* = \tau_N^{(1)} \wedge \tau_n^{(3)} \wedge \tau_m^{(4)}$ and

$$\Psi(t) = \sup_{s\in[0,t]} |u(s) - u^n(s)|^2 + \int_0^t \|u(s) - u^n(s)\|^2 ds.$$

Since a.s. $u \in \mathcal{C}([0,T],H)$ and $\int_0^T \|u(s)\|^2\,ds < +\infty$, the map Ψ is a.s. continuous and

$$\{\tau_* < T\} \subset \{\tau_N^{(1)} < T\} \cup \{\tau_n^{(3)} < T\} \cup \{\tau_m^{(4)} < \tau_N^{(1)}\}$$
$$\subset \Big\{ \sup_{s\in[0,\tau_N^{(1)}]} |u(s)|^2 + \int_0^{\tau_N^{(1)}} \|u(s)\|^2 ds \geq N \Big\} \cup \Big\{ \sup_{s\in[0,\tau_N^{(1)}]} \|u(s)\|_{\mathcal{H}} \geq m \Big\} \cup \Omega_n(T)^c,$$

where $\Omega_n(t)$ is given by (2.14). Therefore, by Chebyshov's inequality, from (2.32) and (3.1) we deduce that

$$\mathbb{P}\big(\tau_* < T\big) \leq C_1 N^{-1} + C_2(N) m^{-q} + \mathbb{P}(\Omega_n(T)^c)\ .$$

Hence, given $\epsilon > 0$, one may choose N and then m large enough to have $C_1 N^{-1} + C_2(N) m^{-q} < \frac{\epsilon}{2}$. Using Lemma 2.1 we deduce that there exists $n_0 \geq 1$ such that for all integers $n \geq n_0$, $\mathbb{P}(\Omega_n(T)^c) < \frac{\varepsilon}{2}$. Thus Lemma 4.1 shows that in order to prove (3.2) in Theorem 3.1, we only need to prove the following: Fix $N, m > 0$; for every $\lambda > 0$,

$$\lim_{n\to\infty} \mathbb{P}\Big(\sup_{t\in[0,\tau_n]} |u(t) - u^n(t)|^2 + \int_0^{\tau_n} \|u(s) - u^n(s)\|^2 ds \geq \lambda \Big) = 0, \tag{4.5}$$

where τ_n is defined by (4.1). To check this convergence, it is sufficient to prove

$$\lim_{n\to\infty} \Big[\mathbb{E}\Big(\sup_{t\in[0,\tau_n]} |u(t) - u^n(t)|^2 \Big) + \mathbb{E} \int_0^{\tau_n} \|u(s) - u^n(s)\|^2 ds \Big] = 0. \tag{4.6}$$

The proof of this last convergence result is given in Section 6. It relies on some precise control of times increments which is proven in the next section.

5. Time increments

Let $h \in \mathcal{A}_M$, ξ be an $\mathcal{F}_0$-measurable H-valued random variable such that $\mathbb{E}|\xi|^4 < \infty$ and let u be the solution to (2.31). For any integer $N \geq 1$ and $t \in [0,T]$ set

$$\widetilde{G}_N(t) = \Big\{ \sup_{0\leq s\leq t} |u(s)| \leq N \Big\}. \tag{5.1}$$

The following lemma refines the estimates proved in [9], Lemma 4.3 and the ideas are similar; see also [12], Lemma 4.2.

Proposition 5.1. *Let $\phi_n, \psi_n : [0,T] \mapsto [0,T]$ be non-decreasing piecewise continuous functions such that*

$$0 \vee \big(s - k_0 T 2^{-n}\big) \leq \phi_n(s) \leq \psi_n(s) \leq \big(s + k_1 T 2^{-n}\big) \wedge T \tag{5.2}$$

for some integers $k_0, k_1 \geq 0$. Assume that $h \in \mathcal{A}_M$ and ξ is a $\mathcal{F}_0$-measurable, H-valued random variable such that $\mathbb{E}|\xi|^4 < \infty$. Let $\widetilde{G}_N(t)$ be given by (5.1) and u be the solution to (2.31). There exists a constant $C(N,M,T)$ such that

$$I_n = \mathbb{E} \int_0^T 1_{\widetilde{G}_N(\psi_n(s))} \, |u(\psi_n(s)) - u(\phi_n(s))|^2 \, ds \leq C(N,M,T) 2^{-3n/4} \tag{5.3}$$

for every $n = 1, 2, \ldots$

Proof. We at first consider the case $\phi_n(s) = 0 \vee (s - k_0 T 2^{-n})$ for some $k_0 \geq 0$; then

$$I_n = \mathbb{E} \int_0^{t_{k_0}} 1_{\widetilde{G}_N(\psi_n(s))} \, |u(\psi_n(s)) - \xi|^2 \, ds + I_n'$$

where $t_{k_0} = k_0 T 2^{-n}$ and

$$I_n' = \mathbb{E} \int_{t_{k_0}}^T 1_{\widetilde{G}_N(\psi_n(s))} \, |u(\psi_n(s)) - u(\phi_n(s))|^2 \, ds.$$

Therefore, using the definition (5.1) one can see that

$$I_n \le C_{N,T} 2^{-n} + I'_n. \tag{5.4}$$

Furthermore, Itô's formula yields

$$|u(\psi_n(s)) - u(\phi_n(s))|^2 = 2\int_{\phi_n(s)}^{\psi_n(s)} (u(r)-u(\phi_n(s)), du(r)) + \int_{\phi_n(s)}^{\psi_n(s)} |(\sigma+\tilde\sigma)(u(r))|_{L_Q}^2 dr,$$

so that $I'_n = \sum_{1\le i\le 6} I_{n,i}$, where

$$\begin{aligned}
I_{n,1} &= 2\mathbb{E}\Big(\int_{t_{k_0}}^T ds 1_{\widetilde{G}_N(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} \big(u(r)-u(\phi_n(s)), (\sigma+\tilde\sigma)(u(r))dW(r)\big)\Big),\\
I_{n,2} &= \mathbb{E}\Big(\int_{t_{k_0}}^T ds 1_{\widetilde{G}_N(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} |(\sigma+\tilde\sigma)(u(r))|_{L_Q}^2\, dr\Big),\\
I_{n,3} &= 2\,\mathbb{E}\Big(\int_{t_{k_0}}^T ds 1_{\widetilde{G}_N(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} \big(G(u(r))\, h(r)\,,\, u(r)-u(\phi_n(s))\big)\, dr\Big),\\
I_{n,4} &= -2\,\mathbb{E}\Big(\int_{t_{k_0}}^T ds 1_{\widetilde{G}_N(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} \big\langle A\, u(r)\,,\, u(r)-u(\phi_n(s))\big\rangle\, dr\Big),\\
I_{n,5} &= -2\,\mathbb{E}\Big(\int_{t_{k_0}}^T ds 1_{\widetilde{G}_N(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} \big\langle B(u(r))\,,\, u(r)-u(\phi_n(s))\big\rangle\, dr\Big),\\
I_{n,6} &= -2\,\mathbb{E}\Big(\int_{t_{k_0}}^T ds 1_{\widetilde{G}_N(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} \big(R(u(r))\,,\, u(r)-u(\phi_n(s))\big)\, dr\Big).
\end{aligned}$$

Clearly $\widetilde{G}_N(\psi_n(s)) \subset \widetilde{G}_N(r)$ for $r \le \psi_n(s)$. This means that $|u(r)| \vee |u(\phi_n(s))| \le N$ in the above integrals. We use this observation in the considerations below.

The Burkholder-Davis-Gundy inequality and (2.16) yield

$$\begin{aligned}
|I_{n,1}| &\le 6\int_{t_{k_0}}^T ds\, \mathbb{E}\Big(\int_{\phi_n(s)}^{\psi_n(s)} |(\sigma+\tilde\sigma)(u(r))|_{L_Q}^2 1_{\widetilde{G}_N(r)}\, |u(r)-u(\phi_n(s))|^2\, dr\Big)^{\frac12}\\
&\le 6\sqrt{2(K_0+K_1N^2)} \int_{t_{k_0}}^T ds\, \mathbb{E}\Big(\int_{\phi_n(s)}^{\psi_n(s)} 1_{\widetilde{G}_N(r)}\, |u(r)-u(\phi_n(s))|^2\, dr\Big)^{\frac12}.
\end{aligned} \tag{5.5}$$

This implies that

$$|I_{n,1}| \le C_N \int_0^T |\psi_n(s)-\phi_n(s)|^{1/2}\, ds \le C_N T\sqrt{k_0+k_1}\, 2^{-\frac{n}{2}}. \tag{5.6}$$

In a similar way using (2.16) again we deduce that

$$|I_{n,2}| \le C_N \int_0^T |\psi_n(s)-\phi_n(s)|\, ds \le C_N T(k_0+k_1)2^{-n}. \tag{5.7}$$

The growth condition (2.27) yields

$$|I_{n,3}| \le C_N \int_0^T ds \int_{\phi_n(s)}^{\psi_n(s)} |h(r)|_0 dr \le C_N \int_0^T ds \int_{0\vee(s-k_0T2^{-n})}^{(s+k_1T2^{-n})\wedge T} |h(r)|_0 dr,$$

and Fubini's theorem implies

$$|I_{n,3}| \le C_N \int_0^T |h(r)|_0 dr 2^{-n} \le C(N,T,M)2^{-n}. \tag{5.8}$$

Using Schwarz's inequality we deduce that

$$I_{n,4} \;\leq\; 2\mathbb{E}\Big(\int_{t_{k_0}}^T ds 1_{\widetilde{G}_N(\psi_n(s))}\int_{\phi_n(s)}^{\psi_n(s)} dr\big[-\|u(r)\|^2+\|u(r)\|\|u(\phi_n(s))\|\big]\Big). \tag{5.9}$$

The antisymmetry relation (2.2) and inequality (2.7) yield

$$\begin{aligned}\big|\big\langle B(u(r)),u(r)-u(\phi_n(s))\big\rangle\big| &= \big|\big\langle B(u(r)),u(\phi_n(s))\big\rangle\big|\\ &\leq \frac{1}{2}\|u(r)\|^2+C|u(r)|^2\|u(\phi_n(s))\|_{\mathcal{H}}^4.\end{aligned}$$

Therefore,

$$\begin{aligned}|I_{n,5}|\leq \mathbb{E}\Big(\int_{t_{k_0}}^T ds 1_{\widetilde{G}_N(\psi_n(s))}\int_{\phi_n(s)}^{\psi_n(s)}\|u(r)\|^2\,dr\Big)\\ +\,2C\mathbb{E}\Big(\int_0^T ds 1_{\widetilde{G}_N(\psi_n(s))}\|u(\phi_n(s))\|_{\mathcal{H}}^4\int_{\phi_n(s)}^{\psi_n(s)}|u(r)|^2\,dr\Big).\end{aligned}$$

Using this inequality, (5.9) and (2.3), we deduce:

$$\begin{aligned}I_{n,4}+I_{n,5}&\leq \mathbb{E}\int_{t_{k_0}}^T 1_{\widetilde{G}_N(\psi_n(s))}\|u(\phi_n(s))\|^2\;|\psi_n(s)-\phi_n(s)|ds\\ &\quad+\,C_N\mathbb{E}\int_{t_{k_0}}^T 1_{\widetilde{G}_N(\psi_n(s))}\|u(\phi_n(s))\|_{\mathcal{H}}^4\;|\psi_n(s)-\phi_n(s)|ds\\ &\leq C(N,T)2^{-n}\mathbb{E}\int_{t_{k_0}}^T 1_{\widetilde{G}_N(\psi_n(s))}\|u(\phi_n(s))\|^2 ds = C(N,T)2^{-n}\mathbb{E}\int_{t_{k_0}}^T\|u(s-t_{k_0})\|^2 ds.\end{aligned}$$

Hence, this last inequality and (2.32) imply

$$I_{n,4}+I_{n,5}\leq C(N,T)\,2^{-n}. \tag{5.10}$$

A similar easier computation based on the growth condition (2.28) on R yields

$$|I_{n,6}|\leq 4R_0N(1+N)\int_0^T|\psi_n(s)-\phi_n(s)|\,ds\leq C(T,N)\,(k_0+k_1)\,2^{-n}. \tag{5.11}$$

Thus by (5.4), (5.6)–(5.8), (5.10) and (5.11), when $\phi_n(s)=0\vee(s-k_0T2^{-n})$ we obtain:

$$I_n=\mathbb{E}\int_0^T 1_{\widetilde{G}_N(\psi_n(s))}\,|u(\psi_n(s))-u(\phi_n(s))|^2\,ds\leq C(N,M,T)2^{-n/2}. \tag{5.12}$$

In order to obtain (5.3), we need to improve the bound for $I_{n,1}$ (see (5.6)). We make it using (5.12) and we again assume at first that $\phi_n(s)=0\vee(s-k_0T2^{-n})$. Let us denote by $\chi_{i,n}(s)=\underline{(s+(i+1)T2^{-n})}_n$ the step function defined with the help of relations (2.11) and set

$$\begin{aligned}\mathcal{I}_n^{(i,-)}&=\int_{t_{k_0}}^{T-t_{k_1}} ds\;\mathbb{E}\int_{s+iT2^{-n}}^{\chi_{i,n}(s)}1_{\widetilde{G}_N(r)}\,|u(r)-u(\phi_n(s))|^2\,dr,\\ \mathcal{I}_n^{(i,+)}&=\int_{t_{k_0}}^{T-t_{k_1}} ds\;\mathbb{E}\int_{\chi_{i,n}(s)}^{s+(i+1)T2^{-n}}1_{\widetilde{G}_N(r)}\,|u(r)-u(\phi_n(s))|^2\,dr.\end{aligned}$$

The inequality (5.5) implies that

$$|I_{n,1}|\;\leq\; C_N\int_{t_{k_0}}^T ds\;\mathbb{E}\Big(\int_{\phi_n(s)}^{\psi_n(s)}1_{\widetilde{G}_N(r)}\,|u(r)-u(\phi_n(s))|^2\,dr\Big)^{\frac{1}{2}}$$

$$\leq \quad C_{N,T} 2^{-n} + C_{N,T} \Big[\sum_{-k_0 \leq i < k_1} \Big(\mathcal{I}_n^{(i,-)} + \mathcal{I}_n^{(i,+)} \Big) \Big]^{1/2}. \tag{5.13}$$

For any r from the interval $[s + iT2^{-n}, \chi_{i,n}(s))[$ we have $\bar{r}_n = \chi_{i,n}(s))$; therefore

$$\mathcal{I}_n^{(i,-)} \quad \leq \quad 2 \int_{t_{k_0}}^{T-t_{k_1}} ds \, \mathbb{E}\Big[\int_{s+iT2^{-n}}^{\chi_{i,n}(s)} 1_{\widetilde{G}_N(r)} \, |u(r) - u(\bar{r}_n)|^2 \, dr$$
$$+ T2^{-n} 1_{\widetilde{G}_N(\chi_{i,n}(s))} |u(\chi_{i,n}(s)) - u(\phi_n(s))|^2 \Big].$$

Thus, using Fubini's theorem and (5.12) we can conclude that

$$\mathcal{I}_n^{(i,-)} \leq C(N,M,T) 2^{-3n/2}.$$

Similarly

$$\mathcal{I}_n^{(i,+)} \quad \leq \quad 2 \int_{t_{k_0}}^{T-t_{k_1}} ds \, \mathbb{E}\Big[\int_{\chi_{i,n}(s)}^{s+(i+1)T2^{-n}} 1_{\widetilde{G}_N(r)} \, |u(r) - u(\underline{r}_n)|^2 \, dr$$
$$+ T2^{-n} 1_{\widetilde{G}_N(\chi_{i,n}(s))} |u(\chi_{i,n}(s)) - u(\phi_n(s))|^2 \Big] \leq C(N,M,T) 2^{-3n/2}.$$

Hence (5.13) implies $I_{n,1} \leq C(N,M,T) 2^{-3n/4}$. This inequality and the above upper estimates for $I_{n,i}$ with $i \neq 1$ prove (5.3) in the case $\phi_n(s) = \phi_n^*(s) := 0 \vee (s - k_0 T 2^{-n})$. In the general case, we can write

$$1_{\widetilde{G}_N(\psi_n(s))} \, |u(\psi_n(s)) - u(\phi_n(s))|^2$$
$$\leq 2 \Big(1_{\widetilde{G}_N(\psi_n(s))} \, |u(\psi_n(s)) - u(\phi_n^*(s))|^2 + 1_{\widetilde{G}_N(\phi_n(s))} \, |u(\phi_n(s)) - u(\phi_n^*(s))|^2 \Big);$$

this concludes the proof of Proposition 5.1 for functions ϕ_n and ψ_n which satisfy (5.2). □

We also need a similar for the time increments of the approximate solutions u^n.

Proposition 5.2. *Let $\phi_n, \psi_n : [0,T] \mapsto [0,T]$ be non-decreasing piecewise continuous functions such that condition (5.2) is satisfied for some positive integers k_0 and k_1. Fix $M > 0$, let $h \in \mathcal{A}_M$ and ξ be a $\mathcal{F}_0$-measurable, H-valued random variable such that $\mathbb{E}|\xi|^4 < \infty$. Let u^n be the solution to (2.33); for $N > 0$ set*

$$G_N^n(t) = \Big\{ \sup_{0 \leq s \leq t} |u^n(s)| \leq N \Big\} \cap \Big\{ \int_0^t \|u^n(s)\|^2 ds \leq N \Big\} \cap \Omega_n(t), \tag{5.14}$$

where $\Omega_n(t)$ is defined in (2.14) and let τ_n be the stopping time defined in (4.1). There exists a constant $C(N,M,T)$ such that

$$\tilde{I}_n = \mathbb{E} \int_0^{\tau_n} 1_{G_N^n(\psi_n(s))} \, |u^n(\psi_n(s)) - u^n(\phi_n(s))|^2 \, ds \leq C(N,M,T) n^{3/2} 2^{-3n/4} \tag{5.15}$$

for every $n = 1, 2, \ldots$

Proof. We use the same idea as in the proof of Proposition 5.1 and at first suppose that $\phi_n(s) = 0 \vee (s - k_0 T 2^{-n})$ for some $k_0 \geq 0$. Let $t_{k_0} = k_0 T 2^{-n}$; then we have

$$\tilde{I}_n \leq C_{N,T} 2^{-n} + \tilde{I}_n', \tag{5.16}$$

where

$$\tilde{I}_n' = \mathbb{E} \int_{\tau_n \wedge t_{k_0}}^{\tau_n} 1_{G_N^n(\psi_n(s))} \, |u^n(\psi_n(s)) - u^n(\phi_n(s))|^2 \, ds. \tag{5.17}$$

Itô's formula applied to $|u^n(.) - u^n(\phi_n(s))|^2$ implies that $\tilde{I}'_n = \sum_{1\le i\le 6} \tilde{I}_{n,i}$, where

$$\begin{aligned}
\tilde{I}_{n,1} &= 2\mathbb{E}\int_{\tau_n\wedge t_{k_0}}^{\tau_n} ds 1_{G_N^n(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} \big(u^n(r) - u^n(\phi_n(s))\,,\,\sigma(u^n(r))dW(r)\big),\\
\tilde{I}_{n,2} &= \mathbb{E}\int_{\tau_n\wedge t_{k_0}}^{\tau_n} ds 1_{G_N^n(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} |\sigma(u^n(r))|^2_{L_Q} dr,\\
\tilde{I}_{n,3} &= 2\,\mathbb{E}\int_{\tau_n\wedge t_{k_0}}^{\tau_n} ds 1_{G_N^n(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} \Big(u^n(r) - u^n(\phi_n(s))\,,\,\tilde{\sigma}(u^n(r))\dot{\widetilde{W}}^n(r)\Big)\,dr,\\
\tilde{I}_{n,4} &= 2\,\mathbb{E}\int_{\tau_n\wedge t_{k_0}}^{\tau_n} ds 1_{G_N^n(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} \big(u^n(r) - u^n(\phi_n(s))\,,\,G(u^n(r))h(r)\big)\,dr,\\
\tilde{I}_{n,5} &= 2\,\mathbb{E}\int_{\tau_n\wedge t_{k_0}}^{\tau_n} ds 1_{G_N^n(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} \Big(u^n(r) - u^n(\phi_n(s)), (\varrho + \frac{1}{2}\tilde{\varrho} - R)(u^n(r))\Big)\,dr,\\
\tilde{I}_{n,6} &= -2\,\mathbb{E}\int_{\tau_n\wedge t_{k_0}}^{\tau_n} ds 1_{G_N^n(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} \big(u^n(r) - u^n(\phi_n(s))\,,\,Au^n(r) + B(u^n(r))\big)\,dr.
\end{aligned}$$

Estimates for $\tilde{I}_{n,2}$, $\tilde{I}_{n,4}$, $\tilde{I}_{n,5}$ are obvious. Indeed, we can first extend outward integration to the time interval $[t_{k_0}, T]$ and then use the growth conditions (2.16), (2.24) and (2.26). This yields the estimate

$$|\tilde{I}_{n,i}| \le C(N,T)\,2^{-n}, \quad i = 2,4,5. \tag{5.18}$$

Note that (5.18) holds as soon as $0 \le \psi_n(s) - \phi_n(s) \le C2^{-n}$ for some constant $C > 0$, and does not require the specific form of ϕ_n. Schwarz's inequality and Condition **(B)** imply

$$\begin{aligned}
-(u^n(r) - u^n(\phi_n(s))\,,\,Au^n(r) + B(u^n(r))) &\le C_1\|u(\phi_n(s))\|^2 + C_2|u(r)|^2\|u(\phi_n(s))\|^4_{\mathcal{H}}\\
&\le C_0\|u(\phi_n(s))\|^2\left[1 + |u(r)|^2|u(\phi_n(s))|^2\right].
\end{aligned}$$

Therefore, if $\phi_n(s) = (s - t_{k_0}) \vee 0$ we deduce

$$\begin{aligned}
|\tilde{I}_{n,6}| &\le C(N,T)\mathbb{E}\int_{\tau_n\wedge t_{k_0}}^{\tau_n} ds 1_{G_N^n(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} \|u^n(\phi_n(s))\|^2 dr\\
&\le C(N,T)2^{-n}\mathbb{E}\int_{\tau_n\wedge t_{k_0}}^{\tau_n} 1_{G_N^n(\psi_n(s))}\|u^n(s - t_{k_0})\|^2\,ds\\
&\le C(N,T)2^{-n}\mathbb{E}\int_{(\tau_n\wedge t_{k_0} - t_{k_0})^+}^{(\tau_n - t_{k_0})^+} \|u^n(s)\|^2 ds\\
&\le C(N,T)2^{-n}\mathbb{E}\int_0^{\tau_n} \|u^n(s)\|^2 ds \le C(N,T)2^{-n}.
\end{aligned} \tag{5.19}$$

Using (4.1) and the upper bound of $\psi_n(s) - \phi_n(s)$ (and not the specific form of ϕ_n), we deduce

$$|\tilde{I}_{n,3}| \le C_N n 2^{-\frac{n}{2}}\mathbb{E}\int_{\tau_n\wedge t_{k_0}}^{\tau_n} ds 1_{G_N^n(\psi_n(s))} \int_{\phi_n(s)}^{\psi_n(s)} |u^n(r) - u^n(\phi_n(s))| dr \le C(T,N) n 2^{-\frac{n}{2}}. \tag{5.20}$$

Since for $s \le t$ we have $G_N^n(t) \subset G_N^n(s)$, the local property of stochastic integrals, the linear growth condition (2.16) and Schwarz's inequality imply that

$$|\tilde{I}_{n,1}| \le 2\sqrt{T}\Big(\mathbb{E}\int_{t_{k_0}\wedge\tau_n}^{\tau_n} ds 1_{G_N^n(\psi_n(s))}$$

$$\times \Big| \int_{\phi_n(s)\wedge\tau_n}^{\psi_n(s)\wedge\tau_n} 1_{G_N^n(r)} \big(\sigma(u^n(r))\,,\, u^n(r)-u^n(\phi_n(s))\big)\, dW(r)\Big|^2\Big)^{\frac12}$$

$$\le 2\sqrt{T}\Big(\int_0^T ds \mathbb{E}\int_{\phi_n(s)}^{\psi_n(s)} 1_{G_N^n(r)}\, 1_{[0,\tau_n]}(r)\, |\sigma(u^n(r))|^2_{L_Q} |u^n(r)-u^n(\phi_n(s))|^2\, dr\Big)^{\frac12} \quad (5.21)$$

$$\le C(N,T)\Big(\int_0^T ds \int_{\phi_n(s)}^{\psi_n(s)} dr\Big)^{\frac12} = C(N,T)\, 2^{-n/2}. \quad (5.22)$$

The inequalities (5.16) - (5.22) yield for $\phi_n(s)=(s-t_{k_0})\vee 0$:

$$\tilde{I}_n = \mathbb{E}\int_0^{\tau_n} 1_{G_N^n(\psi_n(s))}\, |u^n(\psi_n(s))-u^n(\phi_n(s))|^2\, ds \le C(N,M,T) n 2^{-n/2}. \quad (5.23)$$

In order to obtain the final estimate in (5.15) we need to improve the upper estimates of $\tilde{I}_{n,1}$ and $\tilde{I}_{n,3}$. This can be done in a way similar to that used in the proof of previous Proposition. One can easily see that (5.23) holds when $\phi_n\le\psi_n$ satisfy the assumptions of the Proposition and ϕ_n is piece-wise constant. Then (5.20) and Schwarz's inequality obviously imply that

$$|\tilde{I}_{n,3}| \le C_N n^{3/2} 2^{-3n/4}.$$

Thus, to conclude the proof we need to deal with the improvement of $I_{n,1}$. Let the function ϕ_n be piece-wise constant; then given $r\in[\phi_n(s),\psi_n(s)]$, we have $\phi_n(s)\in\{\underline{r-i2^{-n}} : 0\le i\le k_0\}$. Therefore, using the inequality (5.21), Fubini's theorem and (5.23) applied with the functions $\phi_{n,i}(r)=\underline{r-iT2^{-n}}$ and $\psi_n(r)=r$ we deduce

$$|\tilde{I}_{n,1}| \le C(N,T)\Big(\sum_{0\le i\le k_0} \mathbb{E}\int_0^{\tau_n} dr 1_{G_N^n(r)} |u^n(\psi_n(r))-u^n(\phi_{n,i}(r)))|^2 \int_r^{(r+k_0T2^{-n})\wedge T} ds\Big)^{\frac12}$$
$$\le (k_0+1)^{\frac12}\, C(N,M,T) n^{\frac12} 2^{-3n/4};$$

this concludes the proof of (5.15) when ϕ_n is piece-wise constant. To deduce that this inequality holds for arbitrary functions ϕ_n and ψ_n satisfying (5.2), apply (5.15) for $\tilde{\phi}_n(s) = (s-t_{k_0})_n$ and either $\tilde{\psi}_n=\phi_n$ or $\tilde{\psi}_n=\psi_n$; this concludes the proof. □

Proposition 5.2 implies that

$$\mathbb{E}\int_0^{\tau_n} 1_{G_N^n(s)}\Big(|u^n(s)-u^n(s_n)|^2 + |u^n(s)-u^n(\underline{s}_n)|^2\Big)\, ds \le C(T,N,M) n^{3/2} 2^{-3n/4} \quad (5.24)$$

where $G_N^n(t)$ is defined by (5.14). This is precisely what we need below.

6. Proof of convergence result

The aim of this section is to prove Theorem 3.1. For every integer $n\ge 1$, τ_n is the stopping time defined by (4.1) and we prove (4.6). In the estimates below, constants may change from line to line, but we indicate their dependence on parameters when it becomes important.

From equations (2.33) and (2.31) we deduce:

$$\begin{aligned} u^n(t)-u(t) \;=\; & -\int_0^t \Big[A[u^n(s)-u(s)] + B(u^n(s)) - B(u(s)) + R(u^n(s)) - R(u(s))\Big]\, ds \\ & + \int_0^t \big[G(u^n(s)) - G(u(s))\big] h(s)\, ds + \int_0^t \big[\sigma((u^n(s)) - \sigma(u(s))\big]\, dW(s) \\ & + \int_0^t \Big[\tilde{\sigma}(u^n(s_n)) \dot{\widetilde{W}}^n(s)\, ds - \tilde{\sigma}(u(s))\, dW(s)\Big] - \int_0^t \Big(\varrho + \frac12 \tilde{\varrho}\Big)(u^n(s))\, ds \end{aligned}$$

$$+\int_0^t \big[\tilde\sigma(u^n(s))-\tilde\sigma(u^n(s_n))\big]\dot{\widetilde{W}}^n(s)\,ds. \tag{6.1}$$

Let $\underline{t}_n$ and $\bar t_n$ be defined by (2.11), let $\mathcal{E}_n$ denote projector in $L^2(0,T)$ on the subspace of step functions defined by

$$(\mathcal{E}_n f)(t)=\left(T^{-1}2^n\int_{\underline{t}_n}^{\bar t_n} f(s)ds\right)\cdot 1_{[\underline{t}_n,\bar t_n[}(t)$$

and let $\delta_n\,:\,L^2(0,T)\mapsto L^2(0,T)$ denote the shift operator defined by

$$(\delta_n f)(t)=f\left((t+T2^{-n})\wedge T\right)\quad\text{for}\quad t\in[0,T].$$

Using (2.12) and (2.13) we deduce

$$\int_0^t \tilde\sigma(u^n(s_n))\dot{\widetilde{W}}^n(s)ds=\int_0^t \mathcal{E}_n\left[\left(\delta_n[1_{[0,t]}\right)(s)\tilde\sigma(u^n(\underline{s}_n))\circ\Pi_n\right]dW(s). \tag{6.2}$$

Hence

$$\begin{aligned}u^n(t)-u(t)=&-\int_0^t\Big[A\big(u^n(s)-u(s)\big)+B(u^n(s))-B(u(s))+R(u^n(s))-R(u(s))\Big]ds\\&+\int_0^t\big[G(u^n(s))-G(u(s))\big]h(s)\,ds+\int_0^t\Big(\big[\sigma+\tilde\sigma\big](u^n(s))-\big[\sigma+\tilde\sigma\big](u(s))\Big)dW(s)\\&+\int_0^t\big[\tilde\sigma(u^n(s))-\tilde\sigma(u^n(s_n))\big]\dot{\widetilde{W}}^n(s)\,ds-\int_0^t\Big(\varrho+\frac12\tilde\varrho\Big)(u^n(s))\,ds\;+\int_0^t\widetilde\Sigma_n(s)\,dW(s),\end{aligned}$$

where

$$\widetilde\Sigma_n(s)=\mathcal{E}_n\left[\left(\delta_n 1_{[0,t]}\right)(s)\tilde\sigma(u^n(\underline{s}_n))\circ\Pi_n\right]-\tilde\sigma(u^n(s)). \tag{6.3}$$

Itô's formula implies that

$$\begin{aligned}|u^n(t)-u(t)|^2+2\int_0^t\|u^n(s)-u(s)\|^2ds=&-2\int_0^t\langle B(u^n(s))-B(u(s))\,,\,u^n(s)-u(s)\rangle\,ds\\&+2\int_0^t\Big(\left[G(u^n(s))-G(u(s))\right]h(s)-\left[R(u^n(s))-R(u(s))\right]\,,\,u^n(s)-u(s)\Big)ds\\&+2\int_0^t\Big(\big[\tilde\sigma(u^n(s))-\tilde\sigma(u^n(s_n))\big]\dot{\widetilde{W}}^n(s)-\Big(\varrho+\frac12\tilde\varrho\Big)(u^n(s)),u^n(s)-u(s)\Big)ds\\&+\int_0^t|\Sigma_n(s)|^2_{L_Q}\,ds+2\int_0^t\Big(\Sigma_n(s)dW(s)\,,\,u^n(s)-u(s)\Big),\end{aligned}$$

where

$$\Sigma_n(s)=\widetilde\Sigma_n(s)+\big(\sigma+\tilde\sigma\big)(u^n(s))-\big(\sigma+\tilde\sigma\big)(u(s)). \tag{6.4}$$

Using (2.15) and (4.1), we have $\dot{\widetilde{W}}_n(s)=\dot W_n(s)$ on the set $\{s\le\tau_n\}$; let

$$\begin{aligned}Z_n^{(0)}(t)&=\int_0^t\big(\Sigma_n(s)dW(s)\,,\,u^n(s)-u(s)\big), &(6.5)\\Z_n^{(1)}(t)&=\int_0^t\left|\mathcal{E}_n\left[\left(\delta_n 1_{[0,t]}\right)(s)\tilde\sigma(u^n(\underline{s}_n))\circ\Pi_n\right]-\tilde\sigma(u^n(s))\right|^2_{L_Q}ds,\\Z_n^{(2)}(t)&=\int_0^t\left(\left[\tilde\sigma(u^n(s))-\tilde\sigma(u^n(s_n))\right]\dot W^n(s)-\Big(\varrho+\frac12\tilde\varrho\Big)(u^n(s))\,,\,u^n(s)-u(s)\right)ds.\end{aligned}$$

The equation (2.8) with $\eta=1/2$ and condition **(GR)** yield

$$|u^n(t\wedge\tau_n)-u(t\wedge\tau_n)|^2+\int_0^{t\wedge\tau_n}\|u^n(s)-u(s)\|^2ds\le 2\sum_{0\le i\le 2}Z_n^{(i)}(t\wedge\tau_n) \tag{6.6}$$

$$+2\int_0^{t\wedge\tau_n}\Big(2L+\sqrt{L}|h(s)|_0+R_1+C_{1/2}\|u(s)\|_{\mathcal H}^4\Big)\,|u^n(s)-u(s)|^2\,ds.$$

For every integer $n\geq 1$ and every $t\in[0,T]$, set

$$T_n(t) \;=\; \sup_{0\leq s\leq t\wedge\tau_n}|u^n(s)-u(s)|^2+\int_0^{t\wedge\tau_n}\|u^n(s)-u(s)\|^2ds. \tag{6.7}$$

Using (4.4) and Gronwall's lemma we conclude that for all $t\in[0,T]$

$$\mathbb{E}T_n(t)\leq C\sum_{0\leq i\leq 2}\mathbb{E}\Big(\sup_{s\leq t\wedge\tau_n}\Big|Z_n^{(i)}(s)\Big|\Big). \tag{6.8}$$

6.1. **Estimate for $Z_n^{(0)}$.** The Burkholder-Davies-Gundy inequality, equations (6.3), (6.4) and (2.17) imply that for any $\eta>0$ there exists $C_\eta>0$ such that

$$\begin{aligned}
\mathbb{E}\Big(\sup_{s\leq t\wedge\tau_n}\Big|Z_n^{(0)}(s)\Big|\Big) &\leq 3\,\mathbb{E}\Big\{\int_0^{t\wedge\tau_n}|u^n(s)-u(s)|^2\,|\Sigma_n(s)|_{L_Q}^2\,ds\Big\}^{1/2}\\
&\leq 3\,\mathbb{E}\Big\{\sup_{s\leq t\wedge\tau_n}|u^n(s)-u(s)|\,\Big[\int_0^{t\wedge\tau_n}|\Sigma_n(s)|_{L_Q}^2\,ds\Big]^{1/2}\Big\}\\
&\leq \eta\,\mathbb{E}T_n(t)+C_\eta\,\mathbb{E}\Big(\int_0^{t\wedge\tau_n}|\Sigma_n(s)|_{L_Q}^2\,ds\Big)\\
&\leq \eta\,\mathbb{E}T_n(t)+2\,C_\eta\,\mathbb{E}Z_n^{(1)}(t\wedge\tau_n)+4\,L\,C_\eta\int_0^t\mathbb{E}\,|u^n(s\wedge\tau_n)-u(s\wedge\tau_n)|^2\,ds.
\end{aligned}$$

Thus if $\eta=\frac{1}{2}$, (6.8) and Gronwall's lemma imply that for some constant C which does not depend on n,

$$\mathbb{E}T_n(t)\leq C\,\Big(\mathbb{E}\sup_{s\leq t\wedge\tau_n}Z_n^{(1)}(s)+\mathbb{E}\sup_{s\leq t\wedge\tau_n}\big|Z_n^{(2)}(s)\big|\Big). \tag{6.9}$$

6.2. **Estimate of $Z_n^{(1)}$.** The convergence of $Z_n^{(1)}$ is stated in the following assertion.

Lemma 6.1. *Fix $M>0$, $h\in\mathcal{A}_M$ and let $Z_n^{(1)}(t)$ be defined by (6.5); then for fixed N and m we have:*

$$\lim_{n\to\infty}\mathbb{E}\Big(\sup_{t\in[0,T]}Z_n^{(1)}(t\wedge\tau_n)\Big)=0. \tag{6.10}$$

Proof. For $k=0,\dots,2^n-1$ let $\Omega_{n,k}=\{t_k<t\wedge\tau_n\leq t_{k+1}\}$, where as above we set $t_k=kT2^{-n}$. We consider $Z_n^{(1)}(t\wedge\tau_n)$ separately on each set $\Omega_{n,k}$.

We start with the case $\omega\in\Omega_{n,k}$ for $k\geq 2$. Then $\big(\delta_n 1_{[0,t\wedge\tau_n]}\big)(s)=1$ for $s\leq t_{k-1}$ and

$$\begin{aligned}
Z_n^{(1)}(t\wedge\tau_n) \;&=\; \int_0^{t\wedge\tau_n}\Big|\mathcal{E}_n\left[\big(\delta_n[1_{[0,t\wedge\tau_n]}\big)(s)\tilde\sigma(u^n(\underline{s}_n))\circ\Pi_n\right]-\tilde\sigma(u^n(s))\Big|_{L_Q}^2\,ds\\
&\leq\; \sum_{0\leq i\leq k-2}\int_{t_i\wedge\tau_n}^{t_{i+1}\wedge\tau_n}\left|\mathcal{E}_n\left[\tilde\sigma(u^n(\underline{s}_n))\circ\Pi_n\right]-\tilde\sigma(u^n(s))\right|_{L_Q}^2\,ds\\
&\quad +2\int_{t_{k-1}\wedge\tau_n}^{t_{k+1}\wedge t\wedge\tau_n}\Big(|\tilde\sigma(u^n(\underline{s}_n))\circ\Pi_n|_{L_Q}^2+|\tilde\sigma(u^n(s))|_{L_Q}^2\Big)\,ds.
\end{aligned}$$

Thus using (2.16) and (4.2) we deduce that for some constant $C=C(K_0,K_1,N,T)$ which does not depend on n:

$$Z_n^{(1)}(t\wedge\tau_n)\leq C2^{-n}+\sum_{0\leq i\leq k-2}\int_{t_i\wedge\tau_n}^{t_{i+1}\wedge\tau_n}|\tilde\sigma(u^n(\underline{s}_n))\circ\Pi_n-\tilde\sigma(u^n(s))|_{L_Q}^2\,ds$$

$$\le C\,2^{-n} + 2T \sup_{|u|\le 2(N+1)} |\tilde\sigma(u)\circ\Pi_n - \tilde\sigma(u))|^2_{L_Q}$$
$$+ 2\sum_{0\le i\le k-2} \int_{t_i\wedge\tau_n}^{t_{i+1}\wedge\tau_n} |\tilde\sigma(u^n(t_i)) - \tilde\sigma(u^n(s))|^2_{L_Q}\, ds$$
$$\le C\,2^{-n} + 2T \sup_{|u|\le 2(N+1)} |\tilde\sigma(u)\circ\Pi_n - \tilde\sigma(u))|^2_{L_Q} + 2L\int_0^{t\wedge\tau_n} |u^n(\underline{s}_n) - u^n(s)|^2\, ds$$
$$\le C2^{-n} + 2T \sup_{|u|\le 2(N+1)} |\tilde\sigma(u)\circ\Pi_n - \tilde\sigma(u))|^2_{L_Q} + 2L\int_0^{\tau_n} 1_{G_N^n(s)}\, |u^n(\underline{s}_n) - u^n(s)|^2\, ds,$$

where $G_N^n(s)$ is defined by (5.14). Furthermore, given $\omega\in\Omega_{n,1}\cup\Omega_{n,2}$ we have:

$$Z_n^{(1)}(t\wedge\tau_n) \;\le\; 2\int_0^{t_2\wedge t\wedge\tau_n} \Big(|\tilde\sigma(u^n(\underline{s}_n))\circ\Pi_n|^2_{L_Q} + |\tilde\sigma(u^n(s))|^2_{L_Q}\Big)\, ds \le C2^{-n},$$

where $C = C(K_0, K_1, N, T)$ does not depend on n. This yields

$$\mathbb{E}\Big(\sup_{t\in[0,T]} Z_n^{(1)}(t\wedge\tau_n)\Big) \;\le\; C\,2^{-n} + 2T \sup_{|u|\le 2(N+1)} |\tilde\sigma(u)\circ\Pi_n - \tilde\sigma(u))|^2_{L_Q}$$
$$+2L\mathbb{E}\int_0^{\tau_n} 1_{G_N^n(s)}\, |u^n(\underline{s}_n) - u^n(s)|^2\, ds;$$

therefore, (6.10) follows from (2.18) and (5.24). □

6.3. **Estimate of $Z_n^{(2)}$.**

6.3.1. **Main splitting.** The identities (2.19), (2.13) and (2.15) yield

$$Z_n^{(2)}(t\wedge\tau_n) \;=\; \sum_{1\le j\le n} \int_0^{t\wedge\tau_n} \Big([\tilde\sigma_j(u^n(s)) - \tilde\sigma_j(u^n(s_n))]\,\dot\beta_j^{\,n}(s)\,,\, u^n(s) - u(s)\Big)ds$$
$$-\int_0^{t\wedge\tau_n} \Big(\big(\varrho + \frac12\tilde\varrho\big)(u^n(s))\,,\, u^n(s) - u(s)\Big)ds. \tag{6.11}$$

For every $j = 1,\cdots,n$ Taylor's formula implies that

$$\tilde\sigma_j(u^n(s)) - \tilde\sigma_j(u^n(s_n)) = D\tilde\sigma_j(u^n(s_n))[u^n(s) - u^n(s_n)]$$
$$+\int_0^1 (1-\mu)d\mu \Big\langle D^2\tilde\sigma_j\big(u^n(s_n) + \mu[u^n(s) - u^n(s_n)]\big); u^n(s) - u^n(s_n), u^n(s) - u^n(s_n)\Big\rangle,$$

where $\langle D^2\tilde\sigma_j(v); v_1, v_2\rangle$ denotes the value of the second Fréchet derivative $D^2\tilde\sigma_j(v)$ on elements v_1 and v_2. Therefore condition (2.21) and the bound (4.2) imply that for every $t\in[0,T]$,

$$\Big|Z_n^{(2)}(t\wedge\tau_n)\Big| \le T_n(t,1) + \Big|\widetilde Z_n^{(2)}(t)\Big|, \tag{6.12}$$

where

$$T_n(t,1) = C_2(2N+1)\sum_{1\le j\le n} \int_0^{t\wedge\tau_n} |u^n(s) - u^n(s_n)|^2\, |\dot\beta_j^{\,n}(s)|\, |u^n(s) - u(s)|\, ds,$$

and

$$\widetilde Z_n^{(2)}(t) \;=\; \sum_{1\le j\le n} \int_0^{t\wedge\tau_n} \Big(D\tilde\sigma_j(u^n(s_n))[u^n(s) - u^n(s_n)]\,,\, u^n(s) - u(s)\Big)\,\dot\beta_j^{\,n}(s)ds$$
$$-\int_0^{t\wedge\tau_n} \Big(\big(\varrho + \frac12\tilde\varrho\big)(u^n(s))\,,\, u^n(s) - u(s)\Big)ds. \tag{6.13}$$

For $G_N^n(t)$ defined by (5.14), one has

$$T_n(t,1) \le C_N \sum_{1\le j\le n} \int_0^{t\wedge\tau_n} 1_{G_N^n(s)}\, |u^n(s)-u^n(s_n)|^2\, |\dot\beta_j^{\,n}(s)|\, |u^n(s)-u(s)|\, ds.$$

Therefore, (4.1), the inequalities (4.3) and (5.24) yield for some constant $C:=C(N,M,T)$

$$\mathbb{E}\Big(\sup_{t\in[0,T]} T_n(t,1)\Big) \le \tilde C_N n^{\frac32} 2^{\frac n2}\,\mathbb{E}\int_0^{\tau_n} 1_{G_N^n(s)}\, |u^n(s)-u^n(s_n)|^2\, ds \le C n^3 2^{-\frac n4}. \tag{6.14}$$

To bound $\widetilde Z_n^{(2)}$, rewrite $u^n(s)-u^n(s_n)$ in (6.13) using the evolution equation (2.33). This yields the following decomposition:

$$\widetilde Z_n^{(2)}(t) = \sum_{2\le i\le 6} T_n(t,i), \tag{6.15}$$

where

$$\begin{aligned} T_n(t,2) \;&=\; \sum_{1\le j\le n} \int_0^{t\wedge\tau_n} \Big(D\tilde\sigma_j(u^n(s_n))\mathcal{I}_n(s,s_n)\dot\beta_j^{\,n}(s)\,,\, u^n(s)-u(s)\Big) ds \\ &\quad - \int_0^{t\wedge\tau_n} \Big(\big(\varrho+\frac12\tilde\varrho\big)(u^n(s))\,,\, u^n(s)-u(s)\Big) ds \end{aligned} \tag{6.16}$$

with

$$\mathcal{I}_n(s,s_n) := \int_{s_n}^s \sigma(u^n(r))dW(r) + \int_{s_n}^s \tilde\sigma(u^n(r))\widetilde{\dot W}^n(r)dr, \tag{6.17}$$

$$T_n(t,3) = -\sum_{1\le j\le n} \int_0^{t\wedge\tau_n} \Big(D\tilde\sigma_j(u^n(s_n))\Big[\int_{s_n}^s Au^n(r)dr\Big]\dot\beta_j^{\,n}(s)\,,\, u^n(s)-u(s)\Big) ds,$$

$$T_n(t,4) = -\sum_{1\le j\le n} \int_0^{t\wedge\tau_n} \Big(D\tilde\sigma_j(u^n(s_n))\Big[\int_{s_n}^s B(u^n(r))dr\Big]\dot\beta_j^{\,n}(s)\,,\, u^n(s)-u(s)\Big) ds,$$

$$T_n(t,5) = \sum_{1\le j\le n} \int_0^{t\wedge\tau_n} \Big(D\tilde\sigma_j(u^n(s_n))\Big[\int_{s_n}^s G(u^n(r))h(r)dr\Big]\dot\beta_j^{\,n}(s)\,,\, u^n(s)-u(s)\Big) ds,$$

$$T_n(t,6) = -\sum_{1\le j\le n} \int_0^{t\wedge\tau_n} \Big(D\tilde\sigma_j(u^n(s_n))\Big[\int_{s_n}^s \widetilde R(u^n(r))dr\Big]\dot\beta_j^{\,n}(s)\,,\, u^n(s)-u(s)\Big) ds,$$

with $\widetilde R(u) = R(u)+\varrho(u)+\frac12\tilde\varrho(u)$. The most difficult term to deal with is $T_n(t,2)$ and therefore we devote several separate subsections below to upper estimate it. Let us start with the easier case $3\le i\le 6$.

6.3.2. **Bound for $T_n(t,i)$, $3\le i\le 6$.** By duality we obtain

$$|T_n(t,3)| = \sum_{1\le j\le n} \Big| \int_0^{t\wedge\tau_n} \dot\beta_j^{\,n}(s)\Big(\int_{s_n}^s A^{1/2}u^n(r)dr\,,\, A^{1/2}\big[D\tilde\sigma_j(u^n(s_n))\big]^*[u^n(s)-u(s)]\Big) ds\Big|.$$

Therefore, using (2.22), (4.2) and (4.3) we deduce that for every $\tilde t\in[0,T]$

$$\sup_{t\in[0,\tilde t]} |T_n(t,3)| \le C_3(2N+2)\,\alpha\, n^{3/2}\, 2^{n/2} \int_0^{\tilde t\wedge\tau_n} \Big(\int_{s_n}^s \|u^n(r)\|dr\Big)\, \|u^n(s)-u(s)\| ds.$$

For any $\eta>0$, Schwarz's inequality yields

$$\sup_{t\in[0,\tilde t]} |T_n(t,3)| \le \eta \int_0^{\tilde t\wedge\tau_n} \|u^n(s)-u(s)\|^2 ds + C n^3 \int_0^{\tau_n}\int_{s_n}^s \|u^n(r)\|^2 dr ds.$$

for some constant $C := C(N,T,\eta)$. Finally, Fubini's theorem and (4.2) imply that

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |T_n(t,3)|\Big) \leq \eta T_n(T) + C(N,T,\eta)\, n^3\, 2^{-n}. \tag{6.18}$$

Similarly, using (2.6) and (4.3) we obtain

$$|T_n(t,4)| = \sum_{1\leq j\leq n} \Big| \int_0^{t\wedge\tau_n} \dot{\beta}_j^{\,n}(s) \Big(\Big[\int_{s_n}^s B(u^n(r))dr \Big], [D\tilde{\sigma}_j(u^n(s_n))]^*[u^n(s)-u(s)] \Big) ds \Big|$$

$$\leq C\alpha\, n^{3/2} 2^{n/2} \int_0^{t\wedge\tau_n} ds \int_{s_n}^s \|u^n(r)\|^2_{\mathcal{H}}\, dr \sup_{1\leq j\leq n} \|[D\tilde{\sigma}_j(u^n(s_n))]^*[u^n(s)-u(s)]\|\,.$$

Thus the inequalities (2.22), (4.4) and (4.2) yield that for some constant $C := C(N,m,T)$:

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |T_n(t,4)|\Big) \leq C_3(2N+2)\alpha m^2\, n^{\frac{3}{2}} 2^{-\frac{n}{2}} \int_0^{\tau_n} \|u^n(s)-u(s)\|\, ds \leq C n^{\frac{3}{2}} 2^{-\frac{n}{2}}. \tag{6.19}$$

Using (2.20), (4.2) and (4.3) we deduce

$$|T_n(t,5)| \leq C_1(2N+2)\, \alpha\, n^{3/2}\, 2^{n/2} \int_0^{t\wedge\tau_n} \Big(\int_{s_n}^s |G(u^n(r))h(r)|\, dr \Big) |u^n(s)-u(s)| ds.$$

Therefore, (2.27), (4.2) and Fubini's theorem yield for some constant $C := C(N,M,T)$

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |T_n(t,5)|\Big) \leq C_N\, n^{3/2}\, 2^{n/2} \mathbb{E} \int_0^{\tau_n} \int_{s_n}^s |h(r)|\, dr ds \leq C\, n^{3/2}\, 2^{-n/2}. \tag{6.20}$$

Similarly, relying on (2.24), (2.28), (4.3) and (4.2) we deduce

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |T_n(t,6)|\Big) \leq C_{K,R_0,N} n^{3/2} 2^{-n/2}. \tag{6.21}$$

Thus, collecting the relations in (6.9)–(6.21), and choosing $\eta > 0$ small enough in (6.18), we obtain the following assertion:

Proposition 6.2. *Let the assumptions of Theorem 3.1 be satisfied, $T_n(t)$ be defined by* (6.7)*; then we have:*

$$\mathbb{E}T_n(T) \leq \gamma_n(N,M,m,T) + C\mathbb{E}\Big(\sup_{t\in[0,T]} |T_n(t,2)|\Big),$$

where $\lim_{n\to\infty} \gamma_n(N,M,m,T) = 0$ *and $T_n(t,2)$ is defined by* (6.16).

6.3.3. **Splitting of $T_n(t,2)$.** Let $T_n(t,2)$ be defined by (6.16); then we have the following decomposition:

$$T_n(t,2) = \sum_{1\leq i\leq 7} S_n(t,i), \tag{6.22}$$

where

$$S_n(t,1) = \sum_{j=1}^n \int_0^{t\wedge\tau_n} \dot{\beta}_j^{\,n}(s) \Big(D\tilde{\sigma}_j(u^n(s_n))\mathcal{I}_n(s,s_n), [u^n(s)-u(s)] - [u^n(s_n)-u(s_n)] \Big) ds,$$

$$S_n(t,2) = -\int_0^{t\wedge\tau_n} \Big(\big(\varrho + \frac{1}{2}\tilde{\varrho}\big)(u^n(s)) - \big(\varrho_n + \frac{1}{2}\tilde{\varrho}_n\big)(u^n(s))\,,\, u^n(s)-u(s) \Big) ds,$$

$$S_n(t,3) = -\int_0^{t\wedge\tau_n} \Big(\big(\varrho_n + \frac{1}{2}\tilde{\varrho}_n\big)(u^n(s)) - \big(\varrho_n + \frac{1}{2}\tilde{\varrho}_n\big)(u^n(s_n)), u^n(s)-u(s) \Big) ds\,,$$

$$S_n(t,4) = -\int_0^{t\wedge\tau_n} \Big(\big(\varrho_n + \frac{1}{2}\tilde{\varrho}_n\big)(u^n(s_n)), [u^n(s)-u(s)] - [u^n(s_n)-u(s_n)] \Big) ds\,,$$

$$S_n(t,5) = \sum_{1\le j\le n}\int_0^{t\wedge\tau_n} \dot{\beta}_j^{\,n}(s)\Big(D\tilde\sigma_j(u^n(s_n))\Big[\int_{s_n}^s [\sigma(u^n(r))-\sigma(u^n(s_n))]\,dW(r) + \int_{s_n}^s [\tilde\sigma(u^n(r))-\tilde\sigma(u^n(s_n))]\,\dot{\widetilde{W}}^n(r)dr\Big]\,,\,u^n(s_n)-u(s_n)\Big)ds,$$

$$S_n(t,6) = \int_0^{t\wedge\tau_n}\Big(\sum_{1\le j\le n}\dot{\beta}_j^{\,n}(s)D\tilde\sigma_j(u^n(s_n))\big[\sigma(u^n(s_n))\big(W(s)-W(s_n)\big)\big]-\varrho_n(u^n(s_n))\,,\; u^n(s_n)-u(s_n)\Big)ds, \tag{6.23}$$

$$S_n(t,7) = \int_0^{t\wedge\tau_n}\Big(\sum_{1\le j\le n}\dot{\beta}_j^{\,n}(s)D\tilde\sigma_j(u^n(s_n))\Big[\tilde\sigma(u^n(s_n))\Big(\int_{s_n}^s \dot{\widetilde{W}}^n(r)dr\Big)\Big]-\frac{1}{2}\tilde\varrho_n(u^n(s_n))\,,\; u^n(s_n)-u(s_n)\Big)ds. \tag{6.24}$$

The most difficult terms to deal with are $S_n(t,6)$ and $S_n(t,7)$. We start with the simpler ones $S_n(t,i)$, $i=1,...,5$.

6.3.4. **Bound for $S_n(t,1)$.** Let $\mathcal{I}_n(s,s_n)$ be defined in (6.17); using (2.20), (4.2) and (4.3) we obtain

$$|S_n(t,1)|\le C_1(2N+2)\,\alpha\,n^{3/2}\,2^{n/2}\int_0^{t\wedge\tau_n}|\mathcal{I}_n(s,s_n)|\,(|u^n(s)-u^n(s_n)|+|u(s)-u(s_n)|)\,ds.$$

Thus for $\mathcal{N}_n=\int_0^{\tau_n}\big(|u^n(s)-u^n(s_n)|^2+|u(s)-u(s_n)|^2\big)\,ds$, Schwarz's inequality yields

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|S_n(t,1)|\Big)\le C_N\,n^{3/2}\,2^{n/2}\;[\mathbb{E}\mathcal{N}_n]^{1/2}\;\Big[\mathbb{E}\int_0^{\tau_n}|\mathcal{I}_n(s,s_n)|^2\,ds\Big]^{1/2}. \tag{6.25}$$

Since

$$\mathbb{E}\mathcal{N}_n\le\mathbb{E}\int_0^{\tau_n}1_{G_N^n(s)}\,\big(|u^n(s)-u^n(s_n)|^2+|u(s)-u(s_n)|^2\big)\,ds$$

with $G_N^n(s)$ defined by (5.14), using (5.3) with $\phi_n(s)=s_n$ and $\psi_n(s)=s$ we deduce:

$$\mathbb{E}\mathcal{N}_n\le C(N,M,T)\,n^{3/2}\,2^{-3n/4}. \tag{6.26}$$

Furthermore, the local property of the stochastic integral and (4.3) yield

$$\begin{aligned}\mathbb{E}&\int_0^{\tau_n}|\mathcal{I}_n(s,s_n)|^2\,ds\\ &\le C\int_0^T\Big[\mathbb{E}\Big|\int_{s_n}^s 1_{G_N^n(r)}\sigma(u^n(r))dW(r)\Big|^2+\mathbb{E}\Big|\int_{s_n}^s 1_{G_N^n(r)}\tilde\sigma(u^n(r))\dot{W}^n(r)dr\Big|^2\Big]\,ds\\ &\le C\int_0^T\Big[\mathbb{E}\int_{s_n}^s 1_{G_N^n(r)}\,|\sigma(u^n(r))|^2_{L_Q}\,dr+\alpha^2\,n^2\,\mathbb{E}\int_{s_n}^s 1_{G_N^n(r)}\,|\tilde\sigma(u^n(r))|^2_{L_Q}\,dr\Big]\,ds.\end{aligned}$$

Thus by (2.16) and the definition of the set $G_N^n(s)$ given in (5.14), we deduce:

$$\mathbb{E}\int_0^{\tau_n}|\mathcal{I}_n(s,s_n)|^2\,ds\le C(N,M,T)\,n^2\,2^{-n}. \tag{6.27}$$

Consequently the inequalities (6.25) – (6.27) yield

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|S_n(t,1)|\Big)\le C(N,M,T)\,n^{13/4}\,2^{-3n/8}. \tag{6.28}$$

6.3.5. **Bound for $S_n(t,2)$.** The inequality (4.2) implies that

$$\sup_{t\in[0,T]} |S_n(t,2)| \leq C(N)T \sup_{|u|\leq 2(N+1)} \{|\varrho_n(u)-\varrho(u)| + |\tilde{\varrho}_n(u)-\tilde{\varrho}(u)|\}.$$

Therefore, the locally uniform convergence (2.26) of ρ_n to ρ and $\tilde{\rho}_n$ to $\tilde{\rho}$ respectively yields

$$\lim_{n\to\infty} \mathbb{E}\Big(\sup_{t\in[0,T]} |S_n(t,2)|\Big) = 0. \tag{6.29}$$

6.3.6. **Bound for $S_n(t,3)$.** The local Lipschitz property (2.25) and (4.2) imply

$$\begin{aligned}|S_n(t,3)| &\leq 2\,\bar{C}_{2N+2}\sqrt{N+1}\int_0^{\tau_n} |u^n(s)-u^n(s_n)|\,ds\\ &\leq C(N,T)\Big[\int_0^{\tau_n} 1_{G_N^n(s)}\,|u^n(s)-u^n(s_n)|^2\,ds\Big]^{1/2},\end{aligned}$$

where $G_N^n(s)$ is defined by (5.14). Thus (5.24) yields

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |S_n(t,3)|\Big) \leq C(N,M,T)\, n^{3/4}\, 2^{-3n/8}. \tag{6.30}$$

6.3.7. **Bound for $S_n(t,4)$.** The local growth condition (2.24), relations (5.3) and (5.24), and also Schwarz's inequality imply

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\in[0,T]} |S_n(t,4)|\Big) &\leq 2\bar{K}_{2N+2}\mathbb{E}\int_0^{\tau_n} (|u^n(s)-u^n(s_n)| + |u(s)-u(s_n)|)\,ds\\ &\leq 2\bar{K}_{2N+2}\sqrt{T}\,\Big[\mathbb{E}\int_0^{\tau_n}\Big(1_{G_N^n(s)}|u^n(s)-u^n(s_n)|^2 + 1_{\widetilde{G}_N(s)}|u(s)-u(s_n)|^2\Big)\,ds\Big]^{1/2}\\ &\leq C(N,M,T)\, n^{3/4}\, 2^{-3n/8}.\end{aligned} \tag{6.31}$$

6.3.8. **Bound for $S_n(t,5)$.** The local bound (2.20) together with the inequalities (4.2) and (4.3) yield

$$\begin{aligned}|S_n(t,5)| \;\leq\; & C_1(2N+2)\,\alpha\, n^{3/2} 2^{n/2}\int_0^{\tau_n}\Big\{\Big|\int_{s_n}^s [\sigma(u^n(r))-\sigma(u^n(s_n))]\,dW(r)\Big|\\ &+\Big|\int_{s_n}^s [\tilde{\sigma}(u^n(r))-\tilde{\sigma}(u^n(s_n))]\,\dot{\widetilde{W}}^n(r)dr\Big|\Big\}\,ds.\end{aligned}$$

Using Schwarz's inequality, (2.17) and (4.3), we deduce

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\in[0,T]} |S_n(t,5)|\Big) &\leq C_N\, n^{3/2}\, 2^{n/2}\Big\{\int_0^T \mathbb{E}\Big|\int_{s_n\wedge\tau_n}^{s\wedge\tau_n} 1_{G_N^n(r)}\,[\sigma(u^n(r))-\sigma(u^n(s_n))]\,dW(r)\Big|^2 ds\\ &\quad + \alpha^2\, n^2\, 2^n\, L\,\mathbb{E}\int_0^{\tau_n}\Big|\int_{s_n}^s |u^n(r)-u^n(s_n)|\,dr\Big|^2 ds\Big\}^{1/2}\\ &\leq C_N\, n^{3/2}\, 2^{n/2}\sqrt{L}\Big\{\int_0^T ds\,\mathbb{E}\int_{s_n\wedge\tau_n}^{s\wedge\tau_n} 1_{G_N^n(r)}\;|u^n(r)-u^n(s_n)|^2\;dr\\ &\quad + \alpha^2\, n^2\, T\,\mathbb{E}\int_0^{\tau_n} ds\int_{s_n}^s |u^n(r)-u^n(s_n)|^2\,dr\Big\}^{1/2}\\ &\leq C_N n^{3/2} 2^{n/2}\sqrt{L(1+\alpha^2 n^2 T)}\,\Big\{\int_0^T ds\,\mathbb{E}\int_{s_n\wedge\tau_n}^{s\wedge\tau_n} 1_{G_N^n(r)}\,|u^n(r)-u^n(s_n)|^2\,dr\Big\}^{1/2}.\end{aligned}$$

Fubini's theorem and (5.24) imply that

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|S_n(t,5)|\Big) \le \sqrt{L}C(N,T)\,n^{5/2}\,2^{n/2}$$
$$\times\Big(\mathbb{E}\int_0^{\tau_n} 1_{G_N^n(r)}\big[\,|u^n(r)-u^n(r_n)|^2+|u^n(r)-u^n(\underline{r}_n)|^2\,\big]\,2T2^{-n}\,dr\Big)^{\frac12}$$
$$\le C(N,M,T)\,n^{13/4}\,2^{-3n/8}. \tag{6.32}$$

Proposition 6.2 and the relations in (6.28)–(6.32) imply the following assertion:

Proposition 6.3. *Let the assumptions of Theorem 3.1 be satisfied and let $T_n(t)$ be defined by* (6.7)*; then we have:*

$$\mathbb{E}T_n(T) \le \gamma_n^*(N,M,m,T) + C\mathbb{E}\Big(\sup_{t\in[0,T]}|S_n(t,6)| + \sup_{t\in[0,T]}|S_n(t,7)|\Big),$$

where $\lim_{n\to\infty}\gamma_n^*(N,M,m,T)=0$, $S_n(t,6)$ and $S_n(t,7)$ are defined by (6.23) *and* (6.24).

The upper estimates of $S_n(t,6)$ and $S_n(t,7)$ are the key ingredients of the proof; they justify the drift correction term in the definition of u^n.

6.3.9. **Bound for $S_n(t,6)$.**

Lemma 6.4. *Let the assumptions of Theorem 3.1 be satisfied and $S_n(t,6)$ be given by* (6.23)*. Then there exists a constant $C(N,T)$ such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|S_n(t,6)|\Big) \le C(N,T)\,n\,2^{-\frac{n}{2}}. \tag{6.33}$$

Proof. For $t\in[0,T]$ set

$$U_j^n(s) = \dot{\beta}_j^{\,n}(s)\,D\tilde{\sigma}_j(u^n(s_n))\big(\sigma(u^n(s_n))\,[W(s)-W(s_n)]\,\big)\,,$$
$$\Delta_n(s) = \Big(\sum_{1\le j\le n} U_j^n(s) - \varrho_n(u^n(s_n))\,,\,u^n(s_n)-u(s_n)\Big).$$

We also have an obvious decomposition

$$\sum_{1\le j\le n} U_j^n(s) - \varrho_n(u^n(s_n)) = \sum_{1\le i\le 3} V_n^{(i)}(s),$$

where (2.10) yields

$$V_n^{(1)}(s) = \sum_{1\le j\le n} D\tilde{\sigma}_j(u^n(s_n))\sigma(u^n(s_n))\,[W(s)-W(\underline{s}_n)]\,\dot{\beta}_j^{\,n}(s),$$
$$V_n^{(2)}(s) = \sum_{1\le j\le n}\sum_{l\ne j} D\tilde{\sigma}_j(u^n(s_n))\sigma_l(u^n(s_n))\,[\beta_l(\underline{s}_n)-\beta_l(s_n)]\;\;2^nT^{-1}\,[\beta_j(\underline{s}_n)-\beta_j(s_n)]\,,$$
$$V_n^{(3)}(s) = \sum_{1\le j\le n} D\tilde{\sigma}_j(u^n(s_n))\sigma_j(u^n(s_n))\,\Big[2^nT^{-1}\Big(\beta_j(\underline{s}_n)-\beta_j(s_n)\Big)^2-1\Big]\,.$$

The obvious identity

$$1_{\{s\le\tau_n\}} = 1_{\{s_n\le\tau_n\}} - 1_{\{s_n\le\tau_n<s\}} \tag{6.34}$$

yields the following decomposition, where $G_N^n(t)$ is defined by (5.14):

$$S_n(t,6) = \int_0^{t\wedge\tau_n}\Delta_n(s)\,1_{G_N^n(s_n)}\,ds = \sum_{1\le i\le 3} S_n^{(i)}(t) - S_n^{(4)}(t), \tag{6.35}$$

with

$$S_n^{(i)}(t) = \int_0^t 1_{\{s_n\leq \tau_n\}} 1_{G_N^n(s_n)} \big(V_n^{(i)}(s)\,,\, u^n(s_n)-u(s_n)\big)\, ds, \qquad i=1,2,3,$$
$$S_n^{(4)}(t) = \int_0^t 1_{\{s_n\leq \tau_n<s\}} 1_{G_N^n(s_n)} \Delta_n(s) ds.$$

We note that $S_n^{(i)}(t)=0$ for every $i=1,2,3$ and $t\leq t_2$.

Bound for $S_n^{(4)}$. Set $t_{-1}=t_0=0$; using twice Schwarz's inequality, we deduce

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\in[0,T]}|S_n^{(4)}(t)|\Big) &\leq \sum_{0\leq k<2^n} \mathbb{E}\int_{t_k}^{t_{k+1}} 1_{\{t_{k-1}\leq \tau_n \leq t_{k+1}\}} 1_{G_N^n(s_n)}\; |\Delta_n(s)|\, ds\\ &\leq \Big\{2\sum_{0\leq k<2^n} \mathbb{E}1_{\{t_k\leq\tau_n\leq t_{k+1}\}}\Big\}^{1/2} \Big\{\sum_{0\leq k<2^n}\mathbb{E}\Big(\int_{t_k}^{t_{k+1}} 1_{G_N^n(s_n)}\, |\Delta_n(s)|\, ds\Big)^2\Big\}^{1/2}\\ &\leq \sqrt{2}\,\Big\{T2^{-n}\mathbb{E}\int_0^T 1_{G_N^n(s_n)}\, |\Delta_n(s)|^2\, ds\Big\}^{1/2}.\end{aligned}$$

Schwarz's inequality, (2.16), (2.20), (2.24) and the definition (5.14) of the set $G_N^n(s_n)$ yield

$$\begin{aligned}1_{G_N^n(s_n)}\, |\Delta_n(s)|^2 &\leq C(N)\Big(1+\sum_{1\leq j\leq n} |W(s)-W(s_n)|_0\, |\dot{\beta}_j^{\,n}(s)|\Big)^2\\ &\leq C(N)\,\Big(1+n\,|W(s)-W(s_n)|_0^2 \sum_{1\leq j\leq n} |\dot{\beta}_j^{\,n}(s)|^2\Big).\end{aligned}$$

Therefore, Schwarz's inequality implies that for some constants $C(N,T)$, c_1, c_2, one has:

$$\begin{aligned}&\mathbb{E}\Big(\sup_{t\in[0,T]}|S_n^{(4)}(t)|\Big)\\ &\leq C(N,T)\,2^{-n/2}\,\Big\{1+n\sum_{1\leq j\leq n}\int_0^T \Big[\mathbb{E}\,|W(s)-W(s_n)|_0^4\Big]^{1/2}\Big[\mathbb{E}|\dot{\beta}_j^{\,n}(s)|^4\Big]^{1/2} ds\Big\}^{1/2}\\ &\leq C(N,T)2^{-n/2}\Big\{1+n^2\Big[c_1\frac{T^2}{2^{2n}}\Big]^{1/2}\Big[T^{-4}2^{4n}c_2\frac{T^2}{2^{2n}}\Big]^{1/2}\Big\}^{1/2} \leq C(N,T)\,n\,2^{-\frac{n}{2}}. \qquad (6.36)\end{aligned}$$

Bound for $S_n^{(1)}$. Using duality and Fubini's theorem, we can write

$$\begin{aligned}S_n^{(1)}(t) &= \sum_{1\leq j\leq n}\int_0^t 1_{\{s_n\leq\tau_n\}} 1_{G_N^n(s_n)}\, \dot{\beta}_j^{\,n}(s)\\ &\quad\times\int_{\underline{s}_n}^s \big(\,[D\tilde{\sigma}_j(u^n(s_n))\sigma(u^n(s_n))]^*\,(u^n(s_n)-u(s_n))\,,\, dW(r)\big)ds\\ &= \sum_{1\leq j\leq n}\int_0^t \Big(\int_r^{\bar{r}_n} 1_{\{s_n\leq\tau_n\}} 1_{G_N^n(s_n)}\, \dot{\beta}_j^{\,n}(s)\\ &\quad\times [D\tilde{\sigma}_j(u^n(s_n))\sigma(u^n(s_n))]^*\,(u^n(s_n)-u(s_n))\, ds\,,\, dW(r)\Big).\end{aligned}$$

Since $\dot{\beta}_j^n(s)$ is $\mathcal{F}_{\underline{s}_n}=\mathcal{F}_{\underline{r}_n}$ adapted for $r\leq s\leq \bar{r}_n$, the process $S_n^{(1)}$ is a martingale. Therefore, the Burkholder-Davies-Gundy and Schwarz inequalities, (2.16), (2.20) and (5.14) imply

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|S_n^{(1)}(t)|\Big)\leq c_0\,\mathbb{E}\Big\{\int_0^T\Big|\sum_{1\leq j\leq n}\int_r^{\bar{r}_n} 1_{\{s_n\leq\tau_n\}} 1_{G_N^n(s_n)}\,\dot{\beta}_j^{\,n}(s)$$

$$\times \left[D\tilde{\sigma}_j(u^n(s_n))\sigma(u^n(s_n))\right]^* (u^n(s_n)-u(s_n))ds\Big|^2 dr\Big\}^{1/2}$$

$$\leq \frac{C(N,T)\sqrt{n}}{2^{n/2}}\mathbb{E}\Big\{ \int_0^T dr \sum_{1\leq j\leq n} \int_r^{\bar{r}_n} 1_{\{s_n\leq \tau_n\}} 1_{G_N^n(s_n)} \, |D\tilde{\sigma}_j(u^n(s_n))\sigma(u^n(s_n))|^2 \, |\dot{\beta}_j^{\,n}(s)|^2 ds\Big\}^{1/2}$$

$$\leq C(N,T)\,\sqrt{n} 2^{-n/2}\Big\{ \int_0^T dr \sum_{1\leq j\leq n} \mathbb{E}\int_r^{\bar{r}_n} |\dot{\beta}_j^{\,n}(s)|^2 ds\Big\}^{1/2} \leq C(N,T)\, n\, 2^{-n/2}. \qquad (6.37)$$

Bound for $S_n^{(2)}$. For $j=1,\cdots,n$, $l\neq j$, $i=1,\cdots,2^n-1$, set

$$\Phi_{j,l}(i) = \Big(D\tilde{\sigma}_j(u^n(t_i))\sigma_l(u^n(t_i))\,,\, u^n(t_i)) - u(t_i)\Big)\, 1_{\{t_i\leq \tau_n\}}\, 1_{G_N^n(t_i)},$$

and for $k=3,\cdots,2^n$, let

$$M_k = \sum_{2\leq i\leq k}\sum_{1\leq j\leq n}\sum_{l\neq j} \Phi_{j,l}(i-1)\,\big(\beta_l(t_i)-\beta_l(t_{i-1})\big)\,\big(\beta_j(t_i)-\beta_j(t_{i-1})\big).$$

Then the random variable $\Phi_{j,l}(i-1)$ is $\mathcal{F}_{t_{i-1}}$ measurable, and since for $l\neq j$ the sigma-field $\mathcal{F}_{t_{i-1}}$ and the random variables $\beta_j(t_i)-\beta_j(t_{i-1})$ and $\beta_l(t_i)-\beta_l(t_{i-1})$ are independent, the process $(M_k,\mathcal{F}_{t_k}, 2\leq k<2^n)$ is a discrete martingale. Furthermore, for the cases (a) $i<i'$ and $l'\neq j'$, (b) $i'<i$ and $l\neq j$ or (c) $i=i'$ and $(\min(j,l),\max(j,l))\neq (\min(j',l'),\max(j',l'))$, one has

$$\mathbb{E}\big[\Phi_{j,l}(i-1)\,\Phi_{j',l'}(i'-1)\,\big(\beta_l(t_i)-\beta_l(t_{i-1})\big)\,\big(\beta_j(t_i)-\beta_j(t_{i-1})\big)$$
$$\times\big(\beta_{l'}(t_{i'})-\beta_{l'}(t_{i'-1})\big)\,\big(\beta_{j'}(t_{i'})-\beta_{j'}(t_{i'-1})\big)\big]=0.$$

Therefore, Schwarz's and Doob's inequalities yield

$$\mathbb{E}\Big(\max_{2\leq k<2^n}|M_k|\Big)^2 \leq \mathbb{E}\Big(\max_{2\leq k<2^n} M_k^2\Big) \leq 4\mathbb{E}\big(M_{2^n-1}^2\big)$$
$$\leq 12\sum_{2\leq i<2^n}\sum_{1\leq j\leq n}\sum_{l\neq j}\mathbb{E}\big(\Phi_{j,l}(i-1)^2\big)\,\mathbb{E}\big(|\beta_l(t_i)-\beta_l(t_{i-1})|^2\big)\,\mathbb{E}\big(|\beta_j(t_i)-\beta_j(t_{i-1})|^2\big).$$

Furthermore, using (2.20), (2.16) and (5.14) we deduce that for every i,j,l

$$\mathbb{E}\big(\Phi_{j,l}(i-1)^2\big)\leq q_l\, C_1(N)^2\,(K_0+K_1N^2)\,(2N)^2,$$

which implies

$$\mathbb{E}\Big(\max_{2\leq k<2^n}|M_k|\Big)\leq C(N,T)\,n\,2^{-n/2}. \qquad (6.38)$$

A similar easier computation shows that

$$\mathbb{E}\Big(\max_{2\leq k<2^n}\sup_{t_k\leq t\leq t_{k+1}}\Big|\sum_{1\leq j\leq n}\sum_{l\neq j}\Phi_{j,l}(k-1)\frac{2^n(t-t_k)}{T}\big(\beta_l(t_k)-\beta_l(t_{k-1})\big)\big(\beta_j(t_k)-\beta_j(t_{k-1})\Big|\Big)$$
$$\leq \mathbb{E}\Big(\max_{2\leq k<2^n}\Big|\sum_{1\leq j\leq n}\sum_{l\neq j}\Phi_{j,l}(k-1)\big(\beta_l(t_k)-\beta_l(t_{k-1})\big)\big(\beta_j(t_k)-\beta_j(t_{k-1})\Big|\Big)$$
$$\leq \Big\{\mathbb{E}\Big(\max_{2\leq k<2^n}\Big|\sum_{1\leq j\leq n}\sum_{l\neq j}\Phi_{j,l}(k-1)\big(\beta_l(t_k)-\beta_l(t_{k-1})\big)\big(\beta_j(t_k)-\beta_j(t_{k-1})\Big|^2\Big)\Big\}^{1/2}$$
$$\leq \Big\{\sum_{2\leq k<2^n}\mathbb{E}\Big|\sum_{1\leq j\leq n}\sum_{l\neq j}\Phi_{j,l}(k-1)\big(\beta_l(t_k)-\beta_l(t_{k-1})\big)\big(\beta_j(t_k)-\beta_j(t_{k-1})\Big|^2\Big)\Big\}^{1/2}$$
$$\leq \Big\{\sum_{2\leq k<2^n}\sum_{1\leq j\leq n}\sum_{l\neq j}\mathbb{E}\big(\Phi_{j,l}(k-1)^2\big)\mathbb{E}\big(|\beta_l(t_k)-\beta_l(t_{k-1})|^2\big)\mathbb{E}\big(|\beta_j(t_k)-\beta_j(t_{k-1})|^2\big)\Big\}^{\frac{1}{2}}$$

$$\leq C(N,T)\, n\, 2^{-n/2}. \tag{6.39}$$

Furthermore,

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\in[0,T]} |S_n^{(2)}(t)|\right) &\leq \mathbb{E}\left(\sup_{k}\ \sup_{t\in[t_k,t_{k+1}]} |S_n^{(2)}(t)|\right)\\
&\leq\ \ \mathbb{E}\sup_{k\geq 3}\Big|\sum_{2\leq i<k}\int_{t_i}^{t_{i+1}} 1_{\{s_n\leq\tau_n\}} 1_{G_N^n(s_n)}(V_n^{(2)}(s), u^n(s_n)-u(s_n))ds\Big|\\
&\quad +\mathbb{E}\sup_{k\geq 2}\Big[\sup_{t\in[t_k,t_{k+1}]}\Big|\int_{t_k}^{t} 1_{\{s_n\leq\tau_n\}} 1_{G_N^n(s_n)}(V_n^{(2)}(s), u^n(s_n)-u(s_n))ds\Big|\Big].
\end{aligned}$$

This inequality, (6.38) and (6.39) immediately yield

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |S_n^{(2)}|\Big) \leq C(N,T)\, n\, 2^{-\frac{n}{2}}. \tag{6.40}$$

Bound of $S_n^{(3)}$. The argument is similar to the previous one, based on a different discrete martingale. For $i=1,\cdots,2^n-1$, $j=1,\cdots,n$, set

$$\Phi_j(i) = T2^{-n}\left(D\tilde\sigma_j(u^n(t_i))\sigma_j(u^n(t_i))\,,\, u^n(t_i)-u(t_i)\right) 1_{\{t_i\leq\tau_n\}}\, 1_{G_N^n(t_i)}.$$

Then $\Phi_j(i-1)$ is $\mathcal{F}_{t_{i-1}}$-measurable and independent of the centered random variable $Y_{ij}=2^n\,T^{-1}\big|\beta_j(t_i)-\beta_j(t_{i-1})\big|^2-1$. Furthermore, for $(i,j)\neq(i',j')$ one has

$$\mathbb{E}\big(\Phi_j(i-1)\, Y_{ij}\, \Phi_{j'}(i'-1)\, Y_{i'j'}\big)=0.$$

Using (2.20), (2.16) and (5.14), we deduce that for all i,j, $\mathbb{E}\big(|\Phi_j(i-1)|^2\big)\leq C_{N,T}\, 2^{-2n}$. For $k=2,\cdots,2^n$, set

$$N_k=\sum_{2\leq i\leq k}\ \sum_{1\leq j\leq n}\Phi_j(i-1)\, Y_{ij}.$$

The process $(N_k,\mathcal{F}_{t_k})$ is a discrete martingale; thus Schwarz's and Doob's inequality yield

$$\begin{aligned}
\mathbb{E}\Big(\max_{2\leq k\leq 2^n}|N_k|\Big) &\leq 2\big\{\mathbb{E}\big(|N_{2^n}|^2\big)\big\}^{\frac12}\\
&\leq 2\Big\{\sum_{2\leq i\leq 2^n}\ \sum_{1\leq j\leq n}\mathbb{E}\big(\Phi_j(i-1)^2\big)\mathbb{E}\big(|Y_{ij}|^2\big)\Big\}^{\frac12}\leq C_{T,N}\, n^{\frac12}\, 2^{-\frac{n}{2}}.
\end{aligned} \tag{6.41}$$

Finally, a similar argument shows that

$$\begin{aligned}
&\mathbb{E}\Big(\max_{1\leq k<2^n}\ \sup_{t_k\leq t\leq t_{k+1}}\Big|2^nT^{-1}(t-t_k)\sum_{1\leq j\leq n}\Phi_j(k-1)Y_{kj}\Big|\Big)\\
&\leq\Big(\mathbb{E}\sum_{1\leq k<2^n}\Big|\sum_{1\leq j\leq n}\Phi_j(k-1)Y_{kj}\Big|^2\Big)^{\frac12}\\
&\leq\Big(\sum_{1\leq k<2^n}\mathbb{E}\sum_{1\leq j\leq n}\big|\Phi_j(k-1)Y_{kj}\big|^2\Big)^{\frac12}\leq C(N,T)n^{\frac12}2^{-\frac{n}{2}}.
\end{aligned} \tag{6.42}$$

The inequalities (6.41) and (6.42) imply that

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|S_n^{(3)}|\Big)\leq C(N,T)\, n^{\frac12}\, 2^{-\frac{n}{2}}. \tag{6.43}$$

Using (6.35) and collecting the upper estimates in (6.35), (6.37), (6.40) and (6.43), we conclude the proof of Lemma 6.4. □

6.3.10. **Bound for** $S_n(t,7)$**.**

Lemma 6.5. *Let the assumptions of Theorem 3.1 be satisfied and* $S_n(t,7)$ *be defined by* (6.24)*. There exists a constant* $C(N,T)$ *such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|S_n(t,7)|\Big)\le C(N,T)\,n^2\,2^{-\frac{n}{2}}. \tag{6.44}$$

Proof. For $s\in[0,T]$, $j=1,\cdots,n$, set

$$\widetilde{U}_j^n(s)=D\tilde\sigma_j(u^n(s_n))\Big[\tilde\sigma(u^n(s_n))\Big(\int_{s_n}^s \widetilde{\dot W}^n(r)dr\Big)\Big]\,\dot\beta_j^{\,n}(s),$$

$$\widetilde\Delta_n(s)=\Big(\sum_{1\le j\le n}\widetilde{U}_j^n(s)-\frac12\tilde\varrho_n(u^n(s_n))\,,\,u^n(s_n)-u(s_n)\Big).$$

We obviously have that

$$\sum_{1\le j\le n}\widetilde{U}_j^n(s)-\frac12\tilde\varrho_n(u^n(s_n))=\sum_{1\le i\le 3}\widetilde{V}_n^{(i)}(s),$$

where

$$\widetilde{V}_n^{(1)}(s)=\sum_{1\le j\le n}D\tilde\sigma_j(u^n(s_n))\tilde\sigma(u^n(s_n))\big[W_n(s_n)-W_n((s_n-T2^{-n})\vee 0)\big]\,\dot\beta_j^{\,n}(s),$$

$$\widetilde{V}_n^{(2)}(s)=\sum_{1\le j\le n}\sum_{l\ne j}D\tilde\sigma_j(u^n(s_n))\tilde\sigma_l(u^n(s_n))(s-\underline{s}_n)\left[\beta_l(\underline{s}_n)-\beta_l(s_n)\right]\frac{2^{2n}}{T^2}\left[\beta_j(\underline{s}_n)-\beta_j(s_n)\right],$$

$$\widetilde{V}_n^{(3)}(s)=\sum_{1\le j\le n}\tilde\sigma_j(u^n(s_n))\tilde\sigma_j(u^n(s_n))\left[\frac{2^{2n}}{T^2}(s-\underline{s}_n)\left[\beta_j(\underline{s}_n)-\beta_j(s_n)\right]^2-\frac12\right].$$

Using (6.34) we deduce the following decomposition of $S_n(t,7)$:

$$S_n(t,7)=\int_0^{t\wedge\tau_n}\tilde\Delta_n(s)ds=\sum_{1\le i\le 3}\tilde S_n^{(i)}(t)-\tilde S_n^{(4)}(t), \tag{6.45}$$

where

$$\tilde S_n^{(i)}(t)=\int_{t_2}^t 1_{\{s_n\le\tau_n\}}1_{G_N^n(s_n)}\big(\widetilde V_n^{(i)}(s),u^n(s_n)-u(s_n)\big)ds,\quad i=1,2,3,$$

$$\tilde S_n^{(4)}(t)=\int_0^t 1_{\{s_n\le\tau_n<s\}}1_{G_N^n(s_n)}\widetilde\Delta_n(s)ds.$$

We note that $\tilde S_n^{(i)}(t)=0$ for $i=1,2,3$ and $t\le t_2$.

Bound for $\tilde S_n^{(4)}$*.* The proof is similar to that of the upper estimate of $S_n^{(4)}$. Schwarz's inequality implies

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|\tilde S_n^{(4)}(t)|\Big)\le\Big\{2T2^{-n}\sum_{0\le k<2^{n-1}}\mathbb{E}\int_{t_k}^{t_{k+1}}1_{G_N^n(t_k)}\left|\widetilde\Delta_n(s)\right|^2ds\Big\}^{1/2}.$$

The inequalities (2.16), (2.20), (2.24), the definition (5.14) of the set $G_N^n(s)$ and Schwarz's inequality yield for $t_k\le s<t_{k+2}$:

$$1_{G_N^n(t_k)}\left|\widetilde\Delta_n(s)\right|^2\le C(N)\Big(1+n\Big|\int_{s_n}^s\widetilde{\dot W}^n(r)dr\Big|^2\sum_{1\le j\le n}|\dot\beta_j^{\,n}(s)|^2\Big).$$

Therefore, Fubini's theorem and Schwarz's inequality imply

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[0,T]}|\tilde{S}_n^{(4)}(t)|\Big)\\
&\quad\le C_{N,T}2^{-n/2}\Big\{T+n\sum_{1\le j\le n}\sum_{0\le k<2^n}\mathbb{E}\int_{t_{k-1}}^{t_{k+1}}\Big|\int_{s_n}^{s}\dot{\widetilde{W}}^n(r)dr\Big|_0^2|\dot{\beta}_j^{\,n}(s)|^2ds\Big\}^{1/2}\\
&\quad\le C_{N,T}2^{-n/2}\Big\{1+n2^{-n}\sum_{1\le j\le n}\sum_{0\le k<2^n}\mathbb{E}\Big[\int_{t_{k-1}\vee 0}^{t_{k+1}}\Big|\dot{\widetilde{W}}^n(r)\Big|_0^2dr\int_{t_k}^{t_{k+1}}|\dot{\beta}_j^{\,n}(s)|^2ds\Big]\Big\}^{1/2}\\
&\quad\le C_{N,T}2^{-\frac{n}{2}}\Big\{1+n2^{-2n}\sum_{1\le j\le n}\sum_{0\le k<2^n}\Big[\int_{t_{k-1}\vee 0}^{t_{k+1}}\mathbb{E}\big|\dot{\widetilde{W}}^n(r)\big|_0^4dr\Big]^{\frac12}\Big[\int_{t_k}^{t_{k+1}}\mathbb{E}|\dot{\beta}_j^{\,n}(s)|^4ds\Big]^{\frac12}\Big\}^{\frac12}.
\end{aligned}$$

Since for every $s\in[0,T]$ we have $\mathbb{E}\big|\dot{\widetilde{W}}^n(s)\big|_0^4\le C(T)\,n^4\,2^{2n}$ and $\mathbb{E}|\dot{\beta}_j^n(s)|^4\le C(T)2^{2n}$, we deduce the existence of some constant $C(N,T)$ such that

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|\tilde{S}_n^{(4)}(t)|\Big)\le C(N,T)\,n^2\,2^{-n/2}. \tag{6.46}$$

Bound for $\tilde{S}_n^{(1)}$. For $j=1,...,n$ let

$$\varphi_j(s)=1_{\{\underline{s}_n\le\tau_n\}}1_{G_N^n(\underline{s}_n)}\Big(D\tilde{\sigma}_j(u^n(\underline{s}_n))\big[\tilde{\sigma}(u^n(\underline{s}_n))\big(W_n(\underline{s}_n)-W_n(s_n)\big)\big]\,,\,u^n(\underline{s}_n)-u(\underline{s}_n)\Big).$$

Then $\varphi_j(s)$ is $\mathcal{F}_{\underline{s}_n}$ measurable and for $t\ge t_2$,

$$\tilde{S}_n^{(1)}(t)=\sum_{1\le j\le n}\int_{t_1}^{t_n}\varphi_j(s)d\beta_j(s)+\sum_{1\le j\le n}\varphi_j(t-T2^{-n})2^nT^{-1}(t-\underline{t}_n)\big[\beta_j(\underline{t}_n)-\beta_j(t_n)\big].$$

For fixed j the process $\big(\varphi_j(t_k)(\beta_j(t_{k+1})-\beta_j(t_k))\,,\,0\le k<2^n\big)$ is a martingale increments. Therefore, the Burkholder and Schwarz inequalities, (2.20), (2.16) and (5.14), yield

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[0,T]}|\tilde{S}_n^{(1)}(t)|\Big)\\
&\quad\le C\Big\{\mathbb{E}\int_0^T\sum_{1\le j\le n}\varphi_j(s)^2ds\Big\}^{\frac12}+C\mathbb{E}\Big(\sum_{1\le j\le n}\max_{1\le k<2^n}|\varphi_j(t_k)|\,|\beta_j(t_{k+1})-\beta_j(t_k)|\Big)\\
&\quad\le C_{N,T}\Big\{n\mathbb{E}\int_0^T|W_n(\underline{s}_n)-W_n(s_n)|_0^2\,ds\Big\}^{\frac12}\\
&\qquad+C_{N,T}\mathbb{E}\Big\{n\sum_{1\le k<2^n}\sum_{1\le j\le n}|W_n(t_k)-W_n(t_{k-1})|_0^2|\beta_j(t_{k+1})-\beta_j(t_k)|^2\Big\}^{\frac12}\\
&\quad\le C_{N,T}\sqrt{n}\Big[2^{-\frac{n}{2}}+\Big\{\sum_{1\le j\le n}\sum_{1\le k<2^n}\mathbb{E}|W_n(t_k)-W_n(t_{k-1})|_0^2\mathbb{E}|\beta_j(t_{k+1})-\beta_j(t_k)|^2\Big\}^{\frac12}\\
&\quad\le C(N,T)\,n\,2^{-n/2}.
\end{aligned} \tag{6.47}$$

Bound for $\tilde{S}_n^{(2)}$. For $i=1,\cdots,2^n-1$, $j=1,\cdots,n$ and $l\neq j$ set

$$\tilde{\Phi}_{j,l}(i)=2^{2n}T^{-2}1_{\{t_i\le\tau_n\}}1_{G_N^n(t_i)}\Big(D\tilde{\sigma}_j(u^n(t_i))\tilde{\sigma}_l(u^n(t_i))\,,\,u^n(t_i)-u(t_i)\Big).$$

Then $\tilde{\Phi}_{j,l}(i)$ is $\mathcal{F}_{t_i}$ measurable and since for $l\neq j$, $\mathcal{F}_{t_{i-1}}$, $\beta_j(t_i)-\beta_j(t_{i-1})$ and $\beta_l(t_i)-\beta_l(t_{i-1})$ are independent, if one sets $Z_{j,l}(i)=\big(\beta_l(t_i)-\beta_l(t_{i-1})\big)\big(\beta_j(t_i)-\beta_j(t_{i-1})$, the following process $(\tilde{M}_k, 2 \le k \le 2^n)$ is a $(\mathcal{F}_{t_k})$ centered martingale:

$$\tilde{M}_k = \sum_{2\le i\le k} \sum_{1\le j\le n} \sum_{l\ne j} \int_{t_i}^{t_{i+1}} \tilde{\Phi}_{j,l}(i-1)(s-t_i) Z_{j,l}(i) ds$$
$$= T^2 2^{-(1+2n)} \sum_{2\le i\le k} \sum_{1\le j\le n} \sum_{l\ne j} \tilde{\Phi}_{j,l}(i-1) Z_{j,l}(i).$$

Furthermore, if $i < i'$ and $l' \ne j'$, or $i' < i$ and $l \ne j$, or $i = i'$ and $\big(\min(j,l), \max(j,l)\big) \ne \big(\min(j',l'), \max(j',l')\big)$, one has $\mathbb{E}\big[\tilde{\Phi}_{j,l}(i-1) Z_{j,l}(i) \tilde{\Phi}_{j',l'}(i'-1) Z_{j',l'}(i')\big] = 0$. Hence Doob's, Schwarz's inequalities together with (2.20), (2.16) and (5.14) yield

$$\begin{aligned}
&\mathbb{E}\Big(\max_{2\le k<2^n} |\tilde{M}_k|\Big)^2 \le \mathbb{E}\Big(\max_{2\le k<2^n} |\tilde{M}_k|^2\Big) \le 4\mathbb{E}\big(\tilde{M}^2_{2^n-1}\big)\\
&\le C_T 2^{-4n} \sum_{2\le i<2^n} \sum_{1\le j\le n} \sum_{l\ne j} \mathbb{E}\big(|\tilde{\Phi}_{j,l}(i-1)|^2\big) \mathbb{E}\big(|\beta_l(t_i) - \beta_l(t_{i-1})|^2\big) \mathbb{E}\big(|\beta_j(t_i) - \beta_j(t_{i-1})|^2\big)\\
&\le C(N,T)\, n\, 2^{-n}. \qquad (6.48)
\end{aligned}$$

A computation similar to that performed in (6.39) proves that

$$\begin{aligned}
&\mathbb{E}\Big(\max_{2\le k<2^n} \sup_{t_k\le t\le t_{k+1}} \Big| \sum_{1\le j\le n} \sum_{l\ne j} \int_{t_k}^t \tilde{\Phi}_{j,l}(k-1)(s-t_k) Z_{j,l}(k) ds \Big|\Big) \qquad (6.49)\\
&\quad \le T^2 2^{-2n} \Big\{ \sum_{2\le k<2^n} \sum_{1\le j\le n} \sum_{l\ne j} 2^{-4n} \mathbb{E}\big(|\tilde{\Phi}_{j,l}(k-1)|^2\big)\\
&\qquad\qquad \times \mathbb{E}\big(|\beta_l(t_k) - \beta_l(t_{k-1})|^2\big) \mathbb{E}\big(|\beta_j(t_k) - \beta_j(t_{k-1})|^2\big) \Big\}^{\frac12} \le C(N,T)\, n\, 2^{-n/2}.
\end{aligned}$$

The inequalities (6.48) and (6.49) yield

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |\tilde{S}^{(2)}_n(t)|\Big) \le C(N,T)\, n\, 2^{-n/2}. \qquad (6.50)$$

Bound for $\tilde{S}^{(3)}_n$. Finally, for $i = 1, \cdots, 2^n - 1$ and $j = 1, \ldots, n$, set

$$\begin{aligned}
\tilde{\Phi}_j(i) &= 1_{\{t_i\le\tau_n\}} 1_{G^n_N(t_i)} \Big(D\tilde{\sigma}_j(u^n(t_i)) \tilde{\sigma}_j(u^n(t_i)) \,,\, u^n(t_i) - u(t_i) \Big),\\
Z_j(i) &= \int_{t_i}^{t_{i+1}} \Big[\frac{2^{2n}}{T^2} (s-t_i) \big(\beta_j(t_{i+1}) - \beta_j(t_i)\big)^2 - \frac12 \Big] ds.
\end{aligned}$$

Then the random variables $Z_j(i)$ and $\tilde{\Phi}_j(i)$ are independent, $\mathbb{E}(Z_j(i)) = 0$ and $\mathbb{E}(Z_j(i)^2) \le C_T 2^{-2n}$. Furthermore, for $(i,j) \ne (i',j')$, $\mathbb{E}\big(\tilde{\Phi}_j(i) Z_j(i) \tilde{\Phi}_{j'}(i') Z_{j'}(i')\big) = 0$. The process defined for $k = 1, \cdots, 2^n - 1$ by $\tilde{N}_k = \sum_{1\le i\le k} \sum_{1\le j\le n} \tilde{\Phi}_j(i) Z_j(i)$ is a discrete $(\mathcal{F}_{t_{k+1}})$ martingale. Doob's and Schwarz's inequalities, (2.20), (2.16) and (5.14) imply that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[0,T]} |\tilde{S}^{(3)}_n(t)|\Big) \le \mathbb{E}\Big(\max_{1\le k<2^n} \big|\tilde{N}_k\big|\Big)\\
&\qquad + \mathbb{E}\Big(\max_{1\le k<2^n} \sup_{t_k\le t\le t_{k+1}} \Big| \sum_{1\le j\le n} \tilde{\Phi}_j(k) \int_{t_k}^t \Big[\frac{2^{2n}}{T^2} (s-t_k) \big(\beta_j(t_{k+1}) - \beta_j(t_k)\big)^2 - \frac12 \Big] ds \Big|\Big)\\
&\le C\mathbb{E}\big(\big|\tilde{N}_{2^n-1}\big|^2\big)^{\frac12} + \Big(2^n n \max_{1\le k<2^n} \max_{1\le j\le n} \mathbb{E}\big(\tilde{\Phi}_j(k)^2\big) \mathbb{E}\big(Z_j(k)^2\big)\Big)^{\frac12}\\
&\le C\Big(n 2^n \max_{2\le k<2^n} \max_{1\le j\le n} \mathbb{E}\big(\tilde{\Phi}_j(k)^2\big) \mathbb{E}\big(Z_j(k)^2\big)\Big)^{\frac12} \le C(N,T)\, n^{\frac12}\, 2^{-\frac{n}{2}}. \qquad (6.51)
\end{aligned}$$

The relations in (6.45) – (6.51) conclude the proof of Lemma 6.5. □

Now using Proposition 6.3, Lemmas 6.4 and 6.5, we obtain (4.6); this completes the proof of Theorem 3.1.

7. Appendix

We consider some additional properties of the solution to (2.31). The aim of this section is to introduce some more properties on the coefficients σ, $\tilde\sigma$, G and R which will ensure that the property (3.1) holds. Let $\bar C$ denote a constant such that

$$|u| \le \bar C \|u\|\,, \forall u \in V. \tag{7.1}$$

7.1. **Exponential moments.**

Proposition 7.1. *Let $h(t) \in S_M$ be deterministic, suppose that the operators G and $\sigma+\tilde\sigma$ are uniformly bounded and that the linear growth of R is small enough, i.e., there exist positive constants K_0, R_0 and $\tilde R_0$ such that:*

$$|G(u)|^2_{L(H_0,H)} \le K_0,\; |(\sigma+\tilde\sigma)(u)|^2_{L_Q} \le K_0,\; |R(u)| \le R_0 + \tilde R_0 |u| \text{ with } \tilde R_0 < \bar C^{-2} \tag{7.2}$$

for every $u \in H$. Let $u(t)$ be the solution to (2.31) such that the initial condition has some exponential moment, i.e., $\mathbb{E}\exp(\alpha_0|\xi|^2) < \infty$ for some $\alpha_0 > 0$. Then there exist constants $\alpha_1 \in]0,\alpha_0]$, $\beta(\alpha) > 0$ and $c_i > 0, i=1,2$ such that for $0 < \alpha < \alpha_1$ and $t \in [0,T]$:

$$\mathbb{E}\exp\left(\alpha|u(t)|^2 + \beta(\alpha)\int_0^t \|u(s)\|^2 ds\right) \le e^{c_1 t + c_2 M}\,\mathbb{E}\exp(\alpha|\xi|^2). \tag{7.3}$$

The same estimate holds for Galerkin approximations u_n of u with constants c_1, c_2 which do not depend on n.

Proof. Let $\sigma_0 = \sigma+\tilde\sigma$, $\Phi_0(t) = \exp\big(\alpha|u(t)|^2\big)$ and $\Phi(t) = \Phi_0(t)\exp\Big(\beta\int_0^t \|u(s)\|^2 ds\Big)$. By Itô's formula we have for every $t\in[0,T]$:

$$d\Phi(t) = \big[\beta\|u(t)\|^2\Phi_0(t)dt + d\Phi_0(t)\big]\exp\left(\beta\int_0^t \|u(s)\|^2 ds\right)$$

and

$$d\Phi_0(t) = \alpha\Phi_0(t)\Big[2(u(t),du(t)) + |\sigma_0(u(t))|^2_{L_Q}dt + 2\alpha|\sigma_0^*(u(t))u(t)|^2_{H_0}dt\Big].$$

Therefore, if $I(t) = 2\alpha\int_0^t \Phi(s)\big(u(s)\,,\,\sigma_0(u(s))dW(s)\big)$,

$$\begin{aligned} d\Phi(t) &= \Phi(t)\Big[-(2\alpha-\beta)\|u(t)\|^2 + 2\alpha\big(-R(u(t)) + G(u(t))h(t), u(t)\big) + \alpha|\sigma_0(u(t))|^2_{L_Q} \\ &\qquad + 2\alpha^2|\sigma_0^*(u(t))u(t)|^2_{H_0}\Big]dt + I(t). \end{aligned}$$

For any integer $n\ge 1$, let $\tau_n = \inf\{t : \sup_{0\le s\le t}|u(s)|^2 + \int_0^t \|u(s)\|^2 ds \ge n\}\wedge T$. Then we have $\mathbb{E}(I(t\wedge\tau_n) = 0$ for $t\in[0,T]$. Since $|u(t)| \le \bar C\|u(t)\|$, if $\tilde R_0$ from (7.2) is such that $\tilde R_0 < \bar C^{-2}$, for $\alpha_1 \le \alpha_0$ small enough and $0<\alpha<\alpha_0$, we have $1-(\tilde R_0 + 2^{-1}\alpha K_0)\bar C^2 > 0$. For $0<\beta<\beta(\alpha)$ with $\beta(\alpha)$ small enough, and for ϵ small enough, Fubini's theorem implies:

$$\begin{aligned} \sup_{0\le s\le t}\mathbb{E}\Phi(s\wedge\tau_n) &\le \exp(\alpha|\xi|^2) + \mathbb{E}\int_0^{t\wedge\tau_n}\Phi(s)\big[R_0\epsilon^{-1} + K_0\epsilon^{-1}|h(s)|_0^2 + \alpha K_0\big]\,ds \\ &\le \exp(\alpha|\xi|^2) + \int_0^t \mathbb{E}\Phi(s\wedge\tau_n)\big[R_0\epsilon^{-1} + K_0\epsilon^{-1}|h(s)|_0^2 + \alpha K_0\big]\,ds. \end{aligned}$$

Since $\Phi(.\wedge\tau_n)$ is bounded, Gronwall's lemma implies that there exist constants c_1, c_2 depending on K_0, R_0 and α such that for every $t\in[0,T]$,

$$\sup_n \sup_{0\le t\le T} \mathbb{E}\Phi(t\wedge\tau_n) \le \exp(\alpha|\xi|^2)\,\exp(c_1T+c_2M).$$

Using (2.32) and the monotone convergence theorem, we conclude the proof by letting $n\to\infty$. □

7.2. **Properties in $\mathcal{H}$.** Now we are in position to state the conditions which guarantee the validity of conditions (i) and (ii) in Theorems 3.1 and 3.2.

Condition (BS+) Let condition **(B)** hold with $\mathcal{H}=Dom(A^{1/4})$ and suppose that there exists a constant $K>0$ such that for $u\in\mathcal{H}$:

$$|A^{\frac14}\sigma(t,u)|^2_{L_Q(H_0,H)} + |A^{\frac14}\tilde\sigma(t,u)|^2_{L_Q(H_0,H)} \le K(1+\|u\|^2_{\mathcal{H}}). \tag{7.4}$$

Condition (GR1) There exist constants $\bar K_0$ and $\bar R_0$ such that for every $u\in\mathcal{H}$:

$$|A^{\frac14}G(u)|^2_{L(H_0,H)} \le \bar K_0(1+\|u\|^2_{\mathcal{H}})\,,\ |A^{\frac14}R(u)| \le \bar R_0(1+\|u\|_{\mathcal{H}})\,. \tag{7.5}$$

Proposition 7.2. *Assume that conditions* **(BS+)**, **(GR1)**, *as well as* (2.16) *and* (2.17) *from condition* **(S)** *are satisfied. Let the hypotheses of Proposition 7.1 be in force and let u be the solution to* (2.31). *Assume in addition that $\mathbb{E}\|\xi\|^2_{\mathcal{H}}<\infty$. Then there exist $q>0$ and $q_*>0$ such that*

$$\mathbb{E}\Big(\operatorname*{ess\,sup}_{[0,T]}\|u(t)\|^q_{\mathcal{H}}\Big) + \mathbb{E}\Big(\Big|\int_0^T |A^{3/4}u(\tau)|^2 d\tau\Big|^{q_*}\Big) < \infty. \tag{7.6}$$

Proof. We consider the Galerkin approximations u_n and, to ease notations, we skip the index n. Let $\sigma_0=\sigma+\tilde\sigma$ and for $t\in[0,T]$ set

$$I(t) := \sup_{0\le s\le t} 2\Big|\int_0^t \big(A^{\frac14}\sigma_0(u(r))dW(r)\,,\,A^{\frac14}u(r)\big)\Big|.$$

Using Itô's formula for $\|u(t)\|^2_{\mathcal{H}}=|A^{1/4}u(t)|^2$ and usual upper estimates, we deduce that

$$\sup_{s\le t}\|u(s)\|^2_{\mathcal{H}} + 2\int_0^t |A^{\frac34}u(s)|^2\,ds \le \|\xi\|^2_{\mathcal{H}} + 2\int_0^t |\langle B(u(s),u(s)),A^{\frac12}u(s)\rangle|\,ds$$
$$+ I(t) + \int_0^t 4K(1+\|u(s)\|^2_{\mathcal{H}})ds + 2\int_0^t |(-R(u(s))+G(u(s))h(s),A^{1/2}u(s))|ds.$$

The inequality (2.6) and condition **(GR1)** imply

$$|\langle B(u,u),A^{\frac12}u\rangle| \le C_0\|u\|_{\mathcal{H}}\|u\||A^{3/4}u| \le |A^{3/4}u|^2 + C_0^2\,2^{-2}\,\|u\|^2_{\mathcal{H}}\|u\|^2\,,$$
$$|(-R(u)+G(u)h,A^{1/2}u)| \le c_0(1+|h|_0)(1+\|u\|^2_{\mathcal{H}}),$$

where c_0 depends on $\bar K_0$ and $\bar R_0$. Hence, for $X(t)=\sup\{\|u(s)\|^2_{\mathcal{H}}\,:\,0\le s\le t\}$, we deduce

$$X(t)+\int_0^t |A^{\frac34}u(s)|^2\,ds \le \|\xi\|^2_{\mathcal{H}} + I(t) + c_1 + c_2\int_0^t \big[1+|h(s)|_0+\|u(s)\|^2\big]\,X(s)\,ds, \tag{7.7}$$

where the constant c_1 depends on $K,\bar K_0,\bar R_0,T,M$ and c_2 depends on $\bar K_0$ and $\bar R_0$. Gronwall's lemma yields

$$X(t) \;\le\; \big[c_1+\|\xi\|^2_{\mathcal{H}}+I(t)\big]\exp\Big(c_2\int_0^t \big[1+|h(s)|_0+\|u(s)\|^2\big]\,ds\Big).$$

This implies that for $\delta>0$:

$$\mathbb{E}|X(t)|^\delta \;\leq\; C(M,T)\,\Big[\mathbb{E}\left(c_1+\|\xi\|^2_{\mathcal H}+I(t)\right)^{2\delta}\Big]^{1/2}\Big[\mathbb{E}\exp\Big(2c_2\delta\int_0^t\|u(s)\|^2ds\Big)\Big]^{1/2}.$$

Thus Proposition 7.1 implies that for δ small enough we have:

$$\mathbb{E}|X(t)|^\delta\le C(M,T)\mathbb{E}\big[\exp(2c_2\delta|\xi|^2)\big]^{\frac12}\;\big[1+\mathbb{E}\|\xi\|^2_{\mathcal H}+\mathbb{E}I(t)\big]^{\frac12}.$$

The Burkholder-Davies-Gundy inequality, relations (7.4) and (2.32) yield

$$\begin{aligned}\mathbb{E}I(t)\;&\leq\;6\,\mathbb{E}\Big\{\int_0^t|A^{1/4}u(r)|^2\,|A^{1/4}[\sigma+\tilde\sigma](u(r))|^2_{L_Q}\,dr\Big\}^{\frac12}\\ &\leq\;6\,\mathbb{E}\Big\{4K\int_0^t\|u(r)\|^2_{\mathcal H}\left(1+\|u(r)\|^2_{\mathcal H}\right)\,dr\Big\}^{\frac12}\le c_4(T,K,C).\end{aligned}$$

Thus there exists constants $q>0$ and $c:=c(K,T,M,C)$ such that

$$\sup_{n\ge1}\mathbb{E}\Big(\sup_{0\le s\le T}\|u_n(s)\|^q_{\mathcal H}\Big)=c<+\infty \tag{7.8}$$

for the Galerkin approximations u_n. As $n\to+\infty$, after limit transition we deduce that the first term in the left hand-side of (7.6) is finite.

To prove that the second term is finite as well, note that (7.7) implies that for every n:

$$\int_0^t|A^{\frac34}u_n(s)|^2\,ds\le C+\|\xi\|^2_{\mathcal H}+I(t)+c_2\mathrm{ess}\sup_{0\le s\le T}\|u_n(s)\|^2_{\mathcal H}\int_0^t\big[1+|h(s)|_0^2+\|u_n(s)\|^2\big]\,ds.$$

Thus we can use (7.8) and complete the proof of (7.6) by a similar argument. □

We prove that the process u solving (2.31) belongs to $\mathcal C([0,T],\mathcal H)$ a.s.

Proposition 7.3. *Let the conditions of Proposition 7.2 be satisfied and let u be the solution to* (2.31). *Then the process u belongs to $\mathcal C([0,T],\mathcal H)$ a.s.*

Proof. Let $\sigma_0=\sigma+\tilde\sigma$; then for fixed $\delta>0$, we have $e^{-\delta A}u\in C([0,T],\mathcal H)$. Indeed, (7.4) and (2.32) imply that $\mathbb{E}\int_0^T|A^{\frac14}e^{-\delta A}\sigma_0(u(s))|^2_{L_Q}\,ds<+\infty$, so that $\int_0^t e^{-\delta A}\sigma_0(u(s))\,dW(s)\in\mathcal C([0,T],\mathcal H)$. Since for $\delta>0$ the operator $e^{-\delta A}$ maps H to V and V' to $\mathcal H$, we deduce that almost surely the maps $A^{\frac14}\,e^{-\delta A}\int_0^t[B((u(s))+R(u(s))]\,ds$ and $A^{\frac14}\,e^{-\delta A}\int_0^t G((u(s))\,h(s)\,ds$ belong to $\mathcal C([0,T],\mathcal H)$. Therefore it is sufficient to prove that

$$\lim_{\delta\to0}\mathbb{E}\Big(\sup_{0\le t\le T}\|u(t)-e^{-\delta A}u(t)\|^{2p}_{\mathcal H}\Big)=0 \tag{7.9}$$

for some $p>0$. Let $T_\delta=Id-e^{-\delta A}$ and apply Itô's formula to $\|T_\delta u(t)\|^2_{\mathcal H}$. This yields

$$\begin{aligned}\|T_\delta u(t)\|^2_{\mathcal H}=&\|T_\delta\xi\|^2_{\mathcal H}-2\int_0^t|A^{\frac34}u(s)|^2ds+2I(t)+\int_0^t|A^{\frac14}T_\delta\sigma_0(u(s))|^2_{L_Q}ds\\ &-2\int_0^t\big\langle B(u(s))+R(u(s))-G(u(s))h(s),A^{\frac12}T_\delta^2u(s)\big\rangle\,ds,\end{aligned} \tag{7.10}$$

where $I(t)=\int_0^t\big(A^{\frac14}T_\delta\sigma_0(u(s))dW(s),A^{\frac14}T_\delta u(s)\big)$. The Burkholder-Davies-Gundy and Schwarz inequalities together with (7.4) imply that for any $p>0$:

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}|I(t)|^p\;&\leq\;C_p\mathbb{E}\left(\int_0^T\|T_\delta u(s)\|^2_{\mathcal H}|A^{\frac14}T_\delta\,\sigma_0(u(s))|^2_{L_Q}ds\right)^{p/2}\\ &\leq\;\frac12\,\mathbb{E}\sup_{0\le t\le T}\|T_\delta u(t)\|^{2p}_{\mathcal H}+\frac{C_p^2}{2}\mathbb{E}\left(\int_0^T|A^{\frac14}T_\delta\sigma_0(u(s))|^2_{L_Q}\,ds\right)^p.\end{aligned}$$

Hence (7.10) yields for $0 < p < 1$ the existence of a constant c_p such that

$$\mathbb{E}\sup_{0\le t\le T}\|T_\delta u(t)\|_{\mathcal{H}}^{2p} \le c_p\Big[\|T_\delta \xi\|_{\mathcal{H}}^{2p} + \mathbb{E}\Big|\int_0^T |A^{\frac14}T_\delta\sigma_0(u(s))|_{L_Q}^2 ds\Big|^p + \mathbb{E}\Big(\int_0^T \Big|\big\langle B(u(s)) + R(u(s)) - G(u(s))h(s), A^{\frac12}T_\delta^2 u(s)\big\rangle\Big|\, ds\Big)^p\Big].$$

Since for every $u\in\mathcal{H}$, $\|T_\delta u\|_{\mathcal{H}}\to 0$ as $\delta\to 0$ and $\sup_{\delta>0}|T_\delta|_{L(\mathcal{H},\mathcal{H})}\le 1$, we deduce that if $\{\varphi_k\}$ denotes an orthonormal basis in H, then $|A^{\frac14}T_\delta\sigma_0(u(s))Q^{1/2}\varphi_k|^2\to 0$ for every k and almost every $(\omega,s)\in\Omega\times[0,T]$. Since $\sup_{\delta>0}\|e^{-\delta A}\|_{L(\mathcal{H})}<+\infty$, (7.4) implies

$$\sup_{\delta>0}|A^{\frac14}T_\delta\sigma_0(u)|_{L_Q}^2 \le C(1+\|u\|_{\mathcal{H}}^2)\in L^1(\Omega\times[0,T]).$$

Therefore, the Lebesgue dominated convergence theorem yields

$$\mathbb{E}\int_0^T |A^{\frac14}T_\delta\sigma_0(u(s))|_{L_Q}^2 ds \to 0 \quad \text{as} \quad \delta\to 0.$$

Furthermore, using (2.6) we deduce

$$\int_0^T \Big|\big\langle B(u(s)), A^{\frac12}T_\delta^2 u(s)\big\rangle\Big|\, ds \le C\int_0^T \|u(s)\|_{\mathcal{H}}\|u(s)\|\,|A^{\frac34}T_\delta^2 u(s)|\, ds \le C\operatorname*{ess\,sup}_{[0,T]}\|u(s)\|_{\mathcal{H}}\Big[\int_0^T \|u(s)\|^2\, ds\Big]^{1/2}\Big[\int_0^T |A^{\frac34}T_\delta^2 u(s)|^2\, ds\Big]^{1/2}.$$

Thus, using Proposition 7.2 for $p>0$ small enough and Hölder's inequality, we obtain

$$\mathbb{E}\Big[\int_0^T \Big|\big\langle B(u(s)), A^{\frac12}T_\delta^2 u(s)\big\rangle\Big|\, ds\Big]^p \le C\Big[\mathbb{E}\operatorname*{ess\,sup}_{[0,T]}\|u(s)\|_{\mathcal{H}}^{2p}\Big]^{1/2}\Big[\mathbb{E}\Big(\int_0^T\|u(s)\|^2\, ds\Big)^{2p}\Big]^{1/4} \times\Big[\mathbb{E}\Big(\int_0^T |A^{\frac34}T_\delta^2 u(s)|^2\, ds\Big)^{2p}\Big]^{1/4} \le C\Big[\mathbb{E}\Big(\int_0^T |A^{\frac34}T_\delta^2 u(s)|^2\, ds\Big)^{2p}\Big]^{1/4}.$$

Given $u\in Dom(A^{\frac34})$ we have $|A^{\frac34}T_\delta^2 u|\to 0$ as $\delta\to 0$ while $|A^{\frac34}T_\delta^2 u|\le 2|A^{\frac34}u|$. Hence the dominated convergence theorem yields $\mathbb{E}\Big[\int_0^T\Big|\big\langle B(u(s)), A^{\frac12}T_\delta^2 u(s)\big\rangle\Big|\, ds\Big]^p\to 0$ as $\delta\to 0$. A similar argument can be applied to the term $\int_0^T\Big|\big\langle R(u(s)) - G(u(s))h(s), A^{\frac12}T_\delta^2 u(s)\big\rangle\Big|\, ds$. Thus we obtain that (7.9) holds with $p>0$ small enough. □

7.3. **Examples of models.** In Remark 3.3 we have already shown that Theorems 3.1 and 3.2 can be applied to periodic stochastic 2D Navier-Stokes equations and also to some shell models of turbulence. The corresponding arguments involve either the additional symmetry of the bilinear operator B (see (3.5)) or some additional regularity provided by the discrete structure of shell type models. These properties are not true for other 2D hydrodynamical problems which we have in mind (see Section 2.1 in [9]). However the properties stated in (7.2) and also in Conditions **(BS+)** and **(GR1)** provide us with another set of sufficient hypotheses on the operators in (2.31) which guarantee the requirements (i) and (ii) concerning solutions in Theorem 3.1. They allow us to cover several important cases which include:

- 2D Navier-Stokes equations with Dirichlet boundary conditions,
- 2D Boussinesq model for the Bénard convection,
- 2D MHD equations and 2D magnetic Bénard problem in bounded domains.

For more details concerning the models mentioned in this section we refer to [9] and to the references therein. In all these cases a direct analysis based on results of interpolation of intersections ([28]) makes it possible to prove that $Dom(A^{1/4})$ is embedded into L_4 type spaces and thus (due to the considerations in [9]) the basic hypotheses in Condition **(B)** holds with $\mathcal{H} = Dom(A^{1/4})$. Thus we can apply Theorems 3.1 and 3.2 assuming the additional properties (7.2), (7.4) and (7.5) concerning R, G, σ and $\tilde{\sigma}$.

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