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Fragmentation Energy

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Abstract

Motivated by a problem arising in mining industry, we estimate the energy $\mathcal{E}(\eta)$ which is needed to reduce a unit mass to fragments of size at most η in a fragmentation process, when $\eta \to 0$. We assume that the energy used by the instantaneous dislocation of a block of size s into a set of fragments $(s_1, s_2, ...)$, is $s^\beta \varphi(s_1/s, s_2/s, ..)$, where φ is some cost-function and β a positive parameter. Roughly, our main result shows that if $\alpha > 0$ is the Malthusian parameter of an underlying CMJ branching process (in fact $\alpha = 1$ when the fragmentation is mass-conservative), then $\mathcal{E}(\eta) \sim c\eta^{\beta-\alpha}$ whenever $\beta < \alpha$. We also obtain a limit theorem for the empirical distribution of fragments with size less than η which result from the process. In the discrete setting, the approach relies on results of Nerman for general branching processes; the continuous setting follows by considering discrete skeletons. We also provide a direct approach to the continuous setting which circumvents restrictions induced by the discretization.

1 Introduction

One of the main goals in mining industry is to liberate the metal contained into mineral blocks and then separate it from non valued content. In this direction, fragmentation is carried out in a series of steps: the first one being blasting, after the material is transported to crushers, grinders, and further, to mills. Now, at each step the particles are screened, so, if they are smaller than the diameter of the mesh of a grid, they are forwarded to the next step. The process finishes when the material attains a size sufficiently small for the mining purposes (more precisely to enable efficient physical-chemical processing). In crushers, grinders and mills the material is broken by a repetitive mechanism, finishing when the particles can go across the classifying-grid. In the intermediate steps, that is blasting and crushers, output sizes are known to be not optimal in terms of the global

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energy cost. Obviously, one of the problems that faces mining industry is to minimize the energy used in these processes.

To get an idea of the magnitudes involved in fragmentation in the mining industry, at Chuquicamata (the biggest open pit copper mine on earth) located at the north of Chile, one blasts and transports nearly 600 thousands of tons of material per day. Out of this, nearly 200 thousands tons are defined as mineral, the rest is waste or sterile. The average grade of copper content of mineral is around 1 per cent. After blasting, there are two circuits for mechanically decrease the mineral size, the conventional process uses three types of crushers (primary, secondary and tertiary) and the particles are forward to the mills once they are smaller than 1/2 inches. The mills are of two types: bars and balls which are the steel breakage media. The fragmentation process is stopped once the particles have a size smaller than $20\mu m$ and in the other, the SAG mills are fed with particles of size smaller than 8 inches with a similar end product of $20\mu m$.

In this work, we consider discrete and continuous models of fragmentation processes in which particles break independently of each other, and in a self-similar way (see the forthcoming Sections 2 and 3 for precise assumptions). The self-similarity hypothesis agrees with observations made by the mining industry; see e.g. [8, 11, 12, 15]. We assume that the energy required to break a block of size s into a set of smaller blocks of sizes $(s_1, s_2, ...)$ (in one unit of time in the discrete case, and instantaneously in the continuous case), is of the form $s^\beta \varphi(s_1/s, s_2/s, \ldots)$, where φ is a so-called *cost function* and $\beta > 0$ a fixed parameter. To keep generality, we do not impose φ to be nonnegative, although this would be physically relevant.

When $\varphi(s_1, s_2, \ldots) = \sum_{n=1}^{\infty} s_n^\beta - 1$, the energy [1] is called a *potential*, because the total energy needed to break a series of masses $(m_1, m_2, \ldots)$ summing up to 1 into a finer distribution $(m_1', m_2', \ldots)$ is given by $\sum_i m_i'^\beta - \sum_j m_j^\beta$, and this is independent of the fragmentation path leading from the initial configuration to the final one. In this case, if $\{x_i\}$ is the set of the sizes of fragments obtained by removing from the process every fragment at the time when it becomes smaller than certain bound η, the energy needed to attain this state starting from the unit mass is $\mathcal{E}(\eta) = \sum_i x_i^\beta - 1$. We mention that the potential energy is one of the most used models in mining. This corresponds to the laws of Charles, Walker and Bond: $dE(r) = C\frac{dr}{r^\gamma}$, where $dE(r)$ is the specific energy to change the size of a particle from r to $r - dr$; cf. [6, 7, 14].

In Section 2, we will deal with the discrete setting. Our approach to the study of the energy problem relies on a coding of the fragmentation chain by a general (i.e. Crump-Mode-Jagers) branching process in which the time parameter corresponds to the logarithm of the sizes of the fragments. This coding enables us to apply theorems of Nerman [10] from which estimates for the energy follow. The same technique also yields interesting limit theorems for the empirical distribution of the fragments with size less than η resulting from the process and suitably normalized. In Section 3, the continuous setting is treated first by considering a discrete skeleton and estimating errors, and second by a direct approach based on moment estimates via renewal theory for conservative fragmentations that circumvents some restrictions on the exponent β which are imposed by the discretization technique. Finally, we briefly derive conclusions of our study in Section 4.

[1] Observe that this quantity is always nonnegative whenever $\beta < 1$ and $\sum_{n=1}^{\infty} s_n = 1$.

In the sequel, we shall implicitly assume that the process starts from a single block with unit size, except when mentioned explicitly else.

2 Discrete framework

In this section, we consider the following simple setting where the system has a discrete genealogical structure. Specifically, let us denote by

$$\mathcal{I} := \bigcup_{n=0,1,\ldots} \mathbb{N}^n$$

the infinite regular tree, with the usual convention that $\mathbb{N}^0 = \{\emptyset\}$. In the sequel $\mathcal{I}$ will often be referred to as the *genealogical tree*; its elements are called *individuals*. For each $i = (i_1, \ldots, i_n) \in \mathcal{I}$, we call n the *generation* of i and write $|i| = n$, with the obvious convention $|\emptyset| = 0$.

Individuals are used to label the fragments produced by a fragmentation chain; more precisely we mark each individual $i \in \mathcal{I}$ by a real number σ_i, where $\mathrm{e}^{-\sigma_i}$ is the size of the fragment with label i. In other words, the marks assigned to the individuals are given by minus the logarithm of the size of the corresponding fragments. More precisely, recall that we start from a single block of unit size, and denote the sequence of the sizes of the fragments resulting from the dislocation of this block by $(s_1, s_2, \ldots)$. We then assign the mark $\sigma_\emptyset = 0$ to the ancestor and $\sigma_i = -\log s_i$ to each individual $i \in \mathbb{N}^1$ of the first generation. Marks are assigned similarly for the next generations.

Let us now specify the dynamics. We assume that each fragment with size $s > 0$ breaks into a sequence of smaller fragments with sizes $s_1, s_2, \ldots$, where the sequence of ratios $s_1/s, s_2/s, \ldots$ has a fixed distribution ν (i.e. ν does not depend on the size s). So ν is a probability measure on $[0,1]^{\mathbb{N}}$ which will be referred to as the *dislocation law*. We shall always implicitly assume that the dislocation law is not geometric, in the sense that there is no real number $r \in]0,1[$ such that $s_n \in \{1, r, r^2, \ldots\}$ for all n, a.s.

The fragmentation is called *dissipative* if $\sum_{n=1}^{\infty} s_n \leq 1$ ν-a.s. and the inequality is strict with positive probability (this occurs for instance when size means mass and some dust, i.e. infinitesimal fragments, can be produced). It is called *conservative* if $\sum_{n=1}^{\infty} s_n = 1$ ν-a.s. Note that the model also includes the case when $\sum_{n=1}^{\infty} s_n \geq 1$ with positive probability, which occurs for instance when the size of a fragment means its diameter.

Finally, we assume that different fragments evolve independently. In other words, the measure-valued process

$$Z^n(dx) \;=\; \sum_{i \in \mathbb{N}^n} \delta_{\sigma_i}(dx)\,, \qquad x \in [0, \infty[$$

is a *branching random walk* in the sense of Biggins [5], whose reproduction intensity is given in terms of the dislocation law by

$$\mathbb{E}\left(\langle Z^1, f\rangle\right) \;=\; \int_{[0,1]^{\mathbb{N}}} \sum_{n=1}^{\infty} f(-\log s_n)\nu(d\mathbf{s})\,,$$

where $\mathbf{s} = (s_n, n \in \mathbb{N})$ denotes a generic sequence in $[0,1]$ and $f : [0,\infty[\to \mathbb{R}$ a generic bounded measurable function. We stress that the random point measure Z^n does not take into account the fragments with size zero (i.e. for which $\sigma_i = \infty$).

Throughout this section, we shall assume that the *Malthusian hypothesis* is fulfilled, that is there exists $\alpha > 0$, called the *Malthusian parameter*, such that

$$\mathbb{E}\left(\langle Z^1, \mathrm{e}^{-\alpha\cdot}\rangle\right) \;=\; \int_{[0,1]^{\mathbb{N}}} \left(\sum_{n=1}^{\infty} s_n^{\alpha}\right) \nu(d\mathbf{s}) \;=\; 1\,,$$

and that the $L \log L$ condition holds, namely

$$\mathbb{E}\left(\langle Z^1, \mathrm{e}^{-\alpha\cdot}\rangle \log^+\langle Z^1, \mathrm{e}^{-\alpha\cdot}\rangle\right) \;=\; \int_{[0,1]^{\mathbb{N}}} \left(\sum_{n=1}^{\infty} s_n^{\alpha}\right) \log^+ \left(\sum_{n=1}^{\infty} s_n^{\alpha}\right) \nu(d\mathbf{s}) \;<\; \infty\,.$$

Then, it is well-known (see Biggins [5]) that the process

$$W_n \;:=\; \langle Z^n, \mathrm{e}^{-\alpha\cdot}\rangle \;=\; \sum_{i\in\mathbb{N}^n} \mathrm{e}^{-\alpha\sigma_i}$$

is a uniformly integrable martingale, which converges a.s. and in $L^1(\mathbb{P})$ to W_∞, and that $W_\infty > 0$ a.s. conditionally non-extinction (i.e. on $Z^n \neq 0$ for all n). Observe that in the conservative case, the Malthusian parameter is $\alpha = 1$, and the martingale W is constant.

Recall now from the Introduction the problem which motivates this work. Fix $\eta > 0$ and let the process evolve until every fragment in the system has size less than η, in the sense that each fragment with size less than η is instantaneously frozen in the system. Suppose that at each step, the energy cost of the dislocation of a block with size s producing a sequence of fragments with sizes $s_1, s_2, \ldots$ is $s^{\beta}\varphi(s_1/s, s_2/s, \ldots)$, where β is a positive parameter and $\varphi : [0,1]^{\mathbb{N}} \to \mathbb{R}$ a measurable symmetric function, called the *cost function*. The total energy cost for the process stopped when all the fragments have size less than η is thus given by

$$\mathcal{E}(\eta) \;:=\; \sum_{i\in\mathcal{I}} \mathbf{1}_{\{\sigma_i \le -\log\eta\}} \mathrm{e}^{-\beta\sigma_i} \varphi(\Delta_i)\,,$$

where $\Delta_i = (\exp(\sigma_i - \sigma_{i,n}), n \in \mathbb{N})$ is the sequence of ratios of the sizes of the fragments resulting from the dislocation of the fragment labelled by the individual i and the size of that fragment.

We are now able to state the main result in the discrete setting.

Theorem 1 *Suppose $\beta < \alpha$ and $\int_{[0,1]^{\mathbb{N}}} |\varphi(\mathbf{s})|\nu(d\mathbf{s}) < \infty$, and set*

$$m(\alpha) \;=\; \int_{[0,1]^{\mathbb{N}}} \left(\sum_{n=1}^{\infty} s_n^{\alpha} \log 1/s_n\right) \nu(d\mathbf{s})\,.$$

Then under the preceding assumptions,

$$\lim_{\eta\to 0} \eta^{\alpha-\beta}\mathcal{E}(\eta) = \frac{W_\infty}{(\alpha-\beta)m(\alpha)} \int_{[0,1]^{\mathbb{N}}} \varphi(\mathbf{s})\nu(d\mathbf{s}) \qquad \textit{in } L^1(\mathbb{P})\,.$$

Proof: The idea of the proof consists in expressing the energy functional in terms of a general (i.e. Crump-Mode-Jagers) branching process which arises naturally in this setting, and applying to the latter convergence results of Nerman [10].

We think of the mark σ_i which has been assigned to each individual i of the genealogical tree $\mathcal{I}$ as a birth-time. Every individual which is born has an infinite lifetime, and gives birth to children according to a random point process ξ_i. More precisely, if $s_1 \geq s_2 \geq \ldots$ denote the sequence ranked in the decreasing order of the sizes of the fragments that result from the first dislocation of a block with size s (recall that we assume that $s \geq s_1$, i.e. dislocations always produce fragments of smaller sizes), the nth child of that individual is born at time $-\log s_n$.

The particle system which is obtained in this way is then a general branching process with reproduction intensity

$$\mu(t) \;=\; \mathbb{E}(\xi_\emptyset([0,t])) \;=\; \int_{[0,1]^{\mathbb{N}}} \sum_{n=1}^{\infty} \mathbf{1}_{\{s_n \geq \mathrm{e}^{-t}\}} \nu(d\mathbf{s})\,, \qquad t \geq 0.$$

Note that if we index this particle system by generations instead of time, we plainly recover the branching random walk Z^n. In this setting, the martingale W_n coincides with the so-called intrinsic martingale; and if we introduce

$$Y_t \;:=\; \sum_{i \in \mathcal{I}(t)} \mathrm{e}^{-\alpha\sigma_i}\,, \qquad t \geq 0\,,$$

where $\mathcal{I}(t)$ denotes the stopping line composed of the individuals born after time t, whose mothers are born before or at t, then $(Y_t, t \geq 0)$ is a uniformly integrable martingale which has terminal value W_∞. See Jagers [9].

Next, consider a block with size s, and let $(s_1, \ldots)$ be the ranked sequence of the sizes of the fragments that result from its first dislocation. This corresponds to an individual i born at time $\sigma_i = -\log s$ and which has offspring described by the point process

$$\xi_i([0,t]) \;=\; \#\left\{n \in \mathbb{N} : s_n \geq \mathrm{e}^{-t}\right\}\,.$$

For every $t \geq 0$, we set

$$\phi_i(t) \;=\; \mathrm{e}^{\beta t}\varphi(s_1/s, s_2/s, \ldots)\,,$$

and $\phi_i(t) = 0$ for $t < 0$. Note that the energy cost of the dislocation is $\mathrm{e}^{-\beta\sigma_i}\phi_i(0)$, and hence

$$\eta^{-\beta}\mathcal{E}(\eta) \;=\; \sum_{i \in \mathcal{I}} \phi_i(t - \sigma_i)\,, \qquad t = -\log\eta.$$

According to Theorem 3.1 and Corollary 3.3 in Nerman [10], we have

$$\lim_{t\to\infty} \mathrm{e}^{-\alpha t} \sum_{i \in \mathcal{I}} \phi_i(t - \sigma_i) \;=\; m_\infty^\phi\, W_\infty\,,$$

with

$$m_\infty^\phi \;=\; \frac{\int_0^\infty \mathrm{e}^{-\alpha t}\mathbb{E}(\phi_\emptyset(t))dt}{\int_0^\infty t\mathrm{e}^{-\alpha t}\mu(dt)} \qquad \text{in } L^1(\mathbb{P}),$$

see Equation (2.7) in Proposition 2.2 of [10]. Immediate calculations yield

$$m_\infty^\phi \;=\; \frac{\int_{[0,1]^{\mathbb{N}}} \varphi(\mathbf{s})\nu(d\mathbf{s})}{(\alpha-\beta)m(\alpha)}\,.$$

Putting the pieces together, this completes the proof. □

Remarks. (a) In the case $\beta > \alpha$, the total energy needed to reduce entirely a block to dust is $\mathcal{E}(0+) < \infty$ a.s., and thus the asymptotics of $\mathcal{E}(\eta)$ are trivial.

(b) Under a rather mild condition on the dislocation law, one can reinforce Theorem 1 and get almost-sure convergence. See Condition 5.1 and Theorem 5.4 in [10].

(c) In a general branching process, an individual i may be endowed with a so-called *life career* ω_i which contains in particular the information of the ages at which the individual begets its children, but may also have a much richer structure (the crucial hypothesis is that the life careers of different individuals are i.i.d.). Theorem 3.1 of Nerman [10] that provides the key to the present Theorem 1, can be extended to this more general setting; see Theorem 7.3 in Jagers [9]. This allows us to extend Theorem 1 to the more general situation where the energy cost of fragmentation during one unit of time does not only depend on the size of the initial block and that of the resulting fragments, but more generally on further information given by the life career ω_i of the corresponding individual i. In other words, the energy used for a block of size m with life career ω_i is $m^\beta\varphi(\omega_i)$, where the cost-function φ now depends on the whole life career. Then Theorem 1 still holds after replacing $\int_{[0,1]^{\mathbb{N}}} \varphi(\mathbf{s})\nu(d\mathbf{s})$ by $\mathbb{E}(\varphi(\omega_\emptyset))$. This immediate generalization will play a crucial role in the next section where we shall tackle the continuous framework by discretization.

Theorem 1 yields interesting asymptotics for the empirical distribution of the fragments resulting from the process stopped at the time when they become smaller than η. More precisely, consider the random finite measure ρ_η defined by

$$\langle \rho_\eta, f\rangle := \sum_j x_{\eta,j}^\alpha f(x_{\eta,j}/\eta)\,,$$

where $f : [0,1] \to \mathbb{R}_+$ denotes a generic measurable function and $\{x_{\eta,j}\}_{j\in\mathbb{N}}$ stands for the set of fragments with size less than η that result from the stopped process.

Corollary 2 *Assumptions are the same as in Theorem 1. Then for every continuous function $f : [0,1] \to \mathbb{R}_+$*

$$\lim_{\eta\to 0}\langle \rho_\eta, f\rangle = W_\infty\langle \rho, f\rangle \qquad \text{in } L^1(\mathbb{P})\,,$$

where ρ is the deterministic probability measure given by

$$\langle \rho, f\rangle = \frac{1}{m(\alpha)}\int_0^1 f(u)\left(\int_{[0,1]^{\mathbb{N}}}\sum_{n=1}^\infty \mathbf{1}_{\{u>s_n\}}s_n^\alpha\nu(d\mathbf{s})\right)\frac{du}{u}.$$

Proof: The argument is a variation of that in the proof of Theorem 1. We start from the identity

$$\langle \rho_\eta, f\rangle = \sum_{i\in\mathcal{I}} \sum_{n=1}^{\infty} \mathbf{1}_{\{\sigma_i \le t < \sigma_{i,n}\}} \mathrm{e}^{-\alpha\sigma_{i,n}} f(\mathrm{e}^{t-\sigma_{i,n}}),$$

where $\eta = \mathrm{e}^{-t}$ and the notation $\sigma_{i,n}$ refers to the birth-time of the nth child of the individual i.

This incites us to introduce the process $\phi_i(t) = 0$ for $t < 0$ and

$$\phi_i(t) = \sum_{n=1}^{\infty} \mathbf{1}_{\{\sigma_{i,n}-\sigma_i > t\}} \mathrm{e}^{\alpha(t-\sigma_{i,n}+\sigma_i)} f(\mathrm{e}^{t-\sigma_{i,n}+\sigma_i}) \qquad \text{for } t \ge 0,$$

so that

$$\langle \rho_\eta, f\rangle = \mathrm{e}^{-\alpha t} \sum_{i\in\mathcal{I}} \phi_i(t-\sigma_i).$$

Then an easy calculation gives

$$\int_0^\infty \mathrm{e}^{-\alpha t}\mathbb{E}(\phi_\emptyset(t))dt = \int_0^1 f(u) \left(\int_{[0,1]^{\mathbb{N}}} \sum_{n=1}^{\infty} \mathbf{1}_{\{u>s_n\}} s_n^\alpha \nu(d\mathbf{s}) \right) \frac{du}{u},$$

and the rest of the proof mimics that of Theorem 1. □

3 Continuous framework

We now turn our attention to the continuous setting, which is probably more appropriate to model the evolution inside mills or crushers. Indeed, it has been observed in mining industry that a significant proportion of particles that passed through the classifying-grid after screening, have in fact a size much smaller than the mesh of the grid. This suggest that blocks of mineral might be subject to a very large number of very small dislocations, i.e. which affect only a very small portion of the block.

3.1 Preliminaries

The stochastic model that we use is the so-called *homogeneous fragmentations* introduced in [1, 2]. That is we consider a Markov process $X(t) = (X_1(t), X_2(t), \ldots)$ with values in the infinite simplex of decreasing numerical sequences with sum bounded from above by 1,

$$\mathcal{S} = \left\{ \mathbf{s} = (s_1, \ldots) : s_1 \ge s_2 \ge \cdots \ge 0 \quad \text{and} \quad \sum_1^\infty s_i \le 1 \right\}.$$

We endow $\mathcal{S}$ with the uniform distance; it is readily checked that this turns $\mathcal{S}$ into a compact metric space.

A configuration $\mathbf{s} \in \mathcal{S}$ will be called a *mass-partition*; it should be thought of as the ranked family of masses arising from the split of a block with unit mass. Note that

for a technical reason, no creation of mass is allowed, i.e. the fragmentation is always conservative or dissipative. Specifically, a portion of the initial mass may be lost during the split, which corresponds to the situation when $\sum_1^\infty s_n < 1$. Small splits correspond to mass-partitions $\mathbf{s} = (s_1, \ldots)$ such that $1 - s_1$ is small; in other words, a small split produces one large fragment and all the remaining ones are small. In this direction, the mass-partition $(1, 0, \ldots)$ has to be viewed as a zero element in $\mathcal{S}$, since it corresponds to a trivial split. It will be convenient to introduce the notation $\mathcal{S}^* = \mathcal{S}\backslash\{(1, 0, \ldots)\}$ for the subspace of non-trivial mass partitions.

We assume that the process fulfills the branching property, i.e. different fragments have independent evolutions. Specifically, let $\mathbb{P}_r$ stand for the law of X started from $(r, 0, \ldots)$, i.e. at the initial time, there is a single block with mass r. Then for every $t, u \geq 0$, conditionally on $X(t) = (s_1, \ldots)$, $X(t + u)$ has the same law as the variable obtained by ranking in the decreasing order the terms of the sequences $X^{(1)}(u), X^{(2)}(u), \ldots$, where the latter are independent random mass-partitions with values in $\mathcal{S}$, such that $X^{(n)}(u)$ has the same distribution as $X(u)$ under $\mathbb{P}_{s_n}$ for each $n = 1, \ldots$. We further suppose that the process is homogeneous, in the sense that for every $r > 0$, the distribution of $(rX(t), t \geq 0)$ under $\mathbb{P}_1$ is $\mathbb{P}_r$.

It is known that the dynamics of homogeneous fragmentations are characterized by a coefficient of erosion $c \geq 0$ and a so-called *dislocation measure* ν on $\mathcal{S}$ such that $\nu(\{(1, 0, \ldots)\}) = 0$ and

$$\int_{\mathcal{S}} (1 - s_1)\nu(d\mathbf{s}) < \infty\,.$$

Erosion is a deterministic phenomenon (roughly, blocks melt continuously at rate c), and for the sake of simplicity we shall assume in the sequel that $c = 0$. The dislocation measure specifies the rate at which blocks split: informally a block with mass m dislocates into a mass-partition $m\mathbf{s}$ for $\mathbf{s} \in \mathcal{S}^*$ at rate $\nu(d\mathbf{s})$.

When the mass of ν is finite, the total rate of dislocation of a block is thus $\nu(\mathcal{S}) < \infty$, which means that the block remains unchanged for an exponential time with parameter $\nu(\mathcal{S})$ and then splits. In this situation, a logarithmic change of variables transforms the homogeneous fragmentation into a branching random walk in continuous time (see Uchiyama [13]), and the analysis made in the discrete setting can be applied verbatim. The case when $\nu(\mathcal{S}) = \infty$ is thus more interesting as blocks dislocate immediately: informally, on any time-interval, each block is subject to an infinite number of dislocations, all but finitely many are very small.

The cornerstone of the study of homogeneous fragmentation is its Poissonian structure, cf. [1, 2]. Specifically, consider a Poisson point process $((\Delta(t), k(t)), t \geq 0)$ with values in $\mathcal{S} \times \mathbb{N}$, with characteristic measure $\nu \otimes \#$, where $\#$ denotes the counting measure on $\mathbb{N} = \{1, 2, \ldots\}$. This means that for every Borel set $B \subset \mathcal{S}^* \times \mathbb{N}$, the process which counts (as a function of time) the number of instants t such that $(\Delta(t), k(t)) \in B$, is a Poisson process with intensity $\nu\otimes\#(B)$, and to disjoint Borel sets correspond independent counting processes. There exists a unique pure-jump process $(X(t), t \geq 0)$ with values in $\mathcal{S}$, which jumps only at times $t \geq 0$ at which an atom $(\Delta(t), k(t))$ occurs in $\mathcal{S}^* \times \mathbb{N}$. More precisely, the jump (i.e. the dislocation) induced by such a point can be described as follows. The sequence $X(t)$ is obtained from $X(t-)$ by replacing its $k(t)$-th term $X_{k(t)}(t-)$ by the sequence $X_{k(t)}(t-)\Delta(t)$, and ranking all the terms in the decreasing order. For

instance, if $X(t-) = \left(\frac{2}{3}, \frac{1}{4}, \frac{1}{12}, 0, \ldots\right)$, $k(t) = 2$ and $\Delta(t) = \left(\frac{3}{4}, \frac{1}{4}, 0, \ldots\right)$, then we look at the 2-nd largest term in the sequence $X(t-)$, which is $\frac{1}{4}$, and split it according to $\Delta(t)$. This produces two fragments of size $\frac{3}{16}$ and $\frac{1}{16}$, and thus $X(t) = \left(\frac{2}{3}, \frac{3}{16}, \frac{1}{12}, \frac{1}{16}, 0, \ldots\right)$. Of course, it may happen that $X_{k(t)}(t-) = 0$, and in that case we have $X(t) = X(t-)$. The process $(X(t), t \geq 0)$ is then a homogeneous fragmentation with no erosion and dislocation measure ν.

It will be convenient to introduce the critical value

$$\underline{\beta} := \inf\left\{\beta > 0 : \int_{\mathcal{S}} \left|1 - \sum_{n=1}^{\infty} s_n^{\beta}\right| \nu(d\mathbf{s}) < \infty\right\},$$

and then the function $\kappa :]\underline{\beta}, \infty[\to \mathbb{R}$ given by

$$\kappa(\beta) := \int_{\mathcal{S}} \left(1 - \sum_{n=1}^{\infty} s_n^{\beta}\right) \nu(d\mathbf{s}).$$

Plainly, κ is a concave increasing function.

We shall often make use of the fact that for every $\beta > \underline{\beta}$, the process

$$\mathrm{e}^{t\kappa(\beta)} \sum_{n=1}^{\infty} X_n^{\beta}(t), \qquad t \geq 0,$$

is a martingale. See e.g. Section 2.3 in [4]; we point that in the notation of [4], $\underline{\beta} = \underline{p} + 1$ and $\kappa(\beta) = \Phi(\beta + 1)$. This process will be referred to as the *β-martingale* in the sequel.

In the continuous setting, the Malthusian hypothesis takes the following form:

$$\text{there exists } \alpha > \underline{\beta} \text{ such that } \kappa(\alpha) = 0.$$

Then the process $W_t := \sum_{n=1}^{\infty} X_n^{\alpha}(t)$ is martingale which is always uniformly integrable (see the proof of Theorem 2 in [4] for the argument). We call $(W_t, t \geq 0)$ the *intrinsic* martingale and write W_∞ for its terminal value.

Set

$$m(\alpha) := \kappa'(\alpha) \;=\; \int_{\mathcal{S}} \left(\sum_{n=1}^{\infty} s_n^{\alpha} \log 1/s_n\right) \nu(d\mathbf{s}).$$

We shall need the following lemma.

Lemma 3 *For every $t > 0$, we have*

$$\mathbb{E}\left(\sum_{n=1}^{\infty} X_n^{\alpha}(t) \log 1/X_n(t)\right) \;=\; t m(\alpha).$$

Proof: Taking the derivative in the variable β of the β-martingale yields the so-called derivative martingale

$$t\kappa'(\beta)\mathrm{e}^{t\kappa(\beta)} \sum_{n=1}^{\infty} X_n^{\beta}(t) - \mathrm{e}^{t\kappa(\beta)} \sum_{n=1}^{\infty} X_n^{\beta}(t) \log 1/X_n(t)$$

(the martingale property can also be checked directly from the Markov property). Computing expectation gives

$$\mathrm{e}^{t\kappa(\beta)}\mathbb{E}\left(\sum_{n=1}^{\infty} X_n^{\beta}(t)\log 1/X_n(t)\right) \;=\; t\kappa'(\beta)\,.$$

We specify this identity when $\beta = \alpha$ is the Malthusian parameter (so $\kappa(\alpha) = 0$ and $\kappa'(\alpha) = m(\alpha)$) and get the formula of the statement. □

3.2 Discretization

Our goal in this section is to derive by discretization an extension of Theorem 1 for homogeneous fragmentations. Specifically, fix $a > 0$ and consider the discrete skeleton $(X(na), n \in \mathbb{N})$, which is then a fragmentation chain of the type considered in Section 2. More precisely, for each individual (i.e. fragment observed at some time na), we record not only its size and that of its children, but the entire life career ω which governs the evolution of this block on the time interval $[na, (n+1)a]$. In this direction, recall the Poissonian construction of the homogeneous fragmentation presented at the beginning of this section. The life career of an individual i that corresponds to a block with size $\mathrm{e}^{-\sigma_i}$, say at time na, is denoted by $\omega_i = (\delta(t), \ell(t))_{0<t\le a}$, where $\delta(t) \in \mathcal{S}$ and $\ell(t) \in \mathbb{N}$. This means that for every $t \in]0, a]$, at time $na + t$, the $\ell(t)$th largest fragment resulting from this block splits in such a way that the ordered sequence of the resulting size-ratios is $\delta(t)$ (of course, there is no split when $\delta(t) = (1, 0, \ldots)$).

To define the energy used by an instantaneous dislocation, we consider a measurable cost-function $\varphi : \mathcal{S} \to \mathbb{R}$ with $\varphi((1, 0, \ldots)) = 0$. In terms of the discrete skeleton, this leads us to introduce a cost-function which depends on the life career of individuals. Specifically, the energy cost for the fragmentation of a block at time na during a units of time is thus

$$\mathrm{e}^{-\beta\sigma_i}\varphi_a(\omega_i)\,,$$

where i stands for the individual corresponding to that block, ω_i for its life career,

$$\varphi_a(\omega_i) \;=\; \sum_{0<t\le a} s^{\beta}_{\ell(t)}(t-)\varphi(\delta(t))\,,$$

and $s_n(t)\mathrm{e}^{-\sigma_i}$ is the size of the nth largest fragment obtained from that block at time $na + t$. In the special case of a potential energy, i.e. when $\varphi(\mathbf{s}) = \sum_{n=1}^{\infty} s_n^{\beta} - 1$, $\varphi_a(\omega_i)$ only depends on the sizes of the children of the individual i, but in general $\varphi_a(\omega_i)$ depends on the whole life career.

We start by computing the mean of energy for the ancestor $\emptyset$, i.e. the energy spent for fragmenting the initial block with unit size during the time interval $[0, a]$:

$$\varphi_a(\omega_\emptyset) = \sum_{0<t\le a} X^{\beta}_{k(t)}(t-)\varphi(\Delta(t))\,.$$

Lemma 4 *Suppose $\beta > \underline{\beta}$. For every $a > 0$, we have*

$$\mathbb{E}(\varphi_a(\omega_\emptyset)) \;=\; \frac{1-\mathrm{e}^{-a\kappa(\beta)}}{\kappa(\beta)}\int_{\mathcal{S}}\varphi(\mathbf{s})\nu(d\mathbf{s})\,.$$

Proof: The compensation formula for the Poisson point process $(\Delta(u), k(u))_{u\geq 0}$ gives

$$\mathbb{E}\left(\sum_{0<t\leq a} X^{\beta}_{k(t)}(t-)\varphi(\Delta(t))\right) = \mathbb{E}\left(\int_0^a \left(\sum_{n=1}^{\infty} X^{\beta}_n(t)\right) dt\right) \int_{\mathcal{S}} \varphi(\mathbf{s})\nu(d\mathbf{s})\,.$$

Now the mean of the β-martingale is 1, so $\mathbb{E}(\sum_{n=1}^{\infty} X^{\beta}_n(t)) = \mathrm{e}^{-t\kappa(\beta)}$, and the claim follows. □

Next, we fix $\eta > 0$ and write $\mathcal{E}(\eta)$ for the energy spent to reduce a block with unit size to fragments with size less than ε, using the cost function φ. That is

$$\mathcal{E}(\eta) \;=\; \sum_{t\geq 0} \mathbf{1}_{\{X_{k(t)}(t-)\geq\eta\}} X^{\beta}_{k(t)}(t-)\varphi(\Delta(t))\,.$$

We are now able to state the following extension of Theorem 1 to the continuous setting.

Theorem 5 *Suppose $\int_{\mathcal{S}} |\varphi(\mathbf{s})|\nu(d\mathbf{s}) < \infty$ and $\beta \in]\underline{\beta}, \alpha[$. We have that*

$$\lim_{\eta\to 0} \eta^{\alpha-\beta}\mathcal{E}(\eta) \;=\; \frac{W_\infty}{(\alpha-\beta)m(\alpha)} \int_{\mathcal{S}} \varphi(\mathbf{s})\nu(d\mathbf{s}) \qquad \textit{in } L^1(\mathbb{P})\,.$$

Proof: Without loss of generality, we may assume $\varphi \geq 0$. We write $\mathcal{E}_a(\eta)$ for the energy computed for the discrete skeleton and the cost-function φ_a, that is

$$\mathcal{E}_a(\eta) \;=\; \sum_{i\in\mathcal{I}} \mathbf{1}_{\{\sigma_i\leq -\log\eta\}} \mathrm{e}^{-\beta\sigma_i}\varphi_a(\omega_i)\,.$$

Plainly, $\mathcal{E}_a(\eta)$ is an upperbound for $\mathcal{E}(\eta)$ since the former quantity corresponds to the energy spent for the homogeneous fragmentation process when fragments with size less than η are frozen at *discrete times* of the form na for $n \in \mathbb{N}$, instead of instantaneously.

We need to estimate the overcost $\mathcal{E}_a(\eta) - \mathcal{E}(\eta)$, that is the error induced by the discretization. Clearly, if $\{x_n\}_{n\in\mathbb{N}}$ denotes the terminal state of the process when we freeze each fragment at the instant when it size becomes less than η in the fragmentation process, the overcost can be bounded from above by

$$\sum_{n=1}^{\infty} \mathcal{E}(\eta, x_n, a)\,,$$

where $\mathcal{E}(\eta, x, a)$ the energy cost for an initial block with size x and a duration a in the fragmentation process. By self-similarity and Lemma 4, the mean overcost can thus be bounded from above by

$$\begin{aligned}\mathbb{E}\left(\sum_{n=1}^{\infty} \mathcal{E}(\eta, x_n, a)\right) &= \frac{1-\mathrm{e}^{-a\kappa(\beta)}}{\kappa(\beta)} \left(\int_{\mathcal{S}} \varphi(\mathbf{s})\nu(d\mathbf{s})\right) \mathbb{E}\left(\sum_{n=1}^{\infty} x_n^{\beta}\right) \\ &\leq \frac{1-\mathrm{e}^{-a\kappa(\beta)}}{\kappa(\beta)} \left(\int_{\mathcal{S}} \varphi(\mathbf{s})\nu(d\mathbf{s})\right) \eta^{\beta-\alpha}\mathbb{E}\left(\sum_{n=1}^{\infty} x_n^{\alpha}\right).\end{aligned}$$

Now the operation of freezing fragments at the instant when they become less than η is a so-called *frost* in the terminology of [3]; this notion is the continuous analog of that of *stopping lines* in the theory of branching processes. The branching property extends to frosts (see Theorem 1 in [3]), which entails in particular that the terminal value W_∞ of the intrinsic martingale can be expressed in the form

$$W_\infty \,=\, \sum_{n=1}^{\infty} x_n^\alpha W_\infty^{(n)}\,,$$

where $(W_\infty^{(n)}, n \in \mathbb{N})$ is a sequence of i.i.d. copies of W_∞, which is independent of the x_n's. Taking expectation, we deduce that $\mathbb{E}\,(\sum_{n=1}^{\infty} x_n^\alpha) = 1$.

Putting the pieces together, we have shown that

$$\lim_{a\to 0} \eta^{\alpha-\beta}\mathbb{E}\,(|\mathcal{E}(\eta) - \mathcal{E}_a(\eta)|) \,=\, 0\,, \quad \text{uniformly in } \eta\,.$$

On the other hand, the Malthusian hypothesis and the $L\log L$ condition are plainly fulfilled for the discrete skeleton, which allows us to apply the results of the preceding section (see Remark (c) after the proof of Theorem 1). Thanks to Lemmas 3 and 4, we get

$$\lim_{\eta\to 0} \eta^{\alpha-\beta}\mathcal{E}_a(\eta) \,=\, \frac{W_\infty(1-\mathrm{e}^{-a\kappa(\beta)})}{(\alpha-\beta)\kappa(\beta)am(\alpha)} \int_{\mathcal{S}} \varphi(\mathbf{s})\nu(d\mathbf{s}) \qquad \text{in } L^1(\mathbb{P})\,.$$

Since $\frac{1-\mathrm{e}^{-a\kappa(\beta)}}{a\kappa(\beta)}$ tends to 1 as $a \to 0$, this completes the proof of the theorem. □

3.3 The conservative case

If we compare the discrete and the continuous settings, we see that the hypotheses in Theorem 5 are more restrictive that one might expect, as one can only deal with parameters β greater than the critical value $\underline{\beta}$. This restriction stems from the approach by discretization: fragments are frozen at the first discrete time of the form na at which they become smaller than η and not instantaneously. This implies that the fragmentation process keeps running for some time for blocks with size smaller than η, and it turns out that the energy cost for this has a predominant contribution when $\beta \le \underline{\beta}$.

Our purpose in this section is to provide a direct method for the estimation of the energy which circumvent this problem. For the sake of simplicity, we shall assume here that the fragmentation is conservative, i.e.

$$\nu\left(\left\{\mathbf{s}\in\mathcal{S} : \sum_{n=1}^{\infty} s_n \neq 1\right\}\right) \,=\, 0\,.$$

In particular, the Malthusian parameter is $\alpha = 1$ and the intrinsic martingale is trivial. We claim the following:

Theorem 6 *Suppose that the homogeneous fragmentation is conservative, that the cost function φ belongs to $L^1(d\nu) \cap L^2(d\nu)$, and that $\beta < 1$. Then it holds that*

$$\lim_{\eta\to 0} \eta^{1-\beta}\mathcal{E}(\eta) \,=\, \frac{1}{(1-\beta)m(1)} \int_{\mathcal{S}} \varphi(\mathbf{s})\nu(d\mathbf{s}) \qquad \textit{in } L^2(\mathbb{P})\,.$$

Our approach is based on moment calculations, which are performed using so-called *tagged fragments* in the spirit of [4]. In this direction, we shall use an interval representation of the fragmentation, see [3]. That is we identify the initial block with the unit interval $]0,1[$ and work with a Markovian family $(\theta(t), t \geq 0)$ of nested random open subsets of $]0,1[$, in the sense that $\theta(u) \subseteq \theta(t)$ for $u \geq t$. The state of the fragmentation at time t is given by the ranked sequence of the lengths of the interval components of $\theta(t)$; note that the assumption that the fragmentation is conservative entails that $\theta(t)$ has full Lebesgue measure. Let U be a uniform random variable, independent of $(\theta(t), t \geq 0)$. For each $t \geq 0$, we denote by $\chi(t)$ the length of the interval component of $\theta(t)$ that contains U, and call $\chi = (\chi(t), t \geq 0)$ the process of the tagged fragment. The latter provides a powerful tool for computing first moment of additive functionals of the fragmentation; in particular the renewal theorem for subordinators yields the following estimate.

Lemma 7 *Under the same assumptions as in Theorem 6, one has*

$$\lim_{\eta \to 0} \eta^{1-\beta} \mathbb{E}(\mathcal{E}(\eta)) \;=\; \frac{1}{(1-\beta)m(1)} \int_{\mathcal{S}} \varphi(\mathbf{s}) \nu(d\mathbf{s})\,.$$

Proof: Introduce for every $\eta > 0$ the quantity

$$\tilde{\mathcal{E}}(\eta) \;=\; \sum_{t \geq 0} \chi^{\beta-1}(t-) \mathbf{1}_{\{\chi(t-) \geq \eta, k(t) = n(t-)\}} \varphi(\Delta(t))\,,$$

where $n(t)$ stands for the index at time t of the tagged fragment, i.e. $X_{n(t)}(t) = \chi(t)$. So the characteristic function in the sum is one if and only if the tagged fragment splits at time t and its size before the split is not smaller than η. Note that $\tilde{\mathcal{E}}(\eta)$ can be viewed as the $(\beta-1)$-energy required to reduce the tagged fragment alone to a size less than η. Because at each time $t \geq 0$, the tagged fragment is a size-biased picked from the fragments at time t, i.e. $\mathbb{P}(n(t) = \ell \mid X(t)) = X_\ell(t)$, we see that the conditional expectation of $\tilde{\mathcal{E}}(\eta)$ given the sigma-field generated by the sole fragmentation process (i.e. ignoring the uniform random variable U) is precisely $\mathcal{E}(\eta)$, and in particular

$$\mathbb{E}(\tilde{\mathcal{E}}(\eta)) \;=\; \mathbb{E}(\mathcal{E}(\eta))\,.$$

So we need to estimate

$$\mathbb{E}\left(\sum_{t \geq 0} \chi^{\beta-1}(t-) \mathbf{1}_{\{\chi(t-) \geq \eta, k(t) = n(t-)\}} \varphi(\Delta(t)) \right)\,,$$

and in this direction, we use the compensation formula for Poisson point processes. We get

$$\mathbb{E}(\mathcal{E}(\eta)) \;=\; \mathbb{E}\left(\int_0^\infty \chi^{\beta-1}(t) \mathbf{1}_{\{\chi(t) \geq \eta\}} dt \right) \left(\int_{\mathcal{S}} \varphi(\mathbf{s}) \nu(d\mathbf{s}) \right)\,.$$

Next, recall from Theorem 3 in [2] that the process $\xi(t) := -\log \chi(t)$ is a subordinator, that is an increasing process with stationary and independent increments. More precisely, its Laplace exponent

$$\Phi(q) \;=\; -\log \mathbb{E}\left(\exp(-q\xi(1))\right) \;=\; -\log \mathbb{E}\left(\chi(1)^q\right)$$

is given by $\Phi(q) = \kappa(q+1)$. So if $\mathcal{U}$ denotes the renewal measure of ξ, we have

$$\mathbb{E}\left(\int_0^\infty \chi^{\beta-1}(t)\mathbf{1}_{\{\chi(t)\geq\eta\}}dt\right) \;=\; \int_0^{\log 1/\eta} \exp((1-\beta)y)\mathcal{U}(dy)\,. \tag{1}$$

As the subordinator ξ has mean $\Phi'(0) = \kappa'(1) = m(1)$, it now follows from the renewal theorem for subordinators that the quantity above is equivalent to

$$\frac{\eta^{\beta-1}}{m(1)}\int_0^\infty \mathrm{e}^{-(1-\beta)y}dy$$

as $\eta \to 0$. This establishes our claim. □

Now, to complete the proof of Theorem 6, we need second moment estimates. In this direction, the technique of the proof of Lemma 7 lead us to introduce a second independently tagged fragment $\chi' = (\chi'(t), t \geq 0)$. This means that $\chi'(t)$ is the length of the interval component of $\theta(t)$ that contains U', where U' is another independent uniform variable. In the obvious notation with primes referring to the second tagged fragment, we see from the same argument based on size-biased sampling that

$$\mathbb{E}(\tilde{\mathcal{E}}(\eta)\tilde{\mathcal{E}}'(\eta)) \;=\; \mathbb{E}(\mathcal{E}(\eta)^2)\,,$$

where

$$\tilde{\mathcal{E}}'(\eta) \;=\; \sum_{t\geq 0} \chi'^{\beta-1}(t-)\mathbf{1}_{\{\chi'(t-)\geq\eta, k(t)=n'(t-)\}}\varphi(\Delta(t))\,,$$

and $n'(t)$ denotes the index at time t of the second tagged fragment. The proof of Theorem 6 will be complete if we establish the following lemma.

Lemma 8 *Under the same assumptions as in Theorem 6, one has*

$$\lim_{\eta\to 0}\eta^{2-2\beta}\mathbb{E}(\mathcal{E}(\eta)^2) \;=\; \left(\frac{1}{(1-\beta)m(1)}\int_{\mathcal{S}}\varphi(\mathbf{s})\nu(d\mathbf{s})\right)^2\,.$$

Proof: For the sake of simplicity, we shall focus on the case when $\varphi \geq 0$, which induces no loss of generality. Let us denote by T the first instant t at which the two tagged points U and U' belong to two different interval components of $\theta(t)$. Clearly, T is a stopping time, so the branching and self-similar properties enable us to express $\tilde{\mathcal{E}}(\eta)$ and $\tilde{\mathcal{E}}'(\eta)$ in the form (for the sake of notational simplicity, we do not indicate the dependence in the variable η)

$$\tilde{\mathcal{E}}(\eta) = A + x^{\beta-1}B \quad , \quad \tilde{\mathcal{E}}'(\eta) = A + x'^{\beta-1}B'$$

where

$$A \;=\; \sum_{t\leq T} \chi^{\beta-1}(t-)\mathbf{1}_{\{\chi(t-)\geq\eta, k(t)=n(t-)\}}\varphi(\Delta(t))\,,$$

$x = \chi(T)$, $x' = \chi'(T)$, and conditionally on (A, x, x'), B and B' are two independent variables distributed as $\mathcal{E}(\eta/x)$ and $\mathcal{E}(\eta/x')$, respectively. In particular, Lemma 7 yields

$$\lim_{\eta\to 0}\eta^{2-2\beta}\mathbb{E}\left(x^{\beta-1}x'^{\beta-1}BB'\right) \;=\; \left(\frac{1}{(1-\beta)m(1)}\int_{\mathcal{S}}\varphi(\mathbf{s})\nu(d\mathbf{s})\right)^2$$

and

$$\eta^{1-\beta}\mathbb{E}(x^{\beta-1}B + x'^{\beta-1}B' \mid A, x, x') \;\leq\; c \tag{2}$$

for some finite constant c.

Next, by the compensation formula for Poisson point processes, we have

$$\mathbb{E}\,(A) \;=\; \mathbb{E}\left(\int_0^\infty \chi^{\beta-1}(t)\mathbf{1}_{\{\chi(t)\geq\eta\}}\mathbf{1}_{\{t\leq T\}}dt\right)\int_{\mathcal{S}}\varphi(\mathbf{s})\nu(d\mathbf{s})\,.$$

Observe that the event $\{t \leq T\}$ holds if and only if U' belongs to the interval component with length $\chi(t)$ of $\theta(t)$ that contains U. Moreover $\chi(t)$ only depends on U and $\theta(t)$, U' is independent of the latter and has the uniform distribution. It follows that

$$\mathbb{E}\left(\chi^{\beta-1}(t)\mathbf{1}_{\{\chi(t)\geq\eta\}}\mathbf{1}_{\{t\leq T\}}\right) \;=\; \mathbb{E}\left(\chi^{\beta}(t)\mathbf{1}_{\{\chi(t)\geq\eta\}}\right)\,. \tag{3}$$

Recall from the proof of Lemma 7 that the tagged fragment process χ is the exponential of a subordinator ξ with Laplace exponent Φ. Hence

$$\mathbb{E}\left(\chi^{\beta}(t)\mathbf{1}_{\{\chi(t)\geq\eta\}}\right) \leq \mathbb{E}\left(\mathrm{e}^{-\beta\xi(t)}\right) = \mathrm{e}^{-t\Phi(\beta)}\,,$$

and we conclude that

$$\mathbb{E}(A) < \frac{1}{\Phi(\beta)}\int_{\mathcal{S}}\varphi(\mathbf{s})\nu(d\mathbf{s})\,. \tag{4}$$

As a consequence of (2),

$$\lim_{\eta\to 0}\eta^{2-2\beta}\mathbb{E}(A(x^{\beta-1}B + x'^{\beta-1}B')) \;=\; 0\,.$$

Then, we turn our attention to the mean of A^2, a variable which we express in the form $C + 2D$ with

$$\begin{aligned}
C &= \sum_{0<t\leq T}\chi^{2\beta-2}(t-)\mathbf{1}_{\{\chi(t-)\geq\eta,k(t)=n(t-)\}}\varphi(\Delta(t))^2\,,\\
D &= \sum_{t\in]0,T]}\chi^{\beta-1}(t-)\mathbf{1}_{\{\chi(t-)\geq\eta,k(t)=n(t-)\}}\varphi(\Delta(t))\\
&\qquad\times\left(\sum_{u\in]t,T]}\chi^{\beta-1}(u-)\mathbf{1}_{\{\chi(u-)\geq\eta,k(u)=n(u-)\}}\varphi(\Delta(u))\right)\,.
\end{aligned}$$

We compute $\mathbb{E}(C)$ using the compensation formula, and we get

$$\begin{aligned}
\mathbb{E}(C) &= \left(\int_{\mathcal{S}}\varphi(\mathbf{s})^2\nu(d\mathbf{s})\right)\mathbb{E}\left(\int_0^\infty\chi^{2\beta-2}(t)\mathbf{1}_{\{\chi(t)\geq\eta,t<T\}}dt\right)\\
&= \left(\int_{\mathcal{S}}\varphi(\mathbf{s})^2\nu(d\mathbf{s})\right)\mathbb{E}\left(\int_0^\infty\chi^{2\beta-1}(t)\mathbf{1}_{\{\chi(t)\geq\eta\}}dt\right)\,,
\end{aligned}$$

where the last equality follows from (3) with $2\beta-1$ in place of β. Now recall (1). When $\eta\to 0$, this quantity remains bounded for $\beta > 1/2$, is of order $\log 1/\eta$ for $\beta = 1/2$ (by the elementary renewal theorem), and of order $\eta^{2\beta-1}$ for $\beta < 1/2$ (by the renewal theorem).

Finally, we compute $\mathbb{E}(D)$ by conditioning at times t when the tagged fragment splits. We get by an application of the regenerative property of Poisson point processes at times when atoms appear and Lemma 7 that

$$\mathbb{E}(D) \;\leq\; a\eta^{\beta-1}\mathbb{E}\,(A)\;,$$

for some finite constant a. Recall from (4) that $\mathbb{E}(A)$ remains bounded as $\eta \to 0$. Putting the pieces together, we have shown that $\lim_{\eta\to 0}\eta^{2-2\beta}\mathbb{E}(A^2) = 0$, which completes the proof of the statement. □

We now conclude this section with a continuous analog of Corollary 2, which specifies the asymptotics of the empirical distribution of terminal state of the process when fragments are frozen at the instant when they become smaller than η. Introduce the random probability measure ρ_η defined by

$$\langle \rho_\eta, f\rangle := \sum_j x_{\eta,j} f(x_{\eta,j}/\eta)\,,$$

where $f : [0,1] \to \mathbb{R}_+$ denotes a generic measurable function and $\{x_{\eta,j}\}_{j\in\mathbb{N}}$ stands for the set of fragments with size less than η that result from the process.

Corollary 9 *Assumptions are the same as in Theorem 6. Then for every continuous function $f : [0,1] \to \mathbb{R}_+$*

$$\lim_{\eta\to 0}\langle \rho_\eta, f\rangle \;=\; \langle \rho, f\rangle \qquad \textit{in } L^2(\mathbb{P}),$$

where ρ is the deterministic probability measure given by

$$\langle \rho, f\rangle = \frac{1}{m(1)}\int_0^1 f(u)\left(\int_{\mathcal{S}}\sum_{n=1}^{\infty}\mathbf{1}_{\{u>s_n\}}s_n\nu(d\mathbf{s})\right)\frac{du}{u}\,.$$

The proof follows the same type of calculations as for Theorem 6, details are left to the reader. Alternatively, we may also establish Corollary 9 by considering first the case when $f(x) = x^{\beta-1}$ is a power function,

$$\langle \rho_\eta, f\rangle := \eta^{1-\beta}\sum_j x_{\eta,j}^{\beta}\,.$$

As observed in the Introduction, we can express the right-hand side in terms the potential energy, that is for the choice $\varphi(\mathbf{s}) = \sum_{n=1}^{\infty} s_n^{\beta} - 1$. We get

$$\langle \rho_\eta, f\rangle := \eta^{1-\beta}\left(\mathcal{E}(\eta) + 1\right)\,.$$

Then Theorem 6 easily entails Corollary 9 in the case when f is a power function, and the general case follows by standard arguments.

4 Conclusions

In any industry, one strives to minimize the production cost. If the model developed in this work is applied to mining industry, the dislocation measure ν, the cost-function φ, the exponent β and the Malthusian parameter α shall depend essentially on the quality of the mineral and properties of crushers, grinders and mills. These data are given and can hardly be modified: the quality of mineral is given by the pit section being exploited in terms of geological and geophysical characteristics; crushers are heavy and expensive equipment which work for long times and are seldom replaced. However there is some flexibility in the choice of the blasting operation, the size setting of the crushers and the mesh of the grid used for screening. Roughly, our results show that the energy cost depends as certain power function (with a negative exponent $\beta - \alpha$) of the mesh of the grid. A larger mesh induces a smaller fragmentation cost, but if the process produces fragments with larger size, the next step (physical-chemical processing) will also be less efficient. Thus one should also estimate the cost of the physical-chemical processing as a function of the size of the fragments, and then optimize simultaneously both procedures.

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