Sharp optimality for density deconvolution with dominating bias. II

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Abstract

This last part states and proves the fact that the kernel type estimator defined and studied in Part I is optimal in sharp asymptotical minimax sense on $\mathcal{A}$ simultaneously under the pointwise and the $\mathbb{L}_2$-risks. We also discuss some effects of dominating bias, such as superefficiency of minimax estimators.

The notation is preserved and the numbering of sections, results and equations in Part I and II is continued.

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4 Minimax lower bounds, sharp optimality and superefficiency

In this section we establish lower bounds for the risks showing that, under mild additional assumptions, the upper bounds of the previous section cannot be improved (in a minimax sense on the class of densities $\mathcal{A}_{\alpha,r}(L)$) not only among kernel estimators, but also among all estimators. In other words, the estimators suggested in the previous section attain optimal rates of convergence on $\mathcal{A}_{\alpha,r}(L)$ with optimal exact constants.

We suppose that the following assumption holds.

Assumption (ND). *There exist constants $u_1 > 0$, $B > 0$ and $\gamma_1 \in \mathbb{R}$ such that $\Phi^\varepsilon(u)$ is twice continuously differentiable for $|u| \geq u_1$ with the derivatives satisfying*

$$\max\{|(\Phi^\varepsilon(u))'|, |(\Phi^\varepsilon(u))''|\} \leq B|u|^{\gamma_1} \exp(-\beta|u|^s),$$

where $\beta > 0$ and $s > 0$ are the same as in Assumption (N).

Note that this assumption is satisfied for the examples of popular noise densities mentioned in the Introduction.

Theorem 4 *Let $\alpha > 0, L > 0, 0 < r < s \leq 2$, and suppose that Assumption (ND) and the right hand inequality in (1) hold. Then*

$$\liminf_{n\to\infty} \inf_{T_n} R_n(x, T_n, \mathcal{A}_{\alpha,r}(L))\varphi_n^{-2} \geq 1, \quad \forall\, x \in \mathbb{R}, \tag{38}$$

and

$$\liminf_{n\to\infty} \inf_{T_n} R_n(\mathbb{L}_2, T_n, \mathcal{A}_{\alpha,r}(L))\varphi_n^{-2}(\mathbb{L}_2) \geq 1, \tag{39}$$

where $\inf_{T_n}$ denotes the infimum over all estimators and the rates φ_n, $\varphi_n(\mathbb{L}_2)$ are defined in (14) and (15).

Proof of Theorem 4 is given in Section 5.

Theorems 1, 2 and 4 immediately imply the following result on sharp asymptotic minimaxity of the estimators constructed in Section 3 of Part I, see Butucea and Tsybakov (2007).

Theorem 5 *Let $\alpha > 0, L > 0, 0 < r < s \leq 2$, let Assumptions (N), (ND) hold and $\Phi^{\varepsilon}(u) \neq 0, \forall\, u \in \mathbb{R}$. Then the kernel estimator $\hat{f}_n$ with bandwidth defined by (13) (or with bandwidth defined by (20) if $r < s/2$) is sharp asymptotically minimax on $\mathcal{A}_{\alpha,r}(L)$ both in pointwise and in $\mathbb{L}_2$ sense:*

$$\lim_{n\to\infty} R_n(x, \hat{f}_n, \mathcal{A}_{\alpha,r}(L))\varphi_n^{-2} = \lim_{n\to\infty} \inf_{T_n} R_n(x, T_n, \mathcal{A}_{\alpha,r}(L))\varphi_n^{-2} = 1, \quad \forall\, x \in \mathbb{R}, \tag{40}$$

$$\lim_{n\to\infty} R_n(\mathbb{L}_2, \hat{f}_n, \mathcal{A}_{\alpha,r}(L))\varphi_n^{-2}(\mathbb{L}_2) = \lim_{n\to\infty} \inf_{T_n} R_n(\mathbb{L}_2, T_n, \mathcal{A}_{\alpha,r}(L))\varphi_n^{-2}(\mathbb{L}_2) = 1. \tag{41}$$

This is the main result of the paper. It shows that the kernel estimator $\hat{f}_n$ with a properly chosen bandwidth h_n is sharp optimal in asymptotically minimax sense on $\mathcal{A}_{\alpha,r}(L)$ and that for $r < s/2$ the estimator $f_n^{\mathbf{a}}$ is sharp adaptive in asymptotically minimax sense on $\mathcal{A}_{\alpha,r}(L)$. Sharp adaptation is thus obtained by direct tuning of the smoothing parameter without any additional adaptation rule. This is one of the effects of dominating bias. Theorem 5 also provides exact asymptotical expressions for minimax risks on $\mathcal{A}_{\alpha,r}(L)$ under the pointwise and the $\mathbb{L}_2$ losses: it states that they are equal to φ_n^2 and $\varphi_n^2(\mathbb{L}_2)$ respectively.

Thus, φ_n^2 and $\varphi_n^2(\mathbb{L}_2)$ can be chosen as reference values to determine efficiency of estimators. An interesting question is whether there exist superefficient estimators $\tilde{f}_n$, i.e. such that

$$\sup_{x\in\mathbb{R}} E_f\left[|\tilde{f}_n(x) - f(x)|^2\right] = o(\varphi_n^2) \quad \text{and} \quad E_f\left[\|\tilde{f}_n - f\|_2^2\right] = o(\varphi_n^2(\mathbb{L}_2)), \tag{42}$$

as $n \to \infty$, for any fixed $f \in \mathcal{A}_{\alpha,r}(L)$. The answer to this question is positive, as shows the next proposition.

Proposition 3 *Let the conditions of Theorem 1 hold. Let $\tilde{f}_n$ be the kernel estimator $\hat{f}_n$ with bandwidth defined by (13) (or by (20) if $r < s/2$). Then $\tilde{f}_n$ satisfies (42). If, moreover, the conditions of Theorem 5 hold, $\tilde{f}_n$ is superefficient in the sense that*

$$\lim_{n\to\infty} \frac{E_f[|\tilde{f}_n(x) - f(x)|^2]}{\inf_{T_n} \sup_{g\in\mathcal{A}_{\alpha,r}(L)} E_g\left[|T_n(x) - g(x)|^2\right]} = 0, \quad \forall x \in \mathbb{R}, \tag{43}$$

$$\lim_{n\to\infty} \frac{E_f[\|\tilde{f}_n - f\|_2^2]}{\inf_{T_n} \sup_{g\in\mathcal{A}_{\alpha,r}(L)} E_g\left[\|T_n - g\|_2^2\right]} = 0. \tag{44}$$

Proof. Consider the kernel estimator $\hat{f}_n$ with bandwidth defined by (13). Instead of using Proposition 1 to bound the bias term, we apply directly (4) for the pointwise risk and (5) for the $\mathbb{L}_2$-risk which yields that, for any fixed $f \in \mathcal{A}_{\alpha,r}(L)$,

$$\begin{aligned} \sup_{x\in\mathbb{R}} |E_f\hat{f}_n(x) - f(x)|^2 &= o\left(h_*^{r-1}\exp(-2\alpha/h_*^r)\right) = o(\varphi_n^2), \\ \|E_f\hat{f}_n - f\|_2^2 &= o\left(\exp(-2\alpha/h_*^r)\right) = o(\varphi_n^2(\mathbb{L}_2)), \end{aligned}$$

as $n \to \infty$. Now, Proposition 2 and (29) of Lemma 4 in Part I imply that the variance terms are also $o(\varphi_n^2)$ and $o(\varphi_n^2(\mathbb{L}_2))$, as $n \to \infty$, respectively. Hence, (42) follows and implies (43) and (44), in view of Theorem 5. The case where the bandwidth is defined by (20) and $r < s/2$ is treated similarly. □

The result of Proposition 3 is explained by the fact that the value of the minimax risk in the denominator of (44) is attained (up to a $1 + o(1)$ factor) on the densities that depend on n, while in the numerator we have a fixed density f. Such a superefficiency property occurs in other nonparametric problems (see e.g. Brown, Low and Zhao (1997) or Tsybakov (2004), Chapter 3), where it is proved for various adaptive estimators. On the contrary, non-adaptive asymptotically minimax estimators, for example, the Pinsker estimator which is efficient for ellipsoids in gaussian sequence model, are not superefficient and turn out to be inadmissible (Tsybakov (2004), Section 3.8). Compared with that, the result of Proposition 3 is somewhat surprising, because it states that a non-adaptive asymptotically minimax estimator $\hat{f}_n$ with bandwidth defined by (13) is superefficient. This provides a simple counter-example of a superefficient nonparametric estimator which is not adaptive. We conjecture that this is a general property of nonparametric problems with dominating bias.

5 Proof of Theorem 4

5.1 General scheme of the proof

We use the method of proving lower bounds by reduction to the problem of testing two simple hypotheses (cf. e.g. Tsybakov (2004), Chapter 2). Namely, we define two properly chosen probability densities f_{n1} and f_{n2}, depending on n and belonging to $\mathcal{A}_{\alpha,r}(L)$ and we bound the minimax risk as follows

$$\begin{aligned} \inf_{T_n} R_n(T_n, \mathcal{A}_{\alpha,r})\psi_n^{-2} &\geq \inf_{T_n} \max_{f\in\{f_{n1},f_{n2}\}} E_f d^2(T_n, f)\psi_n^{-2} \\ &\geq \inf_{T_n} \max_{f\in\{f_{n1},f_{n2}\}} \left(E_f d(T_n, f)\right)^2 \psi_n^{-2}, \end{aligned} \tag{45}$$

where $R_n(T_n, \mathcal{A}_{\alpha,r}(L))$ is either $R_n(x, T_n, \mathcal{A}_{\alpha,r}(L))$ or $R_n(\mathbb{L}_2, T_n, \mathcal{A}_{\alpha,r}(L))$, ψ_n is defined as φ_n or $\varphi_n(\mathbb{L}_2)$ (cf. (14) and (15)) respectively and $d(T_n, f)$ stands for the distance $|T_n(x) - f(x)|$ at a fixed point x or the $\mathbb{L}_2$-distance $\|T_n - f\|_2$ respectively. Hence, to prove the theorem it remains to show that

$$R \stackrel{\text{def}}{=} \inf_{T_n} \max_{f\in\{f_{n1},f_{n2}\}} E_f d(T_n, f) \geq \psi_n(1 + o(1)), \tag{46}$$

as $n \to \infty$, for both pointwise and $\mathbb{L}_2$ distances $d(\cdot,\cdot)$. This will be done by application of Lemma 8 of the Appendix. According to Lemma 8, (46) is satisfied if the functions f_{n1} and f_{n2} are chosen such that

$$d(f_{n1}, f_{n2}) \geq 2\psi_n(1 + o(1)), \text{ as } n \to \infty, \tag{47}$$

$$\chi^2(P_{f_{n1}}, P_{f_{n2}}) = o(1), \text{ as } n \to \infty, \tag{48}$$

where $\chi^2(P_{f_{n1}}, P_{f_{n2}})$ is the χ^2-divergence between the probability measures $P_{f_{n1}}$ and $P_{f_{n2}}$ (recall that P_f denotes the joint distribution of $Y_1, \dots, Y_n$ when the underlying probability density of X_i's is f). Thus, to prove Theorem 4 it suffices to construct two functions f_{n1} and f_{n2} belonging to $\mathcal{A}_{\alpha,r}(L)$ and satisfying (47) – (48). Since $P_{f_{nj}}$ is a product of n identical probability measures corresponding to the density $f_{nj}^Y = f_{nj} * f^\varepsilon$, for $j = 1, 2$, we have $\chi^2(P_{f_{n1}}, P_{f_{n2}}) \leq Cn\chi^2(f_{n1}^Y, f_{n2}^Y)$ if $\chi^2(f_{n1}^Y, f_{n2}^Y) \leq 1/n$, where C is a finite constant and

$$\chi^2(f_{n1}^Y, f_{n2}^Y) = \int \frac{(f_{n1}^Y - f_{n2}^Y)^2}{f_{n1}^Y}(x)dx$$

(cf. e.g. Tsybakov (2004), p. 72). Therefore, (48) follows from

$$n\chi^2(f_{n1}^Y, f_{n2}^Y) \to 0, \text{ as } n \to \infty. \tag{49}$$

We now proceed to the construction of densities $f_{n1}, f_{n2} \in \mathcal{A}_{\alpha,r}(L)$ satisfying (49) and (47) for pointwise and $\mathbb{L}_2$-distances $d(\cdot,\cdot)$.

Consider a density f_0 of a symmetric stable law whose characteristic function is

$$\Phi_0(u) = \begin{cases} \exp\left(-\left|c_0 u\right|^r\right), & \text{if } 1 < r < 2, \\ \exp\left(-\left|c_0 u\right|\right), & \text{if } 0 < r \le 1, \end{cases}$$

where $c_0 > \max\{\alpha^{1/r}, \alpha\}$. Clearly, for any $0 < a < 1$ there exists $c_0 > 0$ large enough so that $f_0 \in \mathcal{A}_{\alpha,r}(a^2 L)$. In view of Lemma 3, there exists $c_1' > 0$ such that

$$f_0(x) = \frac{1}{c_0} p\left(\frac{x}{c_0}\right) \ge \frac{c_1'}{|x|^{\max\{r+1,2\}} + 1}, \tag{50}$$

for all $x \in \mathbb{R}$, where p is the density of stable symmetric distribution with characteristic function $\exp(-|t|^{\max\{r,1\}})$, $0 < r < 2$. Let $h_+ = h_+(n)$ be the unique solution of the equation

$$\frac{2\alpha}{h_+^r} + \frac{2\beta}{h_+^s} = \log n + (\log\log n)^2. \tag{51}$$

Note that h_+ is analogous to h_* defined by (13) with the only difference that the $(\log\log n)^2$ term changes the sign.

We define the densities f_{n1} and f_{n2} by their characteristic functions

$$\Phi_{n1}(u) = \Phi_0(u) + \Phi^H(u, h_+), \quad \Phi_{n2}(u) = \Phi_0(u) - \Phi^H(u, h_+), \quad u \in \mathbb{R}, \tag{52}$$

where $u \mapsto \Phi^H(u, h)$ with $h > 0$ will be called *perturbation function* and will be defined differently for the pointwise distance and the $\mathbb{L}_2$-distance. The construction of perturbation functions will be based on the following lemma.

Lemma 5 *For any $\delta > 0$ and any $D > 4\delta$ there exists a function $\Phi^G : \mathbb{R} \to [0,1]$ such that*

(i) Φ^G is 3 times continuously differentiable on $\mathbb{R}$ and the first 3 derivatives of Φ^G are uniformly bounded on $\mathbb{R}$,

(ii) Φ^G is compactly supported on $(\delta, D-\delta)$ and

$$I(2\delta \le u \le D - 2\delta) \le \Phi^G(u) \le I(\delta \le u \le D - \delta),$$

for all $u \in \mathbb{R}$.

Proof of Lemma 5. Denote by J_0 the 5-fold convolution of the indicator function $I(|u| \le 1)$ with itself. Let $J : \mathbb{R} \to [0, \infty)$ be a rescaling of J_0 such that the support of J is $(-1,1)$ and $\int J(x)dx = 1$. Then J_0 and J are 3 times continuously differentiable on $\mathbb{R}$. For $\delta > 0$ and $D > 4\delta$ define

$$\Phi^G(u) = \int_{u-D+3\delta/2}^{u-3\delta/2} \frac{2}{\delta} J\left(\frac{2x}{\delta}\right) dx.$$

Clearly, Φ^G is 3 times continuously differentiable on $\mathbb{R}$ and $0 \le \Phi^G(u) \le 1$, $\forall u \in \mathbb{R}$. Moreover, *supp* $\Phi^G = (\delta, D-\delta)$ and for any $u \in (2\delta, D-2\delta)$ we have $\Phi^G(u) = \int_{-1}^{1} J(x)dx = 1$. □

5.2 Lower bound at a fixed point

Without loss of generality, we will prove the lower bound for the distance $d(f,g) = |f(0) - g(0)|$ at the point $x = 0$ (if $x \neq 0$ it suffices to shift the functions f_{n1} and f_{n2} at x). Define the perturbation function

$$\Phi^H(u,h) = \sqrt{2\pi\alpha r L}\; h^{(1-r)/2} \exp\left(\frac{\alpha}{h^r}\right) \exp\left(-2\alpha |u|^r\right) \Phi^G\left(|u|^r - \frac{1}{h^r}\right), \tag{53}$$

where Φ^G is a function satisfying the properties given in Lemma 5 for some $\delta > 0$ and $D > 4\delta$.

Most of the computations below work when Φ^G is replaced by an indicator function of the interval $[0, D]$. However, we obviously need a continuous perturbation function Φ^H that satisfies $\Phi^H(0) = 0$ to ensure that f_{n1} and f_{n2} integrate to 1 and that is smooth enough to allow an appropriate bound on the χ^2-divergence.

Lemma 6 *Let f_{n1} and f_{n2} be the functions defined by their Fourier transforms* (52), (53) *with Φ^G satisfying the properties given in Lemma* 5. *Then we have the following.*

1. *The functions f_{n1} and f_{n2} are probability densities for any n large enough.*
2. *The functions f_{n1} and f_{n2} belong to $\mathcal{A}_{\alpha,r}(L)$ for n large enough if $c_0 > 0$ in the definition of f_0 large enough.*
3. *The distance between f_{n1} and f_{n2} at $x = 0$ satisfies*

   $$|f_{n1}(0) - f_{n2}(0)| \ge 2\varphi_n[e^{-4\alpha\delta} - e^{-2\alpha(D-2\delta)}](1 + o(1)),$$

   as $n \to \infty$.
4. *The χ^2-divergence $\chi^2\left(f_{n1}^Y, f_{n2}^Y\right)$ satisfies* (49).

Proof. 1. Clearly, $\Phi^H(\cdot, h)$ is an even, 3 times continuously differentiable function on $\mathbb{R}$ having a compact support. It is easy to see that the integrals $\int |\Phi^H(u,h)|du$ and $\int |\partial^3\Phi^H(u,h)/\partial u^3|du$ are bounded uniformly over $0 < h \le h_0$ for any $h_0 > 0$. Integration by parts yields that the inverse Fourier transform of $\Phi^H(\cdot, h)$ can be written as

$$H(x,h) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int \cos(xu)\Phi^H(u,h)du = -\frac{1}{2\pi x^3} \int \sin(xu) \frac{\partial^3\Phi^H(u,h)}{\partial u^3} du \tag{54}$$

for all $x \in \mathbb{R}$ and $0 < h \leq h_0$. Thus, there exists a constant $C_H < \infty$ independent of n and such that

$$|H(x, h_+)| \leq C_H(|x|^3 + 1)^{-1}, \text{ for all } x \in \mathbb{R}. \tag{55}$$

Denote by *Dom* the common support of the functions $\Phi^G(|u|^r - 1/h_+^r)$ and $\Phi^H(u, h_+)$:

$$\begin{aligned} Dom &\stackrel{\text{def}}{=} \left\{ u : |u|^r - \frac{1}{h_+^r} \in [\delta, D - \delta] \right\} \\ &= \left\{ u : \left(\delta + \frac{1}{h_+^r}\right)^{1/r} \leq |u| \leq \left(D - \delta + \frac{1}{h_+^r}\right)^{1/r} \right\}. \end{aligned}$$

Using the fact that $\left(\delta + 1/h_+^r\right)^{1/r} \to \infty$, as $n \to \infty$, for any fixed $\delta > 0$ and applying (24) of Lemma 2 in the Appendix of Part I, we find

$$\begin{aligned} \|H(\cdot, h_+)\|_\infty &\stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}} |H(x, h_+)| \leq \frac{1}{2\pi} \int |\Phi^H(u, h_+)| du \\ &\leq \sqrt{\frac{\alpha r L}{2\pi}} h_+^{(1-r)/2} \exp\left(\alpha / h_+^r\right) \int_{Dom} \exp(-2\alpha |u|^r) du \\ &\leq c h_+^{(r-1)/2} \exp(-\alpha / h_+^r) = o(1), \text{ as } n \to \infty, \end{aligned} \tag{56}$$

where $c > 0$ is a finite constant.

Now, $f_{n1}(x) = f_0(x) + H(x, h_+)$, $f_{n2}(x) = f_0(x) - H(x, h_+)$. Choose $A > 0$ large enough so that for $|x| > A$ we have $C_H(|x|^3 + 1)^{-1} < c_1'(|x|^{\max\{r+1,2\}} + 1)^{-1}$ (note that $\max\{r+1, 2\} < 3$). Then, in view of (50) and (55), $f_{nj}(x) > 0$, $j = 1, 2$, for $|x| > A$. Now, if n is large enough, $f_{nj}(x) > 0$ also for $|x| \leq A$ since $\inf_{|x| \leq A} f_0(x) > 0$ (cf. (50)) and (56) holds.

Thus, $f_{nj}(x) > 0$, $j = 1, 2$, for all $x \in \mathbb{R}$ if n is large enough. It remains to note that f_{n1} and f_{n2} integrate to 1 since $\int H(x, h_+) dx = \Phi^H(0, h_+) = 0$ (indeed, $0 \notin supp\ \Phi^H(\cdot, h_+) = Dom$).

2. We have, by (53) and Lemma 5,

$$\begin{aligned} &\int \left|\Phi^H(u, h_+)\right|^2 \exp(2\alpha |u|^r) du \\ \leq\ & 2\pi \alpha r L h_+^{1-r} \exp\left(\frac{2\alpha}{h_+^r}\right) \int_{Dom} \exp(-2\alpha |u|^r) du \\ \leq\ & 4\pi \alpha r L h_+^{1-r} \exp\left(\frac{2\alpha}{h_+^r}\right) \int_{(\delta + 1/h_+^r)^{1/r}}^{\infty} \exp(-2\alpha u^r) du. \end{aligned}$$

By Lemma 6,

$$\begin{aligned}
&\int_{(\delta+1/h_+^r)^{1/r}}^{\infty} \exp(-2\alpha u^r) du \\
= &\ \frac{h_+^{r-1}}{2\alpha r} \exp\left(-\frac{2\alpha}{h_+^r}\right) e^{-2\alpha\delta}(1+\delta h_+^r)^{(1-r)/r}(1+o(1)),
\end{aligned}$$

as $n \to \infty$. We get therefore,

$$\int |\Phi^H(x, h_+)|^2 \exp(2\alpha|u|^r) du \leq 2\pi L \exp(-2\alpha\delta)(1+o(1)), \tag{57}$$

as $n \to \infty$, for any fixed $\delta > 0$. Now, choose $c_0 > 0$ in the definition of f_0 large enough to guarantee that $f_0 \in \mathcal{A}_{\alpha,r}(a^2 L)$ with $a = 1 - e^{-\alpha\delta/2}$. This and (57) imply

$$\begin{aligned}
&\left(\int |\Phi_{nj}(u)|^2 \exp(2\alpha|u|^r) du\right)^{1/2} \\
\leq &\ \|\Phi_0(\cdot) \exp(\alpha|\cdot|^r)\|_2 + \|\Phi^H(\cdot, h_+) \exp(\alpha|\cdot|^r)\|_2 \\
\leq &\ (1 - e^{-\alpha\delta/2})\sqrt{2\pi L} + e^{-\alpha\delta}\sqrt{2\pi L}(1+o(1)) \\
\leq &\ \sqrt{2\pi L}, \quad j = 1, 2,
\end{aligned}$$

for n large enough and any fixed $\delta > 0$.

3. Using the left inequality in (ii) of Lemma 5 we get

$$\begin{aligned}
&|f_{n1}(0) - f_{n2}(0)|^2 \\
= &\ \frac{1}{(2\pi)^2} \left|\int (\Phi_{n1}(u) - \Phi_{n2}(u)) du\right|^2 = \frac{4}{(2\pi)^2} \left|\int \Phi^H(u, h_+) du\right|^2 \\
= &\ \frac{2\alpha r L h_+^{1-r}}{\pi} \exp\left(\frac{2\alpha}{h_+^r}\right) \left|\int \exp(-2\alpha|u|^r) \Phi^G\left(|u|^r - \frac{1}{h_+^r}\right) du\right|^2 \\
\geq &\ \frac{2\alpha r L h_+^{1-r}}{\pi} \exp\left(\frac{2\alpha}{h_+^r}\right) \left|2\int_{(2\delta+1/h_+^r)^{1/r}}^{(D-2\delta+1/h_+^r)^{1/r}} e^{-2\alpha u^r} du\right|^2.
\end{aligned} \tag{58}$$

By (24) of Lemma 2 in the Appendix of Part I,

$$\begin{aligned}
&\int_{(2\delta+1/h_+^r)^{1/r}}^{(D-2\delta+1/h_+^r)^{1/r}} \exp(-2\alpha u^r) du \\
= &\ \frac{h_+^{r-1}}{2\alpha r} \exp\left(-\frac{2\alpha}{h_+^r}\right) \Big[(1+2\delta h_+^r)^{(1-r)/r} e^{-4\alpha\delta}(1+o(1)) \\
&\ -(1+(D-2\delta)h_+^r)^{(1-r)/r} e^{-2\alpha(D-2\delta)}(1+o(1))\Big] \\
= &\ \frac{h_+^{r-1}}{2\alpha r} \exp\left(-\frac{2\alpha}{h_+^r}\right) [e^{-4\alpha\delta} - e^{-2\alpha(D-2\delta)}](1+o(1)),
\end{aligned} \tag{59}$$

as $n \to \infty$. The expression in square brackets here is positive since $D > 4\delta$. Combining (58) and (59) and using (77) of Lemma 9 in the Appendix together with (14) we get

$$\begin{aligned}
& |f_{n1}(0) - f_{n2}(0)|^2 \\
\geq\ & 4\left[\frac{L}{2\pi\alpha r} h_+^{r-1} \exp\left(-\frac{2\alpha}{h_+^r}\right)\right][e^{-4\alpha\delta} - e^{-2\alpha(D-2\delta)}]^2(1+o(1)) \\
=\ & 4\left[\frac{L}{2\pi\alpha r} h_*^{r-1} \exp\left(-\frac{2\alpha}{h_*^r}\right)\right][e^{-4\alpha\delta} - e^{-2\alpha(D-2\delta)}]^2(1+o(1)) \\
=\ & 4\varphi_n^2[e^{-4\alpha\delta} - e^{-2\alpha(D-2\delta)}]^2(1+o(1)),
\end{aligned}$$

as $n \to \infty$.

4. Inequalities (50), (55), (56) and the fact that $r < 2$ imply the existence of a constant $c_2' > 0$ independent of n and such that

$$f_{n1}(x) \geq \frac{c_2'}{|x|^{\max\{r+1,2\}} + 1}, \quad \forall x \in \mathbb{R},$$

for all n large enough. Since f^ε is a probability density, we have $\int_{-M}^{M} f^\varepsilon(x)dx \geq 1/2$ for a constant $M > 1$ large enough. Hence,

$$\begin{aligned}
f_{n1}^Y(x) &\geq \int_{-M}^{M} f_{n1}(x-y)f^\varepsilon(y)dy \geq \frac{c_2'}{2}\inf_{|y|\leq M}\left[\frac{1}{|x-y|^{\max\{r+1,2\}}+1}\right] \\
&\geq c_3' \min\left\{\frac{1}{M^{\max\{r+1,2\}}}, \frac{1}{|x|^{\max\{r+1,2\}}}\right\} \qquad (60)
\end{aligned}$$

where n and M are large enough, $c_3' > 0$ is independent of n, and the last inequality is obtained by considering separately $|x| \leq M$ and $|x| > M$. Thus

$$\begin{aligned}
n\chi^2(f_{n1}^Y, f_{n2}^Y) &= n\int \frac{(f_{n2}^Y - f_{n1}^Y)^2(x)}{f_{n1}^Y(x)}dx = 4n\int \frac{(H * f^\varepsilon)^2(x)}{f_{n1}^Y(x)}dx \\
&\leq \frac{4}{c_3'}\left(nM^{\max\{r+1,2\}}\int_{|x|\leq M}(H * f^\varepsilon)^2(x)dx\right. \\
&\qquad \left. + n\int_{|x|>M}|x|^{\max\{r+1,2\}}(H * f^\varepsilon)^2(x)dx\right) \\
&\leq (4M^3/c_3')(T_{n1} + T_{n2}), \qquad (61)
\end{aligned}$$

for n and M large enough, where $H(x) = H(x, h_+)$ for brevity and

$$T_{n1} = n\|H * f^\varepsilon\|_2^2, \quad T_{n2} = n\int |x|^4(H * f^\varepsilon)^2(x)dx. \qquad (62)$$

Using Plancherel's formula and the right hand inequality in (1) we get, for n large enough,

$$\begin{aligned}
\|H * f^{\varepsilon}\|_2^2 &= \frac{1}{2\pi}\int \left|\Phi^H(u,h_+)\Phi^{\varepsilon}(u)\right|^2 du \\
&\leq b_{\max}^2 \alpha r L h_+^{1-r}\exp\left(\frac{2\alpha}{h_+^r}\right)\int_{Dom} |u|^{2\gamma'}\exp(-4\alpha|u|^r - 2\beta|u|^s)du \\
&\leq 2b_{\max}^2 \alpha r L h_+^{1-r}\exp\left(\frac{2\alpha}{h_+^r}\right)\int_{(\delta+1/h_+^r)^{1/r}}^{\infty} u^{2\gamma'}\exp(-4\alpha u^r - 2\beta u^s)du \\
&\leq 2b_{\max}^2 \alpha r L h_+^{1-r}\exp\left(-\frac{2\alpha}{h_+^r}\right)\int_{1/h_+}^{\infty} u^{2\gamma'}\exp(-2\beta u^s)du. \qquad (63)
\end{aligned}$$

The last integral is evaluated using (24) of Lemma 2 in Appendix, Part I:

$$\int_{1/h_+}^{\infty} u^{2\gamma'}\exp(-2\beta u^s)du = \frac{h_+^{s-2\gamma'-1}}{2\beta s}\exp\left(-\frac{2\beta}{h_+^s}\right)(1+o(1)), \qquad (64)$$

as $n \to \infty$. This, together with (63) and (78) of Lemma 9 in the Appendix, yields

$$\|H * f^{\varepsilon}\|_2^2 \leq C h_+^{s-2\gamma'-r}\exp\left(-\frac{2\alpha}{h_+^r} - \frac{2\beta}{h_+^s}\right) = o\left(\frac{1}{n}\right), \qquad (65)$$

as $n \to \infty$, where $C > 0$ is a constant. Thus,

$$T_{n1} = o(1), \text{ as } n \to \infty. \qquad (66)$$

Now, assume that n is large enough to have $(\delta + 1/h_+^r)^{1/r} > \max(u_0, u_1)$, where $u_0 > 0$, $u_1 > 0$ are the constants in Assumptions (N) and (ND). Then $\Phi^G(|u|^r - 1/h_+^r) = 0$ for $|u| \leq \max(u_0, u_1)$, and thus the function $\Phi^H(\cdot, h_+)\Phi^{\varepsilon}(\cdot)$ is twice continuously differentiable on $\mathbb{R}$. Using Assumption (ND), the right hand inequality in (1) and the fact that Φ^G, together with its first two derivatives, is uniformly bounded on $\mathbb{R}$ we find that there exist constants $B_1 < \infty$ and $a \in \mathbb{R}$ such that, for n large enough and all $u \in \mathbb{R}$,

$$\left|(\Phi^H(u,h_+)\Phi^{\varepsilon}(u))''\right| \leq B_1 h_+^{(1-r)/2}\exp\left(\frac{\alpha}{h_+^r}\right)|u|^a\exp(-2\alpha|u|^r - \beta|u|^s). \qquad (67)$$

Thus, for n large enough, we have, by Plancherel's formula for derivatives and (67),

$$\begin{aligned}
T_{n2} &= \frac{n}{2\pi}\int \left|(\Phi^H(u,h_+)\Phi^\varepsilon(u))''\right|^2 du \\
&\leq \frac{n}{2\pi}B_1^2 h_+^{1-r}\exp\left(\frac{2\alpha}{h_+^r}\right)\int_{Dom} |u|^{2a}\exp(-4\alpha|u|^r - 2\beta|u|^s)du \\
&\leq \frac{n}{\pi}B_1^2 h_+^{1-r}\exp\left(\frac{2\alpha}{h_+^r}\right)\int_{(\delta+1/h_+^r)^{1/r}}^{\infty} u^{2a}\exp(-4\alpha u^r - 2\beta u^s)du \\
&\leq \frac{n}{\pi}B_1^2 h_+^{1-r}\exp\left(-\frac{2\alpha}{h_+^r}\right)\int_{1/h_+}^{\infty} u^{2a}\exp(-2\beta u^s)du. \qquad (68)
\end{aligned}$$

Plugging (64) with $\gamma' = a$ into (68) and using (78) of Lemma 9 in the Appendix we get

$$T_{n2} \leq Cnh_+^{-2a+s-r}\exp\left(-\frac{2\alpha}{h_+^r} - \frac{2\beta}{h_+^s}\right)(1+o(1)) = o(1), \qquad (69)$$

as $n \to \infty$, where $C > 0$ is a constant.

Combining (61), (66) and (69) we get that $n\chi^2(f_{n1}^Y, f_{n2}^Y) \to 0$, as $n \to \infty$.
□

Proof of (38). We use the general scheme of Section 5.1 with $d(f_{n1}, f_{n2}) = |f_{n1}(0) - f_{n2}(0)|$. Choose $c_0 > 0$ in the definition of f_0 large enough to guarantee that assertion 2 of Lemma 6 holds. Lemma 6 implies that (49) and thus (48) are satisfied and that (47) holds with

$$\psi_n = \varphi_n[e^{-4\alpha\delta} - e^{-2\alpha(D-2\delta)}].$$

Therefore, Lemma 8 of the Appendix implies that

$$R \geq \varphi_n[e^{-4\alpha\delta} - e^{-2\alpha(D-2\delta)}](1+o(1)),$$

as $n \to \infty$, where R is defined in (46). This and (45) yield that, as $n \to \infty$,

$$\inf_{T_n} R_n(0, T_n, \mathcal{A}_{\alpha,r}(L))\varphi_n^{-2} \geq [e^{-4\alpha\delta} - e^{-2\alpha(D-2\delta)}](1+o(1)).$$

Taking limits as $n \to \infty$ and then as $D \to \infty$ and $\delta \to 0$ we get (38) for $x = 0$. The proof for $x \neq 0$ is analogous (see the remark at the beginning of this section). □

5.3 Lower bound in $\mathbb{L}_2$

Introduce the perturbation function

$$\Phi^H(u,h) = \sqrt{2\pi\alpha r L(d-1)}\, h^{(1-r)/2} e^{(d-1)\alpha/h^r}\exp\left(-\alpha d|u|^r\right)\Phi^G\left(|u|^r - \frac{1}{h^r}\right), \qquad (70)$$

where Φ^G is a function satisfying the properties given in Lemma 5 and $d = d(\delta) > 1$ is a constant depending on the value δ that appears in the construction of Φ^G. The argument below is similar to that of Section 5.2, modulo the choice of the perturbation function (70) which is slightly different from (53). The argument goes through with d such that $d(\delta) \to \infty$ and $\delta d(\delta) \to 0$ as $\delta \to 0$, but we will set for simplicity $d(\delta) = \delta^{-1/2}$ and assume that $0 < \delta < 1$, which ensures that $d(\delta) > 1$.

Lemma 7 *Let f_{n1} and f_{n2} be the functions defined by their Fourier transforms* (52), (70) *with Φ^G satisfying the properties of Lemma* 5 *and* $0 < \delta < 1$. *Then we have the following.*

1. *The functions f_{n1} and f_{n2} are probability densities for n large enough.*
2. *The functions f_{n1} and f_{n2} belong to $\mathcal{A}_{\alpha,r}(L)$ for n large enough if $c_0 > 0$ in the definition of f_0 large enough.*
3. *The $\mathbb{L}_2$ distance between f_{n1} and f_{n2} satisfies*

$$\|f_{n1}-f_{n2}\|_2 \geq 2\varphi_n(\mathbb{L}_2)\left((1-\sqrt{\delta})[e^{-4\alpha\sqrt{\delta}} - e^{-2\alpha(D-2\delta)/\sqrt{\delta}}]\right)^{1/2}(1+o(1)),$$

   as $n \to \infty$.
4. *The χ^2-divergence $\chi^2\left(f_{n1}^Y, f_{n2}^Y\right)$ satisfies* (49).

Proof. 1. The argument is analogous to the proof of assertion 1 of Lemma 6. In particular, one also has $|H(x,h)| \leq C_H'(|x|^3+1)^{-1}$, $\forall x \in \mathbb{R}$, and $\|H(\cdot,h_+)\|_\infty = o(1)$, as $n \to \infty$, for some constant $C_H' < \infty$. We omit the details.

2. We have by (52) and Lemma 5

$$\begin{aligned}
&\int \left|\Phi^H(u,h_+)\right|^2 \exp\left(2\alpha\,|u|^r\right) du \\
\leq\;& 2\pi\alpha r L(d-1)h_+^{1-r}\exp\left(\frac{2(d-1)\alpha}{h_+^r}\right)\int_{Dom}\exp\left(-2\alpha(d-1)\,|u|^r\right) du \\
\leq\;& 4\pi\alpha r L(d-1)h_+^{1-r}\exp\left(\frac{2(d-1)\alpha}{h_+^r}\right)\int_{(\delta+1/h_+^r)^{1/r}}^{\infty}\exp\left(-2\alpha(d-1)u^r\right) du.
\end{aligned}$$

By Lemma 2 of Part I,

$$\begin{aligned}
&\int_{(\delta+1/h_+^r)^{1/r}}^{\infty}\exp\left(-2\alpha(d-1)u^r\right) du \\
=\;& \frac{h_+^{r-1}}{2\alpha(d-1)r}\exp\left(-\frac{2(d-1)\alpha}{h_+^r}\right)\exp\left(-2\alpha(d-1)\delta\right)\left(1+\delta h_+^r\right)^{(1-r)/r}(1+o(1)),
\end{aligned}$$

as $n \to \infty$. We get therefore,

$$\int |\Phi^H(u, h_+)|^2 \exp(2\alpha |u|^r) du \le 2\pi L \exp(-2\alpha(d-1)\delta)(1 + o(1)),$$

as $n \to \infty$, for any fixed $\delta > 0$. Now, since $d = \delta^{-1/2}$, we get that the last exponent is strictly less than 1 for $0 < \delta < 1$, and thus the argument similar to that after formula (57) can be applied to show that

$$\int |\Phi_{nj}(u)|^2 \exp(2\alpha |u|^r) du \le 2\pi L, \quad j = 1, 2,$$

for n large enough, if $c_0 > 0$ in the definition of f_0 is chosen large enough.

3. The $\mathbb{L}_2$ distance is

$$\begin{aligned}
\|f_{n1} - f_{n2}\|_2^2 &= \frac{1}{2\pi} \int (\Phi_{n1}(u) - \Phi_{n2}(u))^2 du = \frac{4}{2\pi} \int \left|\Phi^H(u, h_+)\right|^2 du \\
&= 4L\alpha r(d-1) h_+^{1-r} \exp\left(\frac{2(d-1)\alpha}{h_+^r}\right) \int e^{-2\alpha d|u|^r} \left|\Phi^G\left(|u|^r - \frac{1}{h_+^r}\right)\right|^2 du \\
&\ge 4L\alpha r(d-1) h_+^{1-r} \exp\left(\frac{2(d-1)\alpha}{h_+^r}\right) \left[2 \int_{(2\delta + 1/h_+^r)^{1/r}}^{(D-2\delta+1/h_+^r)^{1/r}} e^{-2\alpha d u^r} du\right] \quad (71)
\end{aligned}$$

where we used the left inequality in (ii) of Lemma 6. Lemma 2 (Part I) implies that (cf. (59)):

$$\begin{aligned}
&\int_{(2\delta+1/h_+^r)^{1/r}}^{(D-2\delta+1/h_+^r)^{1/r}} \exp\left(-2\alpha d u^r\right) du \\
&= \frac{h_+^{r-1}}{2\alpha d r} \exp\left(-\frac{2\alpha d}{h_+^r}\right) [e^{-4\alpha d\delta} - e^{-2\alpha d(D-2\delta)}](1 + o(1)),
\end{aligned}$$

as $n \to \infty$. Substituting this into (71) and using (77) of Lemma 9 we obtain

$$\begin{aligned}
\|f_{n1} - f_{n2}\|_2^2 &\ge 4L\frac{d-1}{d} \exp\left(-\frac{2\alpha}{h_+^r}\right) [e^{-4\alpha d\delta} - e^{-2\alpha d(D-2\delta)}](1 + o(1)) \\
&= 4L \exp\left(-\frac{2\alpha}{h_*^r}\right) (1 - \sqrt{\delta})[e^{-4\alpha\sqrt{\delta}} - e^{-2\alpha(D-2\delta)/\sqrt{\delta}}](1 + o(1)) \\
&= 4\varphi_n^2(\mathbb{L}_2)(1 - \sqrt{\delta})[e^{-4\alpha\sqrt{\delta}} - e^{-2\alpha(D-2\delta)/\sqrt{\delta}}](1 + o(1)),
\end{aligned}$$

as $n \to \infty$, (cf. the definition of $\varphi_n(\mathbb{L}_2)$ in (15)).

4. Similarly to the proof of assertion 4 of Lemma 6, we obtain

$$n\chi^2(f_{n1}^Y, f_{n2}^Y) \le c_4'(T_{n1} + T_{n2}), \quad (72)$$

for n and M large enough, where T_{n1} and T_{n2} are defined in (62) and $c_4' < \infty$ is a constant. The only difference from the proof of Lemma 6 is that the function $H(x) = H(x, h_+)$ is now defined as the inverse Fourier transform of (53) and not as that of (52). As in (63) – (65), we get, for n large enough,

$$\begin{aligned} & T_{n1} = n\|H * f^\varepsilon\|_2^2 \\ \le\ & b_{\max}^2 \alpha r L(d-1) n h_+^{1-r} \exp\left(\frac{2(d-1)\alpha}{h_+^r}\right) \int_{Dom} |u|^{2\gamma'} e^{-2\alpha d|u|^r - 2\beta|u|^s} du \\ \le\ & c' n h_+^{1-r} \exp\left(-\frac{2\alpha}{h_+^r}\right) \int_{1/h_+}^{\infty} u^{2\gamma'} \exp\left(-2\beta u^s\right) du \\ \le\ & c'' n h_+^{s-2\gamma'-r} \exp\left(-\frac{2\alpha}{h_+^r} - \frac{2\beta}{h_+^s}\right) = o(1), \end{aligned} \tag{73}$$

as $n \to \infty$, where $c' > 0$ and $c'' > 0$ are some finite constants.

Next, similarly to (67), we have, for n large enough and all $u \in \mathbb{R}$,

$$|(\Phi^H(u, h_+)\Phi^\varepsilon(u))''| \le B_2 h_+^{(1-r)/2} \exp\left(\frac{(d-1)\alpha}{h_+^r}\right) |u|^{a'} e^{-2\alpha d|u|^r - \beta|u|^s},$$

where $B_2 < \infty$ and $a' \in \mathbb{R}$ are some constants. This implies, as in (68)–(69), that

$$\begin{aligned} T_{n2} &= \frac{n}{2\pi} \int |(\Phi^H(u, h_+)\Phi^\varepsilon(u))''|^2 du \\ &\le \frac{n}{\pi} B_2^2 h_+^{1-r} \exp\left(-\frac{2\alpha}{h_+^r}\right) \int_{1/h_+}^{\infty} u^{2a'} \exp(-2\beta u^s) du \\ &\le \bar{c} n h_+^{-2a'+s-r} \exp\left(-\frac{2\alpha}{h_+^r} - \frac{2\beta}{h_+^s}\right) = o(1), \end{aligned} \tag{74}$$

as $n \to \infty$, where $\bar{c} > 0$ is finite constant. It remains now to combine (72) – (74).

Proof of (39) is now obtained following the same lines as the proof of (38) in Section 5.2, but with $d(f_{n1}, f_{n2}) = \|f_{n1} - f_{n2}\|_2$ and $\psi_n = \varphi_n(\mathbb{L}_2)\Big((1-\sqrt{\delta})[e^{-4\alpha\sqrt{\delta}} - e^{-2\alpha(D-2\delta)/\sqrt{\delta}}]\Big)^{1/2}$. □

Appendix

Let $(\mathcal{X}, \mathcal{A})$ and $(\Theta, \mathcal{T})$ be measurable spaces and let P_1 and P_2 be two probability measures on $\mathcal{A}$. Let $d : (\Theta \times \Theta, \mathcal{T} \otimes \mathcal{T}) \to (\mathbb{R}_+, \mathcal{B})$ be a non-negative measurable function where $\mathcal{B}$ is the Borel σ-algebra. Define

$$R = \inf_{\hat{\theta}} \max_{i \in \{1,2\}} E_i[d(\hat{\theta}, \theta_i)],$$

where $\inf_{\hat{\theta}}$ denotes the infimum with respect to all the measurable mappings $\hat{\theta} : (\mathcal{X}, \mathcal{A}) \to (\Theta, \mathcal{T})$, E_i denotes the expectation with respect to P_i, and θ_1, θ_2 are two elements of Θ.

Lemma 8 *Suppose that:*

(i) $d(\cdot,\cdot)$ *satisfies the triangle inequality,*

(ii) $\theta_1, \theta_2 \in \Theta$ *are such that* $d(\theta_1, \theta_2) \geq 2\psi$, *for some* $\psi > 0$,

(iii) $P_2 \ll P_1$ *and there exist constants* $\tau > 0$ *and* $0 < \gamma_0 < 1$ *such that*

$$P_1 \left[\frac{dP_2}{dP_1} \geq \tau \right] \geq 1 - \gamma_0.$$

Then

$$R \geq \psi(1 - \gamma_0) \min\{\tau, 1\}. \tag{75}$$

Furthermore, if instead of (iii) we suppose that

(iv) $\chi^2(P_1, P_2) \leq \gamma_0^2$, *where* $0 < \gamma_0 < 1$ *and*

$$\chi^2(P_1, P_2) = \int \left(\frac{dP_2}{dP_1} - 1 \right)^2 dP_1,$$

then

$$R \geq \psi(1 - \gamma_0)(1 - \sqrt{\gamma_0}). \tag{76}$$

Proof. We first show (75). We have

$$\begin{aligned} R &\geq \frac{1}{2} \inf_{\hat{\theta}} \left(E_1[d(\hat{\theta}, \theta_1)] + E_2[d(\hat{\theta}, \theta_2)] \right) \\ &\geq \frac{1}{2} \inf_{\hat{\theta}} \left(E_1[d(\hat{\theta}, \theta_1)] + \tau E_1 \left[I \left(\frac{dP_2}{dP_1} \geq \tau \right) d(\hat{\theta}, \theta_2) \right] \right) \\ &\geq \frac{\min\{\tau, 1\}}{2} \inf_{\hat{\theta}} E_1 \left[I \left(\frac{dP_2}{dP_1} \geq \tau \right) [d(\hat{\theta}, \theta_1) + d(\hat{\theta}, \theta_2)] \right]. \end{aligned}$$

Using here the triangle inequality and $(ii) - (iii)$, we find

$$R \geq \psi \min\{\tau, 1\} P_1 \left[\frac{dP_2}{dP_1} \geq \tau \right] \geq \psi(1 - \gamma_0) \min\{\tau, 1\}.$$

To show (76) it is sufficent to note that, in view of Chebyshev's inequality

$$\begin{aligned} P_1 \left[\frac{dP_2}{dP_1} \geq 1 - \sqrt{\gamma_0} \right] &= 1 - P_1 \left[\frac{dP_2}{dP_1} - 1 < -\sqrt{\gamma_0} \right] \\ &\geq 1 - \frac{1}{\gamma_0} \int \left(\frac{dP_2}{dP_1} - 1 \right)^2 dP_1 \geq 1 - \gamma_0, \end{aligned}$$

and thus (iv) implies (iii) with $\tau = 1 - \sqrt{\gamma_0}$. □

Lemma 9 *Let* $0 < r < s < \infty$ *and let* $h_+ = h_+(n)$ *be the solution of* (51). *Then* $h_+(n) = (\log n/(2\beta))^{-1/s}(1 + o(1))$*,*

$$h_+^a \exp\left(-\frac{2\alpha}{h_+^r}\right) = h_*^a \exp\left(-\frac{2\alpha}{h_*^r}\right)(1 + o(1)), \text{ as } n \to \infty \tag{77}$$

and

$$(\log n)^b n \exp\left(-\frac{2\alpha}{h_+^r} - \frac{2\beta}{h_+^s}\right) = o(1), \tag{78}$$

as $n \to \infty$*, for any* $a \in \mathbb{R}$*,* $b \in \mathbb{R}$*.*

Proof is analogous to that of Lemma 4 in Part I. □

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