STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

LORENZO ZAMBOTTI

For questions, corrections and remarks, please write to lorenzo.zambotti@upmc.fr.

1. Introduction

The aim of these notes is to give an introduction to some aspects of the theory of Stochastic Partial Differential Equations. The focus will not be on generality but on presenting interesting and useful concepts in particular cases.

First consider the SDE in $\mathbb{R}^d$

$$dX = -\frac{1}{2}\nabla U(X)\,dt + dW_t, \qquad X_0 = x \tag{1.1}$$

where $U : \mathbb{R}^d \mapsto \mathbb{R}$ is a smooth function and $W_t = (W_t^1, \dots, W_t^d)$ is a d-dimensional Brownian motion. This equation can be seen as a stochastic perturbation of the standard differential equation $\dot{x} = -\frac{1}{2}\nabla U(x)$ and it can be called a *stochastic gradient system.*

It turns out that $(X_t, t \geq 0)$ is a Markov process, and it is natural to look for its invariant probability measures, i.e. for the probability measures μ on $\mathbb{R}^d$ such that $(X_t, t \geq 0)$ is stationary if X_0 has law μ.

It is a classical remark due to Kolmogorov, that if $Z := \int \exp(-U(z))\,dz < +\infty$, then the solutions of equations like (1.1) have an explicit invariant probability measures

$$\mu(dx) = \frac{1}{Z}\exp(-U(x))\,dx,$$

which turns out to be reversible and the *only* invariant probability measure. We recall that uniqueness of invariant measures implies ergodicity and gives therefore crucial information about the asymptotic behavior of the solutions.

We shall recall that the infinitesimal generator of X:

$$\mathcal{L}f := \frac{1}{2}\Delta f - \frac{1}{2}\langle \nabla U, \nabla f\rangle$$

is self-adjoint in $L^2(\mu)$, we shall introduce the associated Dirichlet form, and we shall show that the link between μ, $\mathcal{L}$ and equation (1.1) is encoded in the *integration by parts formula*

$$\int \langle \nabla f, h\rangle\,d\mu = \int f\,\langle \nabla U, h\rangle\,d\mu, \tag{1.2}$$

where $h \in \mathbb{R}^d$ is constant.

In this course we will show how the previous techniques can be applied to the study of infinite-dimensional processes, including solutions of stochastic partial differential equations.

For instance, consider the space $H := L^2(0,1)$ w.r.t. the Lebesgue measure, and the function $U : D(U) \subset L^2(0,1) \mapsto \mathbb{R}$

$$D(U) := \{h \in H : h' \in H,\, h(0) = h(1) = 0\} = H_0^1(0,1), \qquad U(h) := \int_0^1 |h'|^2\,d\theta.$$

Then we have formally that the gradient of U is $\nabla U(h) = -h''$. If we consider a complete orthonormal basis $\{e_k\}_{k\geq 1}$ in H and a sequence of independent standard Brownian motions $\{W_t^k\}_{k\geq 1}$, then we can write an analog of (1.1) in our infinite-dimensional setting

$$dX = -\frac{1}{2}\nabla U(X)\,dt + dW_t = \frac{1}{2}\,\frac{\partial^2}{\partial\theta^2}X\,dt + \sum_{k\geq 1} e_k\,dW_t^k.$$

This is called the stochastic heat equation. The process X_t lives in a.s. in $L^2(0,1)$: therefore it has two natural parameters, the time $t \geq 0$ and the space $\theta \in [0,1]$. It turns out that there exists a continuous version of this process $X_{t,\theta}$ as function of (t,θ). However the typical behavior of $\theta \mapsto X_{t,\theta}$ is that of Brownian bridge, in particular not more that $\frac{1}{2}$-Hölder continuous. Therefore the second derivative of X does not exist and the equation has to be interpreted in a suitable way. In this case the invariant measure of X is the law of the Brownian bridge over $[0,1]$. More generally, the same techniques allows to treat stochastic reaction-diffusion equations

$$dX = \frac{1}{2}\left(\frac{\partial^2}{\partial\theta^2}X\ - f'(X)\right)dt + dW_t.$$

The integration by parts formula (1.2) can be written also for this infinite-dimensional setting and is the starting point of the Malliavin calculus.

2. SDEs in $\mathbb{R}^d$

2.1. **The equation.** Let $F : \mathbb{R}^d \mapsto \mathbb{R}^d$ be a Lipschitz map, i.e. such that

$$\sup\left\{\frac{|F(x)-F(y)|}{|x-y|} : x,y\in\mathbb{R}^d,\ x\neq y\right\} =: L < \infty.$$

Then we have the classical

Theorem 2.1. *Let $x \in \mathbb{R}^d$, $T \geq 0$ and $w : [0,T] \mapsto \mathbb{R}^d$ continuous. There exists a unique continuous $(X_t(x))_{t\in[0,T]}$ such that*

$$X_t(x) = x + \int_0^t F(X_s(x))\,ds + w_t, \qquad \forall\ t\in[0,T]. \tag{2.1}$$

Moreover we have continuity w.r.t. the initial data:

$$|X_t(x) - X_t(y)| \leq e^{LT}|x-y|, \qquad \forall\ t\in[0,T],\ x,y\in\mathbb{R}^d. \tag{2.2}$$

Proof. Set $B := C([0,T];\mathbb{R}^d)$ with norm $\|f\|_B := \sup_{t\in[0,T]} e^{-Lt}|f_t|$. Then $(B, \|\cdot\|_B)$ is a Banach space. Set $\Gamma : B \mapsto B$:

$$\Gamma(f)_t := x + \int_0^t F(f_s)\,ds + w_t, \qquad \forall\ t\in[0,T].$$

Then it is easy to see that

$$\|\Gamma(f)-\Gamma(g)\|_B \leq (1-e^{-LT})\,\|f-g\|_B, \qquad \forall\, f,g\in B$$

so that Γ is a contraction in B and has a unique fixed point, which is the process X we are looking for. Moreover

$$X = \lim_{n\to\infty}\Gamma^n(\overline{x}) \tag{2.3}$$

in the norm of B, where $\Gamma^n = \Gamma \circ \cdots \circ \Gamma$ (n times) and $\overline{x} : [0,T] \mapsto \mathbb{R}^d$ is the function constant equal to x. Finally, notice that

$$|X_t(x) - X_t(y)| \le |x-y| + L \int_0^t |X_s(x) - X_s(y)|\, ds, \qquad \forall\, t \in [0,T].$$

Then (2.2) follows from Gronwall's Lemma. □

2.2. **Flow property.** Let $\{B^1, \ldots, B^d\}$ be independent standard Brownian motions (BM) in $\mathbb{R}$, and denote by $W = (B^1, \ldots, B^d)$ a d-dimensional BM. We define the σ-algebras

$$\mathcal{F}_{s,t} := \sigma\{W_r - W_u, s < u \le r < r\}, \qquad -\infty \le s \le t \le +\infty.$$

Arguing as in the proof of Theorem 2.1, we obtain a double-index process $(X_{s,t}(x), -\infty < s \le t < +\infty, x \in \mathbb{R})$, unique solution of

$$X_{s,t}(x) = x + \int_s^t F(X_{s,u}(x))\, du + W_t - W_s, \qquad -\infty < s \le t < +\infty, \tag{2.4}$$

where we consider two independent standard BMs (W^1, W^2) in $\mathbb{R}^d$ and define the *two-sided BM*

$$W_t := \begin{cases} W^1_t, & t \ge 0, \\ \\ W^2_{-t}, & t < 0. \end{cases} \tag{2.5}$$

Remark that $(W_t - W_s)_{s,t\in\mathbb{R}, s\le t}$ is a centered Gaussian process such that

- $W_t - W_s$ is independent of $W_r - W_u$ if $]s,t[\cap]u,r[= \emptyset$,
- $W_{t+u} - W_{s+u}$ has same distribution as $W_t - W_s$.

We also notice that $(W_{0,u} = W_u, u \ge 0)$ is a standard BM. Setting $X_t(x) := X_{0,t}(x)$, $t \ge 0$, then this process is the solution of the usual SDE

$$X_t(x) = x + \int_0^t F(X_u(x))\, du + W_t, \qquad t \ge 0. \tag{2.6}$$

Since we can construct $(X_{s,t}(x), -\infty < s \le t < +\infty, x \in \mathbb{R})$ using Theorem 2.1, which is a deterministic construction working for fixed $w \in C([0,T];\mathbb{R}^d)$, then we can consider $X_{s,t} : \mathbb{R}^d \mapsto \mathbb{R}^d$, as a *random map* associating $X_{s,t}(x)$ to x. This is trivial in this case, because w plays the role of a parameter. In the case of a non-constant diffusion coefficient, a proof is required since one has first a construction of $(X_{s,t}(x), -\infty < s \le t < +\infty, x \in \mathbb{R})$ for fixed x and then one has to prove that it is possible to realize the process for all x on the same event of probability 1.

Proposition 2.2. *The family $(X_{s,t}, -\infty < s \le t < +\infty)$ is a stationary stochastic flow, i.e. it satisfies*

1. *For all $-\infty < s \le r \le t < +\infty$: $X_{r,t} \circ X_{s,r} = X_{s,t}$*
2. *For all $-\infty < t_1 \le \cdots \le t_n < +\infty$, the family $\{X_{t_i,t_{i+1}}\}_{i=1,\ldots,n-1}$ is independent*
3. *For all $-\infty < s \le t < +\infty$ and $u \ge 0$, $X_{s+u,t+u}$ has same distribution as $X_{s,t}$.*

Proof. This follows from uniqueness of solutions of (2.4), the above mentioned properties of $(W_t - W_s)_{s,t\in\mathbb{R},s\le t}$ and the fact that $X_{r,t}$ is constructed as the limit (2.3) of $\mathcal{F}_{r,t}$-measurable functions. □

2.3. **The transition semigroup.** We set now $C_b(\mathbb{R}^d) := \{f : \mathbb{R}^d \mapsto \mathbb{R}$ bounded and continuous$\}$, endowed with the norm

$$\|f\|_\infty := \sup_{\mathbb{R}^d} |f|.$$

We set for $f \in C_b(\mathbb{R}^d)$

$$P_t f(x) = \mathbb{E}[f(X_t(x))], \qquad t \geq 0, \ x \in \mathbb{R}^d.$$

Notice that for all $f \in C_b(\mathbb{R}^d)$

$$\|P_t f\|_\infty \leq \|f\|_\infty.$$

By (2.2) we obtain that $P_t : C_b(\mathbb{R}^d) \mapsto C_b(\mathbb{R}^d)$, i.e. (P_t) is a *Feller* semigroup. We recall the following important fact

Proposition 2.3. *The family $(P_t)_{t\geq 0}$ of operators on $C_b(\mathbb{R}^d)$ forms a semigroup, i.e. $P_s P_t = P_{t+s}$, $t, s \geq 0$.*

Proof. Let $f \in C_b(\mathbb{R}^d)$. Notice that, since $(W_{s+t} - W_s, t \geq 0)$ has the same distribution as $(W_t, t \geq 0)$, then $(X_{s,t+s}(y), t \geq 0)$ has the same distribution as $(X_t(y), t \geq 0)$, for any $y \in \mathbb{R}^d$. Then

$$\mathbb{E}[f(X_{s,t+s}(y))] = \mathbb{E}[f(X_t(y))] = P_t f(y).$$

By independence of $\mathcal{F}_{0,s}$ and $\mathcal{F}_{s,s+t}$ we have

$$\mathbb{E}[f(X_{s,t+s}(X_{0,s}(x))) \,|\, \mathcal{F}_{0,s}] = P_t f(X_{0,s}(x)), \qquad \text{a.s.}$$

and therefore

$$\begin{aligned} P_{t+s} f(x) &= \mathbb{E}[f(X_{0,t+s}(x))] = \mathbb{E}[f(X_{s,t+s}(X_{0,s}(x))] = \mathbb{E}[\mathbb{E}[f(X_{s,t+s}(X_{0,s}(x))) \,|\, \mathcal{F}_{0,s}]] \\ &= \mathbb{E}[P_t f(X_{0,s}(x))] = P_s P_t f(x). \end{aligned}$$

□

We could actually prove that X is a *strong Markov* process.

2.4. **Invariant measures.**

The case of a Markov chain. Let E be a countable set and $Q : E \times E \mapsto [0,1]$ a transition probability matrix, i.e. such that $\sum_{y\in E} Q(x,y) = 1$ for all $x \in E$. A probability measure μ on E is said to be *invariant* if

$$\mu Q(x) = \sum_{y\in E} \mu(y)\, Q(y,x) = \mu(x), \qquad \forall\, x \in E.$$

A probability measure μ on E is said to be *reversible* if

$$\mu(x)\, Q(x,y) = \mu(y)\, Q(y,x), \qquad \forall\, x, y \in E.$$

A reversible measure is necessarily invariant. Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix Q and initial law μ. Then

- if μ is invariant, then $(X_n)_{n\geq 0}$ is *stationary*: for all $k \geq 0$, $(X_{n+k}, n \geq 0)$ has same distribution as $(X_n, n \geq 0)$
- if μ is reversible, then $(X_n)_{n\geq 0}$ is *reversible*: for all $k \geq 0$, $(X_{k-n}, n = 0, \ldots, k)$ has same distribution as $(X_n, n = 0, \ldots, k)$.

Back to (2.1). We say that a Borel probability measure μ on $\mathbb{R}^d$ is an invariant measure of X if

$$\int_{\mathbb{R}^d} P_t f \, d\mu = \int_{\mathbb{R}^d} f \, d\mu, \qquad \forall\ t \geq 0,\ f \in C_b(\mathbb{R}^d).$$

We say that μ is a reversible measure if

$$\int_{\mathbb{R}^d} g \cdot P_t f \, d\mu = \int_{\mathbb{R}^d} f \cdot P_t g \, d\mu, \qquad \forall\ t \geq 0,\ f, g \in C_b(\mathbb{R}^d).$$

Notice that a reversible measure is invariant, since for $g \equiv 1$ we have $P_t g \equiv 1$. We have the following

Proposition 2.4. *Let μ be an invariant probability measure of (P_t). If η is a r.v. with distribution μ and independent of $(W_t)_{t\geq 0}$ and wet set $\overline{X}_t := X_t(\eta)$, $t \geq 0$, then the process $(\overline{X}_t)_{t\geq 0}$ is stationary, i.e. $(\overline{X}_{t+h})_{t\geq 0}$ has the same law for any $h \geq 0$. If moreover μ is reversible, $(\overline{X}_t)_{t\geq 0}$ is reversible, i.e. $(\overline{X}_t)_{t\in[0,T]}$ has the same law as $(\overline{X}_{T-t})_{t\in[0,T]}$ for all $T > 0$.*

Proof. It is enough to check equality of finite dimensional distributions. As an exercise, revise the proofs for a Markov chain and translate it in this framework. □

Proposition 2.5. *Let μ be an invariant measure of (P_t). For any $p \geq 1$, P_t has a unique extension to a contraction in $L^p(\mu)$, that we call P_t again:*

$$\|P_t f\|_{L^p(\mu)} \leq \|f\|_{L^p(\mu)}, \qquad \forall\ t \geq 0,\ f \in L^p(\mu).$$

Moreover $(P_t)_{\geq 0}$ is a semigroup in $L^p(\mu)$ and if μ is reversible then P_t is symmetric in $L^2(\mu)$.

Proof. Let $f \in C_b(\mathbb{R}^d) \subset L^p(\mu)$. Then for all $x \in \mathbb{R}^d$, by Hölder's inequality

$$|P_t f(x)|^p = |\mathbb{E}[f(X_t(x))]|^p \leq \mathbb{E}[|f(X_t(x))|^p] = P_t|f|^p(x).$$

Then by invariance of μ

$$\|P_t f\|^p_{L^p(\mu)} = \int |P_t f(x)|^p \, \mu(dx) \leq \int P_t |f|^p(x) \, \mu(dx) = \int |f|^p(x) \, \mu(dx) = \|f\|^p_{L^p(\mu)}.$$

By the density of $C_b(\mathbb{R}^d)$ in $L^p(\mu)$ we conclude. □

2.5. **The infinitesimal generator.** There are several possible definitions of the generator of a Markov process. A convenient one is the following: we define the infinitesimal generator as the set of pairs $(f, Lf) \in C_b(\mathbb{R}^d) \times C_b(\mathbb{R}^d)$ such that

$$f(X_t(x)) - f(x) - \int_0^t Lf(X_s(x)) \, ds, \qquad t \geq 0,$$

is a $(\mathcal{F}_t)$-martingale for all $x \in \mathbb{R}^d$. By the Ito formula, if X is solution of (2.6) and $f \in C^2_b(\mathbb{R}^d)$, then

$$f(X_t(x)) - f(x) - \int_0^t Lf(X_s(x)) \, ds = \int_0^t \langle \nabla f(X_s(x)), dW_s \rangle,$$

where

$$Lf(y) := \frac{1}{2}\Delta f(y) + \langle F(y), \nabla f(y) \rangle, \quad y \in \mathbb{R}^d,$$

so that (f, Lf) belongs to the infinitesimal generator of X.

3. A prototype: The Ornstein-Uhlenbeck process

Let $\lambda > 0$ and $x \in \mathbb{R}$ and let us consider the SDE in $\mathbb{R}$

$$\begin{cases} dX_t = -\lambda\, X_t\, dt + dW_t \\ \\ X_0 = x \end{cases}$$

where $(W_t)_{t\in\mathbb{R}}$ is a two-sided BM as in (2.5). By Itô's formula

$$d(e^{\lambda t} X_t) = e^{\lambda t}\,(\lambda\, X_t\, dt - \lambda\, X_t\, dt + dW_t) = e^{\lambda t}\, dW_t$$

so that

$$e^{\lambda t} X_t = x + \int_0^t e^{\lambda u}\, dW_u$$

and we obtain an explicit formula for the unique solution

$$X_t(x) = e^{-\lambda t} x + \int_0^t e^{-\lambda(t-u)}\, dW_u.$$

3.1. **First properties.**

The law of $X_t(x)$ is easily computed: it is Gaussian with mean and variance:

$$\mathbb{E}(X_t(x)) = e^{-\lambda t} x, \qquad \mathrm{Var}(X_t(x)) = \int_0^t e^{-2\lambda(t-u)}\, du = \frac{1}{2\lambda}(1 - e^{-2\lambda t})$$

so that

$$X_t(x) \sim \mathcal{N}\left(e^{-\lambda t} x, \frac{1}{2\lambda}(1 - e^{-2\lambda t})\right).$$

We recall that the characteristic function of a Gaussian r.v. $V \sim \mathcal{N}(a, q)$ is

$$\mathbb{E}(e^{iV\alpha}) = e^{ia\alpha - \frac{1}{2} q \alpha^2}, \qquad \alpha \in \mathbb{R}.$$

We can see that

$$\lim_{t\to+\infty} \mathbb{E}(X_t(x)) = 0, \qquad \lim_{t\to+\infty} \mathrm{Var}(X_t(x)) = \frac{1}{2\lambda}$$

and therefore we obtain that

$$\mathcal{N}\left(e^{-\lambda t} x, \frac{1}{2\lambda}(1 - e^{-2\lambda t})\right) \rightharpoonup \mathcal{N}\left(0, \frac{1}{2\lambda}\right) =: \mu_\lambda. \tag{3.1}$$

3.2. **Flow property.**

We define now a double-index process $(X_{s,t}(x), -\infty < s \le t < +\infty, x \in \mathbb{R})$, unique solution of

$$X_{s,t} = x - \lambda \int_s^t X_{s,u}\, du + W_t - W_s, \qquad -\infty < s \le t < +\infty,$$

where $W_t - W_s$ is defined as in (2.5). Applying Itô's formula to $t \to (e^{\lambda t} X_t)$ we obtain the explicit formula for the unique solution

$$X_{s,t}(x) = e^{-\lambda(t-s)} x + \int_s^t e^{-\lambda(t-u)}\, dW_u, \qquad -\infty < s \le t < +\infty.$$

One can easily check directly that $X_{s,t}(x)$ defines a stochastic flow. For instance

$$\begin{aligned} X_{r,t}(X_{s,r}(x)) &= e^{-\lambda(t-r)}\left(e^{-\lambda(r-s)}x + \int_s^r e^{-\lambda(r-u)}\,dW_u\right) + \int_r^t e^{-\lambda(t-u)}\,dW_u \\ &= e^{-\lambda(t-r+r-s)}x + \int_s^r e^{-\lambda(t-r+r-u)}\,dW_u + \int_r^t e^{-\lambda(t-u)}\,dW_u \\ &= e^{-\lambda(t-s)}x + \int_s^t e^{-\lambda(t-u)}\,dW_u = X_{s,t}(x). \end{aligned}$$

The other properties are consequences of the independence and stationarity of increments of BM.

3.3. **The invariant measure.** By Proposition 2.3, the family $(P_t)_{t\geq 0}$ of operators on $C_b(\mathbb{R})$

$$P_t f(x) = \mathbb{E}[f(X_t(x))], \qquad t \geq 0,\ x \in \mathbb{R},$$

forms a semigroup, i.e. $P_sP_t = P_{t+s}$, $t,s \geq 0$. Then we have the

Proposition 3.1. *The unique invariant measure of X is $\mu_\lambda = \mathcal{N}(0,(2\lambda)^{-1})$.*

Proof. By the semigroup law and the limit (3.1), for all $f \in C_b(\mathbb{R})$:

$$\int f\,d\mu_\lambda = \lim_{s\to+\infty} P_{t+s}f(x) = \lim_{s\to+\infty} P_sP_tf(x) = \int P_tf\,d\mu_\lambda.$$

Moreover, if ν is another invariant measure

$$\int f\,d\nu = \int P_tf\,d\nu \to \int\left(\int f\,d\mu_\lambda\right)d\nu = \int f\,d\mu_\lambda, \qquad \forall\ f \in C_b(\mathbb{R}).$$

□

3.4. **Stationary O-U process.** An interesting question is the following: since the law of $X_t(x)$ converges, as $t \to +\infty$, to μ_λ, can one expect convergence of the trajectories of $X_t(x)$, as $t \to +\infty$? One can see that, for instance, convergence in L^2 can be ruled out, since

$$\begin{aligned} \mathrm{Var}(X_{t+h}(x) - X_t(x)) &= \int_0^t \left(e^{-\lambda(t+h-u)} - e^{-\lambda(t-u)}\right)^2 du + \int_t^{t+h} e^{-2\lambda(t+h-u)}\,du \\ &\geq \int_t^{t+h} e^{-2\lambda(t+h-u)}\,du = \frac{1-e^{-2\lambda h}}{2\lambda} \end{aligned}$$

which does not tend to 0 as $t \to +\infty$. If $\lim_{t\to+\infty} X_t(x) = \ell \in \mathbb{R}$ for all ω in an event with positive probability, then for all such ω

$$0 = \lim_{t\to+\infty} \frac{X_t(x)}{t} = \lim_{t\to+\infty}\left(\frac{x}{t} - \frac{\lambda}{t}\int_0^t X_{s,u}(x)\,du + \frac{W_t}{t}\right) = -\lambda\,\ell$$

so that $\ell = 0$; but then the limit law μ_λ should give positive probability to 0, which is not the case. Therefore, a.s. the limit of $X_t(x)$ as $t \to +\infty$ does not exist.

However, one can perform a different limit procedure: let us recall that, for fixed $s \geq 0$, $(X_{t+s}, t \geq 0)$ has same distribution as $(X_{-s,t}, t \geq 0)$. Now it is easy to see that

$$\lim_{s\to+\infty} X_{-s,t} = \lim_{s\to+\infty}\left(e^{-\lambda(t+s)}x + \int_{-s}^t e^{-\lambda(t-u)}\,dW_u\right) = \int_{-\infty}^t e^{-\lambda(t-u)}\,dW_u$$

in L^2 for all $t \geq 0$, since

$$\mathrm{Var}\left(\int_{-\infty}^{t} e^{-\lambda(t-u)}\, dW_u - \int_{-s}^{t} e^{-\lambda(t-u)}\, dW_u\right) = \int_{-\infty}^{-s} e^{-2\lambda(t-u)}\, du = \frac{e^{-2\lambda(t+s)}}{2\lambda} \to 0$$

as $s \to +\infty$. Therefore, we define

$$\overline{X}_t := \int_{-\infty}^{t} e^{-\lambda(t-u)}\, dW_u, \qquad t \in \mathbb{R}.$$

It is easy to see that

Proposition 3.2. *$\overline{X}$ is stationary (i.e. $(\overline{X}_{t+s}, t \in \mathbb{R})$ and has same distribution as $(\overline{X}_t, t \in \mathbb{R})$) solution of*

$$dX_t = -\lambda X_t\, dt + dW_t, \qquad t \in \mathbb{R}.$$

The law of $\overline{X}_t$ is μ_λ for all $t \in \mathbb{R}$.

3.5. **Reversibility.**

Proposition 3.3. *The probability measure μ_λ is reversible for the O-U process.*

Proof. The reversibility statement can be rewritten as

$$\mathbb{E}\left[g(Z_1)\, f\left(e^{-\lambda t} Z_1 + \sqrt{1-e^{-2\lambda t}}\, Z_2\right)\right] = \mathbb{E}\left[g\left(e^{-\lambda t} Z_1 + \sqrt{1-e^{-2\lambda t}}\, Z_2\right)\, f(Z_1)\right],$$

for all $f, g \in C_b(\mathbb{R})$, where Z_1, Z_2 are i.i.d. variables with law $\mathcal{N}(0, 1/(2\lambda))$. Let now $\theta \in [0, \pi/2]$ such that

$$e^{-\lambda t} = \cos\theta, \qquad \sqrt{1-e^{-2\lambda t}} = \sin\theta.$$

We introduce the rotation of $\mathbb{R}^2$

$$T_\alpha := \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \qquad \alpha \in \mathbb{R},$$

and we set $\mathcal{Z} = (Z_1, Z_2)$, $\mathcal{Z}(\alpha) := T_\alpha \mathcal{Z}$. Notice that

$$\mathcal{Z} \sim \mathcal{N}\left((0,0), \frac{1}{2\lambda} I\right) \implies \mathcal{Z}(\alpha) \sim \mathcal{N}\left(T_\alpha(0,0), \frac{1}{2\lambda} T_\alpha \cdot I \cdot T_\alpha^t\right) = \mathcal{N}\left((0,0), \frac{1}{2\lambda} I\right)$$

In particular, $\mathcal{Z}(\alpha)$ is equal in law to $\mathcal{Z}$ for all α. The reversibility statement follows easily. □

Exercise 3.4. Show directly the *reversibility* of the process $\overline{X}$: for all $T \geq 0$, the processes $(\overline{X}_{T-t}, t \in [0,T])$ and $(\overline{X}_t, t \in [0,T])$ have the same distribution.

3.6. **Integration by parts.** By Ito's formula, it is easy to see that for all $f \in C_c^2(\mathbb{R})$ we have that (f, Lf) belongs to the infinitesimal generator, where

$$Lf(y) := \frac{1}{2}\frac{d^2 f}{dy^2}(y) - \lambda y \frac{df}{dy}(y), \qquad y \in \mathbb{R}.$$

In this case we have a very important formula: for all $f, g \in C_c^2(\mathbb{R})$

$$\int_{\mathbb{R}} g\, Lf\, d\mu_\lambda = -\frac{1}{2}\int_{\mathbb{R}} g' f'\, d\mu_\lambda = \int_{\mathbb{R}} Lg\, f\, d\mu_\lambda. \tag{3.2}$$

Indeed, by integration by parts we obtain

$$\int_{\mathbb{R}} g(y)\left(\frac{1}{2}\frac{d^2 f}{dy^2}(y)-\lambda y\frac{df}{dy}(y)\right)e^{-\frac{y^2}{2\lambda}}\,dy =$$
$$= \int_{\mathbb{R}} \frac{df}{dy}(y)\left(-\frac{d}{dy}\left(g(y)\,e^{-\frac{y^2}{2\lambda}}\right)-\lambda y g(y)\,e^{-\frac{y^2}{2\lambda}}\right)dy = -\frac{1}{2}\int_{\mathbb{R}}\frac{df}{dy}(y)\frac{dg}{dy}(y)\,e^{-\frac{y^2}{2\lambda}}\,dy$$

and since this expression is symmetric in f, g, we obtain also the last equality in (3.2).

We shall later show that formula (3.2) is a particular case of a class of functional relations, linking generators and *Dirichlet forms*, which will be defined below. For now, notice that (3.2) encodes a symmetry property in $L^2(\mu_\lambda)$, to be made more precise, and this is clearly related to the symmetry of the semigroup, i.e. to the reversibility of μ_λ.

3.7. **Hermite polynomials.** We recall that the space $L^2(\mu_\lambda)$ is a Hilbert space if endowed with the canonical scalar product

$$\langle f, g\rangle_{\mu_\lambda} := \int_{\mathbb{R}} f\,g\,d\mu_\lambda, \qquad f,g\in L^2(\mu_\lambda).$$

The reversibility of μ_λ implies that the operator $P_t : L^2(\mu_\lambda)\mapsto L^2(\mu_\lambda)$ is symmetric in the Hilbert space $L^2(\mu_\lambda)$. We want now to diagonalize it.

Let us define for all $n\in\mathbb{N}$ and $x\in\mathbb{R}$:

$$H_0(x) := 1, \qquad H_n(x) := (-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}, \qquad n\geq 1.$$

Let us set for all $t, x\in\mathbb{R}$

$$F(t,x) := \exp\left(tx-\frac{t^2}{2}\right).$$

Then we have by analiticity in $t\in\mathbb{R}$

$$F(t,x) = e^{x^2/2}\,e^{-(t-x)^2/2} = \sum_{n=0}^{\infty} e^{x^2/2}\,\frac{t^n}{n!}\,\frac{d^n}{dt^n}e^{-(t-x)^2/2}\bigg|_{t=0} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\,H_n(x).$$

From this expression we obtain some useful equalities. First

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\,H_n'(x) = \frac{\partial}{\partial x}F(t,x) = t\,F(t,x) = \sum_{n=0}^{\infty}\frac{t^{n+1}}{n!}\,H_n'(x)$$

and by comparing the coefficients

$$H_n' = nH_{n-1}, \qquad n\geq 1. \tag{3.3}$$

Moreover

$$\sum_{n=1}^{\infty}\frac{t^{n-1}}{(n-1)!}\,H_n(x) = \frac{\partial}{\partial t}F(t,x) = (x-t)\,F(t,x) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\,xH_n'(x) - \sum_{n=0}^{\infty}\frac{t^{n+1}}{n!}\,H_n(x)$$

and by comparing the coefficients

$$H_n = xH_{n-1} - H_{n-1}', \qquad n\geq 1. \tag{3.4}$$

By induction on $n\geq 0$, we can prove from (3.3) that H_n is for all $n\geq 0$ a polynomial of degree and the coefficient of the monomial x^n is equal to 1.

Proposition 3.5. *The family* $(H_n/\sqrt{n!})_{n\geq 0}$ *is a complete orthonormal basis of the Hilbert space* $L^2(\mathbb{R};\mathcal{N}(0,1))$.

Proof. Let $Z \sim \mathcal{N}(0,1)$. Notice that $H_n \in L^2(\mathbb{R}; \mathcal{N}(0,1))$, since Z has moments of all order. Now, for all $s, t \in \mathbb{R}$

$$\mathbb{E}(F(t,Z)F(s,Z)) = \mathbb{E}\left(e^{Z(s+t)-(s^2+t^2)/2}\right) = e^{(s+t)^2/2-(s^2+t^2)/2} = e^{st}$$

so that

$$\sum_{n,m\geq 0} \frac{t^n s^m}{n!\, m!}\, \mathbb{E}(H_n(Z)H_m(Z)) = \sum_{n\geq 0} \frac{(ts)^n}{n!}$$

which implies that

$$\mathbb{E}(H_n(Z)H_m(Z)) = \delta_{nm}\, n!,$$

where δ_{nm} is the Kronecker symbol, i.e. equal to 1 if $n = m$ and to 0 otherwise.

It remains to prove the completeness, i.e. fact that if the 0 function is the only function $f \in L^2(\mathbb{R}; \mathcal{N}(0,1))$ such that

$$\int_{\mathbb{R}} f\, H_n\, d\mathcal{N}(0,1) = 0, \qquad \forall\, n \geq 0.$$

Let us consider a function $f \in L^2(\mathbb{R}; \mathcal{N}(0,1))$ with such property; then

$$\int_{\mathbb{R}} f(x)\, x^n\, \mathcal{N}(0,1)(dx) = 0, \qquad \forall\, n \geq 0.$$

We define the function $G : \mathbb{C} \mapsto \mathbb{C}$

$$G(z) := \int_{\mathbb{R}} f(x)\, e^{xz-x^2/2}\, dx$$

which is entire (i.e. holomorphic all over $\mathbb{C}$), since $z \mapsto e^{uz}$ is entire and, by Cauchy-Schwartz, the function $x \mapsto f(x)(1+|x|)e^{-x^2/2}$ belongs to $L^1(\mathbb{R}; \mathrm{Leb})$. But then

$$G(z) = \sum_{n\geq 0} \int_{\mathbb{R}} f(x)\, \frac{x^n}{n!}\, e^{-x^2/2}\, dx = 0, \qquad \forall\; z \in \mathbb{C}.$$

For $z = it$, $t \in \mathbb{R}$, we obtain that the Fourier transform of $x \mapsto f(x)e^{-x^2/2}$ vanishes, and therefore $f \equiv 0$. □

3.8. **The Ornstein-Uhlenbeck operator.** Let us set

$$H_n^\lambda(x) := H_n\left(x \cdot \sqrt{2\lambda}\right), \qquad x \in \mathbb{R},\, n \geq 0,$$

$$e_n^\lambda := \frac{1}{\sqrt{n!}}\, H_n^\lambda, \qquad n \geq 0. \tag{3.5}$$

Then $(e_n^\lambda)_{n\geq 0}$ is a complete orthonormal basis of $L^2(\mu_\lambda)$. Moreover this family satisfies

$$\frac{d}{dx}\, H_n^\lambda = \sqrt{2\lambda}\, n\, H_{n-1}^\lambda, \qquad n \geq 1, \tag{3.6}$$

$$H_n^\lambda = \sqrt{2\lambda}\, x\, H_{n-1}^\lambda - \frac{1}{\sqrt{2\lambda}} \frac{d}{dx}\, H_{n-1}^\lambda, \qquad n \geq 1. \tag{3.7}$$

Let us set for all $n \geq 0$

$$L^\lambda H_n^\lambda(x) := \frac{1}{2} \frac{d^2}{dx^2}\, H_n^\lambda(x) - \lambda\, x\, \frac{d}{dx}\, H_n^\lambda(x).$$

For $n = 0$ we clearly have $LH_n^\lambda \equiv 0$, while for $n \geq 1$, by using first (3.6) and then (3.7)

$$L^\lambda H_n^\lambda(x) = \frac{n\sqrt{2\lambda}}{2} \left(\frac{d}{dx}\, H_{n-1}^\lambda(x) - 2\lambda\, x\, H_{n-1}^\lambda(x)\right) = -n\lambda\, H_n^\lambda(x)$$

Let us now notice that, by Itô's formula, for any $n \geq 0$

$$dH_n^\lambda(X_t(x)) = L^\lambda H_n^\lambda(X_t(x))\, dt + \left(\frac{d}{dx} H_n^\lambda\right)(X_t(x))\, dW_t,$$

so that

$$H_n^\lambda(X_t(x)) = H_n^\lambda(x) - n\lambda \int_0^t H_n^\lambda(X_u(x))\, ds + \int_0^t \sqrt{2\lambda}\, n\, H_{n-1}^\lambda(X_u(x))\, dW_u,$$

which can be solved explicitly

$$H_n^\lambda(X_t(x)) = e^{-n\lambda t} H_n^\lambda(x) + \int_0^t \sqrt{2\lambda}\, n\, e^{-n\lambda(t-u)}\, H_{n-1}^\lambda(X_u(x))\, dW_u.$$

By taking expectation we obtain

$$\mathbb{E}(H_n^\lambda(X_t(x))) = P_t H_n^\lambda(x) = e^{-n\lambda t} H_n^\lambda(x), \qquad n \geq 0,\ x \in \mathbb{R}.$$

Therefore, we have indeed diagonalized P_t in $L^2(\mu_\lambda)$.

Up to now, the operator L is defined only on the linear span of $\{e_n^\lambda, n \geq 0\}$. Since $(e_n^\lambda)_{n\geq 0}$ is a complete orthonormal basis of $L^2(\mu_\lambda)$, the natural domain of definition of L^λ is

$$D(L^\lambda) = \left\{ h \in L^2(\mu_\lambda) : \sum_{n\geq 0} n^2 \langle h, e_n^\lambda \rangle_{\mu_\lambda}^2 < +\infty \right\}$$

$$L^\lambda h := -\sum_{n\geq 0} \lambda\, n\, \langle h, e_n^\lambda \rangle_{\mu_\lambda}\, e_n^\lambda, \qquad \forall\, h \in D(L^\lambda).$$

3.9. **The Dirichlet form.** Let us denote by $C_b^k(\mathbb{R})$ the space of all k times differentiable $f \in C_b(\mathbb{R})$ such that $\frac{d^j f}{dx^k} \in C_b(\mathbb{R})$ for all $j = 1, \ldots, k$. We notice the following integration by parts formula: for all $f, g \in C_b^1(\mathbb{R})$

$$\int_{\mathbb{R}} f'\, g\, d\mu_\lambda = -\int_{\mathbb{R}} f\, g'\, d\mu_\lambda + 2\lambda \int_{\mathbb{R}} f\, g\, x\, d\mu_\lambda.$$

It follows that for all $f, g \in C_b^2(\mathbb{R})$

$$\int_{\mathbb{R}} (L^\lambda f)\, g\, d\mu_\lambda = \frac{1}{2} \int_{\mathbb{R}} f''\, g\, d\mu_\lambda - \lambda \int_{\mathbb{R}} f'\, g\, d\mu_\lambda = -\frac{1}{2} \int_{\mathbb{R}} f'\, g'\, d\mu_\lambda.$$

Let us read this equality in terms of the series expansion in the Hermite orthogonal basis:

$$\int_{\mathbb{R}} (L^\lambda f)\, g\, d\mu_\lambda = -\sum_{n\geq 0} \lambda\, n\, \langle f, e_n^\lambda \rangle_{\mu_\lambda}\, \langle g, e_n^\lambda \rangle_{\mu_\lambda}$$

and by (3.5) and (3.6)

$$f' = \sum_{n\geq 0} \langle h, e_n^\lambda \rangle_{\mu_\lambda} \frac{d}{dx} e_n^\lambda = \sum_{n\geq 0} \sqrt{2\lambda\, n}\, \langle f, e_n^\lambda \rangle_{\mu_\lambda}\, e_{n-1}^\lambda$$

so that indeed

$$\begin{aligned} -\frac{1}{2} \int_{\mathbb{R}} f'\, g'\, d\mu_\lambda &= -\frac{1}{2} \sum_{n,m\geq 1} 2\lambda \sqrt{n\, m}\, \langle f, e_n^\lambda \rangle_{\mu_\lambda}\, \langle g, e_m^\lambda \rangle_{\mu_\lambda}\, \langle e_{n-1}^\lambda, e_{m-1}^\lambda \rangle_{\mu_\lambda} \\ &= -\sum_{n\geq 0} \lambda\, n\, \langle f, e_n^\lambda \rangle_{\mu_\lambda}\, \langle g, e_n^\lambda \rangle_{\mu_\lambda}. \end{aligned}$$

We define the *positive definite symmetric bilinear form*

$$\mathcal{E}(f,g) := \sum_{n\geq 0} \lambda\, n\, \langle f, e_n^\lambda\rangle_{\mu_\lambda}\, \langle g, e_n^\lambda\rangle_{\mu_\lambda},$$

with domain

$$f,g \in D(\mathcal{E}) := \left\{ h \in L^2(\mu_\lambda) : \sum_{n\geq 0} n\, \langle h, e_n^\lambda\rangle_{\mu_\lambda}^2 < +\infty \right\}.$$

Notice that $D(\mathcal{E})$ is the domain of the *square root* of $(-L^\lambda)$:

$$\sqrt{-L^\lambda}\, h := \sum_{n\geq 0} \sqrt{\lambda\, n}\, \langle h, e_n^\lambda\rangle_{\mu_\lambda}\, e_n^\lambda, \qquad h \in D(\mathcal{E})$$

and that one has correctly

$$\langle \sqrt{-L^\lambda}\, f, \sqrt{-L^\lambda}\, g\rangle_{\mu_\lambda} = \sum_{n\geq 0} \lambda\, n\, \langle f, e_n^\lambda\rangle_{\mu_\lambda}\, \langle g, e_n^\lambda\rangle_{\mu_\lambda} = -\langle L^\lambda f, g\rangle_{\mu_\lambda}.$$

Therefore we obtain, for all $f,g \in C_b^1(\mathbb{R})$:

$$\langle \sqrt{-L^\lambda}\, f, \sqrt{-L^\lambda}\, g\rangle_{\mu_\lambda} = \frac{1}{2}\int_{\mathbb{R}} f'\, g'\, d\mu_\lambda = \frac{1}{2}\, \langle f', g'\rangle_{\mu_\lambda}.$$

Exercise 3.6. Prove that indeed $C_b^1(\mathbb{R}) \subset D(\mathcal{E})$ and $C_b^2(\mathbb{R}) \subset D(L^\lambda)$.

Exercise 3.7. Convince yourself that it is *not* true that $\sqrt{-2L^\lambda}\, f = f'$.

Spectral gap. Notice that for all $f \in L^2(\mu_\lambda)$:

$$P_t f - \int f\, d\mu_\lambda = P_t f - \langle f, 1\rangle_{\mu_\lambda} = \sum_{n\geq 1} e^{-t\lambda n}\, \langle f, e_n^\lambda\rangle_{\mu_\lambda}\, e_n^\lambda$$

so that

$$\| P_t f - \langle f, 1\rangle_{\mu_\lambda} \|_{L^2(\mu_\lambda)}^2 = \sum_{n\geq 1} e^{-2t\lambda n}\, \langle f, e_n^\lambda\rangle_{\mu_\lambda}^2 \leq e^{-2t\lambda}\, \|f\|_{L^2(\mu_\lambda)}^2$$

i.e. we have exponential convergence in $L^2(\mu_\lambda)$ of the semigroup to the invariant measure.

4. White noises

4.1. **Gaussian measures.** Let $a \in \mathbb{R}^d$ and Q a symmetric $d\times d$ matrix with positive eigenvalues. Then we can define $A := -\frac{1}{2}Q^{-1}$, a symmetric matrix with negative eigenvalues. The Gaussian probability measure $\mathcal{N}(a,Q)$ on $\mathbb{R}^d$ is defined by

$$\begin{aligned}\mathcal{N}(a,Q)(dx) &= \frac{1}{\sqrt{(2\pi)^d \det Q}} \exp\left(-\frac{1}{2}\langle Q^{-1}(x-a), x-a\rangle\right) dx \\ &= \frac{1}{Z}\exp\left(\langle A(x-a), x-a\rangle\right) dx\end{aligned}$$

and we have the important formula for the Fourier transform of $\mathcal{N}(a,Q)$: if $X \sim \mathcal{N}(a,Q)$ then

$$\mathbb{E}\left(e^{i\langle X,h\rangle}\right) = \int e^{i\langle x,h\rangle}\, \mathcal{N}(a,Q)(dx) = \exp\left(i\langle a,h\rangle - \frac{1}{2}\langle Qh,h\rangle\right)$$

from which, by derivation, one obtains for all $h,k \in \mathbb{R}^d$

$$\mathbb{E}(\langle X,h\rangle) = \langle a,h\rangle, \qquad \mathrm{Cov}(\langle X,h\rangle, \langle X,k\rangle) = \mathbb{E}(\langle X-a,h\rangle\langle X-a,k\rangle) = \langle Qh,k\rangle.$$

Then a is the mean and Q the covariance operator of X. Moreover one obtains the main property of a Gaussian family $\{X_i\}_{i\in I}$: if $J,K \subset I$, $J \cap I = \emptyset$, and $\mathrm{Cov}(X_j, K_k)$ for all $j \in J$ and $k \in K$, then $\{X_j\}_{j\in J}$ and $\{X_k\}_{k\in K}$ are independent. In particular, if $\{e_1, \dots, e_d\}$ are eigenfunctions of Q with respective eigenvalues $\{\lambda_1, \dots, \lambda_d\}$, then the variables $\{\langle X, e_j\rangle, j = 1, \dots, n\}$ are independent and $\langle X, e_j\rangle \sim \mathcal{N}(\langle a, e_j\rangle, \lambda_j)$.

4.2. **White noises.**

Proposition 4.1. *Let $\mathcal{H}$ be a separable Hilbert space. There exists a process $(W(h), h \in \mathcal{H})$ such that $h \mapsto W(h)$ is linear, $W(h)$ is a centered real Gaussian random variable and*

$$\mathbb{E}(W(h)\,W(k)) = \langle h, k\rangle_{\mathcal{H}}, \qquad \forall\ h, k \in \mathcal{H}.$$

Proof. Let $(Z_i)_i$ be a i.i.d. sequence of real standard Gaussian variables and $(h_i)_i$ a complete orthonormal system in $\mathcal{H}$ and set

$$W^n(h) := \sum_{i=1}^{n} \langle h, h_i\rangle_{\mathcal{H}}\, Z_i, \qquad h \in \mathcal{H}.$$

Then it is easy to see that

$$\mathbb{E}(W^n(h)\,W^n(k)) = \sum_{i=1}^{n} \langle h, h_i\rangle_{\mathcal{H}} \langle k, h_i\rangle_{\mathcal{H}}, \qquad \forall\ h, k \in \mathcal{H}.$$

Moreover for all $n < m$

$$\mathbb{E}\left((W^n(h) - W^m(h))^2\right) = \sum_{i=n+1}^{m} \langle h, h_i\rangle^2 \to 0$$

as $n, m \to +\infty$. The conclusion is standard. □

Notice that the application $\mathcal{H} \ni h \mapsto W(h)$ is an isomorphism of Hilbert spaces between $\mathcal{H}$ and a space of Gaussian random variables.

Let now $(T, \mathcal{B}, m)$ be a separable measurable space, with m a σ-finite measure without atoms. We apply Proposition 4.1 to $\mathcal{H} := L^2(T, \mathcal{B}, m)$. The process $(W(h), h \in \mathcal{H})$ is called a *Gaussian white noise* over $(T, \mathcal{B}, m)$.

If $A \in \mathcal{B}$ and $m(A) < +\infty$, then $1_A \in \mathcal{H}$ and we denote $W(A) := W(1_A)$. If $A, B \in \mathcal{B}$ with $m(A) + m(B) < +\infty$ then

$$\mathbb{E}(W(A)\,W(B)) = m(A \cap B).$$

In particular, if $m(A \cap B) = 0$, then $\{W(A'), A' \subseteq A\}$ and $\{W(B'), B' \subseteq B\}$ are independent.

It is customary to use the notation

$$W(A) = \int_A W(dt), \qquad W(h) = \int_T h(t)\,W(dt).$$

We have the important property, which follows immediately from the fact that $W : L^2(T, m) \mapsto L^2(\Omega)$ is an isometry:

Proposition 4.2. *If $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}$ is such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $m(\cup_n A_n) < +\infty$, then*

$$\lim_{n\to+\infty} \sum_{i=0}^{n} W(A_i) = W(\cup_n A_n) \qquad \text{in } L^2.$$

Moreover, since $A_i \cap A_j = \emptyset$ for $i \neq j$, then the sequence $(W(A_n))_n$ is independent and, since all variables $W(A_n)$ are centered, the sequence is orthogonal in L^2.

Notice however that $W(dt)$ is, in general, *not* a signed measure, as this notation might suggest; indeed, the process $h \mapsto W(h)$ does not always admit a modification such that $W(h)$ is defined on the same set of probability 1 for all h: see Remark 4.3 below; however, it is possible to interpret $W(dt)$ as a (random) distribution in the sense of Schwarz, see subsection 4.10.

4.3. **Finite dimensional white noise.** Let us consider first the easiest case: $T = \{1, \ldots, d\}$ and m is the counting measure. In this case $L^2(T, \mathcal{B}, m) = \mathbb{R}^d$ and the white noise $W(h)$ can be realized as $W(h) = \langle W, h \rangle_{\mathbb{R}^d}$, where $W \sim \mathcal{N}(0, I)$.

4.4. **Brownian motion.** Let now $T = \mathbb{R}$ endowed with the Borel σ-algebra and the Lebesgue measure λ_1. Then for any choice of two intervals $[a, b]$ and $[c, d]$ in $\mathbb{R}$

$$\mathbb{E}(W([a, b])\, W([c, d])) = \lambda_1([a, b] \cap [c, d]).$$

Then the process

$$\hat{W}_t := \begin{cases} W([0, t]), & t \geq 0, \\ \\ W([t, 0]), & t < 0. \end{cases}$$

is a two-sided standard Brownian motion, i.e. has the same law as the process defined in (2.5), and $W(dt)$ is simply called *white noise* over $\mathbb{R}$. In particular, the process $(W([0, t]),\, t \geq 0)$ is a standard BM.

Remark 4.3. In this case we can see very clearly why in general a white noise can no be written as a signed measure. If was the case, this would imply that the BM $(W([0, t]),\, t \geq 0)$ has a.s. paths with bounded variation, which is notoriously false.

4.5. **Multi-dimensional Brownian motion.** Let now $T = \mathbb{R} \times \{1, \ldots, d\}$ endowed with the Borel σ-algebra and the measure $\lambda_1 \otimes m$ where m is the counting measure.

Then for any choice of two intervals $[a, b]$ and $[c, d]$ in $\mathbb{R}$ and for any $i, j \in \{1, \ldots, d\}$

$$\mathbb{E}(W([a, b] \times \{i\})\, W([c, d] \times \{j\})) = \lambda_1([a, b] \cap [c, d])\, \mathbb{1}_{(i=j)}.$$

Then the process $(\hat{W}^1_t, \ldots, \hat{W}^d_t)$, defined by

$$\hat{W}^i_t := \begin{cases} W([0, t] \times \{i\}), & t \geq 0, \\ \\ W([t, 0] \times \{i\}), & t < 0. \end{cases}$$

is a two-sided standard Brownian motion, i.e. has the same law as the process defined in (2.5), and $W(dt)$ is simply called *white noise* over $\mathbb{R}$. In particular, the process $(W^1([0, t]), \ldots, W^d([0, t]))_{t \geq 0}$ is a standard BM in $\mathbb{R}^d$.

4.6. **Brownian sheet.** If $T = \mathbb{R}^2$ endowed with the Borel σ-algebra and the Lebesgue measure λ_2, then

$$\mathbb{E}(W([0, t] \times [0, t'])\; W([0, s] \times [0, s'])) = (t \wedge t')\; (s \wedge s'), \qquad t, t', s, s' \geq 0.$$

The process $(W(t, s) := W([0, t] \times [0, s]), t, s \geq 0)$ is called a *Brownian sheet* and $W(dt, ds)$ a *space-time white noise*. One can also use the notations

$$W(dt, ds) = \frac{\partial^2 W}{\partial t \partial s} = \dot{W}(t, s).$$

Notice that the same construction can be done if $T = \mathbb{R}^d$: this gives a space-time white noise with a d-dimensional space variable.

4.7. **Cylindrical Brownian motion.** Let H be any separable Hilbert space and $(e_i)_{i\geq 1}$ a complete orthonormal basis of H. Let us consider a sequence of independent standard real Brownian motions $(w^i_t, t \geq 0)_i$. We set for all $n \in \mathbb{N}$:

$$W^n_t := \sum_{i=1}^n w^i_t \, e_i, \qquad \langle W^n_t, h\rangle = \sum_{i=1}^n w^i_t \, \langle h, e_i\rangle, \qquad t \geq 0.$$

Now, for all $h \in H$ we have for $n < m$

$$\mathbb{E}\left((\langle W^n_t, h\rangle - \langle W^m_t, h\rangle)^2\right) = \mathbb{E}\left(\left(\sum_{i=n+1}^m w^i_t \, \langle h, e_i\rangle\right)^2\right) \leq \sum_{i=n+1}^m \langle h, e_i\rangle^2 \to 0$$

as $n, m \to \infty$, since $\sum_i \langle h, e_i\rangle^2 < +\infty$. Therefore, for all $t \geq 0$ the series

$$\langle W_t, h\rangle = \sum_{i=1}^\infty w^i_t \, \langle h, e_i\rangle.$$

converges in $L^2(\mathbb{P})$. Notice that for all $h, k \in H$ and $s, t \geq 0$ we have

$$\begin{aligned} \mathbb{E}\left(\langle W_t, h\rangle \, \langle W_s, k\rangle\right) &= \mathbb{E}\left(\sum_{i,j=1}^\infty w^i_t \, w^j_s \, \langle h, e_i\rangle \, \langle k, e_j\rangle\right) = \sum_{i=1}^\infty \mathbb{E}\left(w^i_t \, w^i_s\right) \, \langle h, e_i\rangle \, \langle k, e_i\rangle \\ &= t \wedge s \, \langle h, k\rangle. \end{aligned} \tag{4.1}$$

Formally, the series

$$W_t := \sum_{i=1}^\infty \langle W_t, e_i\rangle \, e_i = \sum_{i=1}^\infty w^i_t \, e_i, \qquad t \geq 0$$

defines a Brownian motion in H. However, this series does not define a H-valued variable. In fact, it can be seen that $\mathbb{P}(W_t \in H) = 0$; one easily notes that

$$\mathbb{E}\left(\|W_t\|^2_H\right) = \sum_{i=1}^\infty \mathbb{E}\left((w^i_t)^2\right) = \sum_{i=1}^\infty t = +\infty, \quad t > 0.$$

Since W_t is not well defined in H, but $\langle W_t, h\rangle$ is for all $h \in H$, the process $(\langle W_t, h\rangle, h \in H)$ is called a *cylindrical Brownian motion*.

4.8. **Fourier construction of space-time white noise.** Let now $H := L^2(0,1)$. Then (4.1) becomes

$$\mathbb{E}\left(\langle W_s, h\rangle \, \langle W_t, k\rangle\right) = s \wedge t \int_0^1 h_x \, k_x \, dx, \qquad \forall \, h, k \in H.$$

In particular, if $1_{[0,y]}$ and $1_{[0,z]}$ denote the indicator functions of two intervals $[0, y]$ and respectively $[0, z]$ in $[0, 1]$, then for all $s, t \geq 0$

$$\mathbb{E}\left(\langle W_s, 1_{[0,y]}\rangle \, \langle W_t, 1_{[0,z]}\rangle\right) = s \wedge t \, \langle 1_{[0,y]}, 1_{[0,z]}\rangle = (s \wedge t) \, (y \wedge z).$$

Therefore, the process $(\langle W_t, 1_{[0,s]}\rangle, t, s \geq 0)$ is a Brownian sheet. Therefore we can use the representation in terms of the space-time white noise:

$$\langle W_t, h\rangle = \int_0^t \int_0^1 h(x) \, W(ds, dx).$$

4.9. **A physicist's description.** Let us start from the white noise in 1 dimension. If $(W_t, t \geq 0)$ is a standard real BM, then the classical formula

$$\mathbb{E}(W_t\, W_s) = t \wedge s, \qquad t, s \geq 0,$$

can be interpreted by saying that

$$\mathbb{E}\left(\dot{W}_t\, \dot{W}_s\right) = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\, t \wedge s = \frac{\partial}{\partial t}\, 1_{[s,+\infty[}(t) = \delta(t-s)$$

where $\delta(t)$ is the Dirac mass at 0. Since $\delta(t-s) = 0$ if $t \neq s$ and $(\dot{W}_t, t \geq 0)$ is a Gaussian process, then $\dot{W}_t$ and $\dot{W}_s$ are independent for $t \neq s$. In the case of the Brownian sheet, we have analogously

$$\mathbb{E}(W(t,s)\, W(t',s')) = (t \wedge t')\,(s \wedge s'), \qquad t, t', s, s' \geq 0,$$

and therefore

$$\mathbb{E}(W(dt,ds)\, W(dt',ds')) = \mathbb{E}\left(\dot{W}(t,s)\, \dot{W}(t',s')\right) = \delta(t-t')\,\delta(s-s').$$

Then, $\dot{W}(t,s)$ and $\dot{W}(t',s')$ are independent, unless $(t,s) = (t',s')$.

The Fourier representation of the space-time white noise reads

$$\dot{W}(t,x) := \frac{\partial}{\partial t}\, W_t(x) = \sum_{i=1}^{\infty} \frac{dw_t^i}{dt}\, e_i(x),$$

where $(e_i)_i$ is *any* complete orthonormal system in $L^2(\mathbb{R}_+, dx)$.

4.10. **Random distribution.** Another possible interpretation of the white noise on $\mathbb{R}^d$ is the random distribution viewpoint. Notice first that the covariance structure implies, e.g. if $s \leq s'$, that

$$\mathbb{E}(|W(t,s) - W(t',s')|^2) = ts + t's' - 2(t \wedge t')\,(s \wedge s') = |t-t'|s + t'|s-s'|.$$

Since $(W(t,s) - W(t',s'))$ is a Gaussian r.v. then there exists a constant $C_{m,T}$ such that

$$\mathbb{E}(|W(t,s) - W(t',s')|^{2m}) \leq C_{m,T}(|t-t'|^m + |s-s'|^m), \qquad \forall\, t, t', s, s' \in [0,T].$$

Therefore by the Kolmogorov criterion the process $(W(s,t), s, t \geq 0)$ has an a.s. continuous modification. The same holds for $(W(t_1, \ldots, t_d), t_1, \ldots, t_d \geq 0)$. Now, if $\varphi \in C_c^\infty(\mathbb{R}^d)$, then

$$W(\varphi) = \int_{\mathbb{R}^d} \varphi(x)\, W(dx) = (-1)^d \int_{\mathbb{R}^d} \frac{\partial^d \varphi}{\partial x_1 \cdots \partial x_d}(x)\, W([0,x])\, dx,$$

where $[0,x] := [0,x_1] \times \cdots \times [0,x_d]$. This expression gives a *measurable modification* $C_c^\infty(\mathbb{R}^d) \ni \varphi \mapsto W(\varphi)(\omega)$, for $\mathbb{P}$-a.e. ω, of the white noise.

4.11. **Couloured noise.** For the sake of completeness, we point out that the cylindrical white noise (or space-time white noise) is by no means the only possible choice. A *coloured noise*, for instance, is defined by a (possibly formal) series

$$B\, W_t := \sum_{i=1}^{\infty} b_i\, w_t^i\, e_i, \qquad t \geq 0,$$

where $b_i \in \mathbb{R}$ and $B : D(B) \subseteq H \mapsto H$ is the linear operator defined by $Be_i := b_i e_i$, $i \in \mathbb{N}$. The covariance structure becomes

$$\mathbb{E}\left(\langle B\, W_t, h\rangle\, \langle B\, W_s, k\rangle\right) = t \wedge s\, \langle B\, h, B\, k\rangle, \qquad h, k \in D(B),$$

to be compared with (4.1) (notice also that $B = B^*$). The noise therefore is still Brownian-like in time, but can be more regular in space (depending on the speed of convergence of $b_i \to 0$ as $i \to +\infty$).

5. The law on path space of Brownian motion and Brownian bridge

5.1. **Some formal computations.** From the independence of increments of Brownian motion we obtain that the law of $(B_{t_1}, \ldots, B_{t_n})$, for $0 := t_0 < t_1 < \cdots < t_n \le 1$ is

$$\frac{1}{Z_n} \exp\left(-\frac{1}{2}\sum_{j=1}^{n} \frac{|x_j - x_{j-1}|^2}{t_j - t_{j-1}}\right) dx_1 \cdots dx_n, \qquad x_0 := 0.$$

If $t_j - t_{j-1} = 1/n$ for all $j = 1, \ldots, n$, then for n large one could think that

$$n \sum_{j=1}^{n} |x_j - x_{j-1}|^2 \sim \int_0^1 (\dot{x}_r)^2 \, dr.$$

As $n \to \infty$, one obtains as a formal limit an expression for the law of the process $X := (B_t)_{t\in[0,1]} \in L^2(0,1)$

$$X = (B_r)_{r\in[0,1]} \sim \frac{1}{Z} \exp\left(-\frac{1}{2}\int_0^1 (\dot{x}_r)^2 \, dr\right) \mathbb{1}_{\{x_0=0\}} \prod_{r\in[0,1]} dx_r. \tag{5.1}$$

However, this formula does not make sense for several reasons:

- the typical trajectory of $(B_r)_{r\in[0,1]}$ has no derivative in $L^2(0,1)$, and the term in the exponential is equal to $-\infty$ a.s. if x is a typical trajectory of B;
- the normalizing constant Z is
$$Z = \lim_{n\to\infty} Z_n = \lim_{n\to\infty} (2\pi/n)^{n/2} = 0;$$
- finally, the measure $dx = \prod_{r\in[0,1]} dx_r$ is not well defined. Indeed, notice that dx, if well defined, would be a translationally-invariant measure on $L^2(0,1)$. However, an infinite-dimensional Hilbert space admits no non-zero translationally-invariant measure being finite on all balls. Indeed, consider a sequence $(e_i)_{i\in\mathbb{N}}$ such that $\|e_i\| = 1$ and $\|e_i - e_j\| \ge 1$, for instance an orthonormal system; if the measure is non-null, there is a ball $B(x,r)$ with $r > 0$ and with positive finite mass; the balls $B(2r\, e_i, r)$ are all contained in $B(0,4r)$, are pairwise disjoint and have the same mass as $B(x,r)$ by invariance under translations. Therefore, the ball $B(0,4r)$ has infinite mass.

5.2. **The covariance operator.** However, something rigorous can be said about formula (5.1). We recall that in $\mathbb{R}^d$

$$\mathcal{N}(0,Q)(dx) = \frac{1}{Z} \exp\left(\langle Ax, x\rangle\right) dx, \tag{5.2}$$

where $A := -\frac{1}{2}Q^{-1}$. Therefore, we can start by studying the covariance operator Q. It is well known that $\mathbb{E}(B_s B_r) = s \wedge r$, and, denoting the canonical scalar product in $H := L^2(0,1)$

$$\langle h, k\rangle := \int_0^1 h_r \, k_r \, dr, \qquad h, k \in H = L^2(0,1),$$

then we obtain

$$\mathbb{E}(\langle B,h\rangle)=0, \qquad \mathbb{E}(\langle B,h\rangle\langle B,k\rangle)=\int_{[0,1]^2} s\wedge r\, h_s\, h_r\, ds\, dr=\langle Qh,k\rangle, \qquad h,k\in H,$$

where we denote

$$Q: H\mapsto H, \qquad (Qh)_r:=\int_{[0,1]} s\wedge r\, h_s\, ds, \quad r\in[0,1]. \tag{5.3}$$

Stated differently, we can write by integrating by parts

$$\langle B,h\rangle=\int_0^1 B_r\, h_r\, dr=\int_0^1\left(\int_r^1 h_u\, du\right) dB_r$$

and this is clearly a real Gaussian variable with 0 mean and covariance

$$\int_0^1\left(\int_r^1 h_u\, du\right)^2 dr=\langle Qh,h\rangle.$$

Therefore

$$\mathbb{E}\left(e^{i\langle B,h\rangle}\right)=\exp\left(-\frac{1}{2}\langle Qh,h\rangle\right),$$

which suggest the natural interpretation $B\sim\mathcal{N}(0,Q)$ by analogy with the finite dimensional case. Now, let us study more in detail the operator Q. Let $h\in H$ and $f:=Qh$. Then we can write

$$f_r=\int_0^r s\, h_s\, ds+r\int_r^1 h_s\, ds, \qquad r\in[0,1],$$

and an easy computation shows that f is the (unique) solution of the equation

$$\begin{cases} -\dfrac{d^2 f}{dr^2}=h, \\ \\ f(0)=\dfrac{df}{dr}(1)=0. \end{cases}$$

Therefore, Q is the inverse of the operator $(-2A)$, where

$$D(A):=\left\{f\in H:\ f''\in H,\ f(0)=\frac{df}{dr}(1)=0\right\}, \qquad Af:=\frac{1}{2}\frac{d^2 f}{dr^2}.$$

Notice also that, again by integrating by parts, for all $f\in D(A)$

$$\langle Af,f\rangle=-\frac{1}{2}\int_0^1 \dot f^2 dr+[f'f]_0^1=-\frac{1}{2}\int_0^1 \dot f^2 dr$$

where the boundary terms vanish because of the boundary conditions in the definition of $D(A)$. Notice that, in the last formula, the left hand side makes sense for $f\in D(A)$, i.e. with $f''\in L^2(0,1)$, while the right hand side is clearly well defined as long as $f'\in L^2(0,1)$. In fact, we can see that the closure of $D(A)$ w.r.t. the norm $\int_0^1 \dot f^2 dr$ is equal to

$$D(\mathcal{E})=\left\{f\in H:\ f'\in H,\ f(0)=0\right\}$$

and we can define the symmetric bilinear form

$$\mathcal{E}(f,g):=\int_0^1 \dot f\, \dot g\, dr, \qquad f,g\in D(\mathcal{E}).$$

Indeed, we can write

$$D(A) := \left\{ f = \int_0^{\cdot} g\, dr \,:\, g' \in H,\, g(1) = 0 \right\}, \qquad \mathcal{E}(f,f) = \|g\|_H^2.$$

Now, the space $\{g : g' \in H,\, g(1) = 0\}$ is dense in H, and therefore the closure of $D(A)$ w.r.t. to the scalar product $\mathcal{E}$ is

$$\left\{ f = \int_0^{\cdot} g\, dr \,:\, g \in H \right\} = D(\mathcal{E}).$$

Therefore, formulae (5.2) and (5.1) are indeed very close. Notice that $D(\mathcal{E})$ is equal to the classical Cameron-Martin space of Brownian motion.

5.3. **A white-noise point of view.** Recall that the white noise $W(dt)$ associated with $L^2(0,1)$ has the property that

$$\mathbb{E}(W(h)W(k)) = \int_0^1 h_r\, k_r\, dr, \qquad h,k \in L^2(0,1).$$

This means that $W(dt)$ has covariance operator equal to the identity, and therefore law $\mathcal{N}(0,I)$ on $L^2(0,1)$. By (5.2), this means that

$$Y = (W(dr))_{r\in[0,1]} \sim \frac{1}{Z} \exp\left(-\frac{1}{2} \int_0^1 y_r^2\, dr \right) dy.$$

Again, this is only formal, but it matches well with (5.1), since $Y = \dot{X}$ and the densities correspond nicely to each other via this transformation: $\int_0^1 y_r^2\, dr = \int_0^1 \dot{x}_r^2\, dr$.

Let $(e_i)_{i\in\mathbb{N}}$ be any complete orthonormal system in $L^2(0,1)$. Then by the property of white noise, the sequence $(W(e_i))_{i\in\mathbb{N}}$ is iid with $\xi_i := W(e_i) \sim \mathcal{N}(0,1)$. Therefore, by the construction of the proof of Proposition 4.1, we obtain that

$$W(h) = \lim_n \sum_{i\le n} \xi_i\, \langle h, e_i\rangle, \qquad \text{in } L^2(\Omega).$$

Since $W(h) = \int_0^1 h_r\, W(dr)$ and $\langle h, e_i\rangle = \int_0^1 h_r\, e_i(r)\, dr$, then we can again interpret formally this construction as

$$W(dr) = \lim_n \sum_{i\le n} \xi_i\, e_i(r)\, dr, \qquad \text{in } L^2(\Omega),$$

i.e. the white noise on $(0,1)$ corresponds to choosing *any* complete orthonormal system in $L^2(0,1)$ and then multiplying every vector of the basis by a standard normal variable from an iid sequence and taking a formal series. This is a Fourier expansion of white noise.

5.4. **Fourier expansion of Brownian motion.** Since Brownian motion is a primitive of a white noise, can we obtain a Fourier expansion for B from the previous subsection? If we consider $h = \mathbb{1}_{[0,t]}$, then we obtain

$$B_t = W([0,t]) = \lim_n \sum_{i\le n} \xi_i \int_0^t e_i(r)\, dr, \qquad \text{in } L^2(\Omega),\ t\in[0,1].$$

However, the vectors $\eta_i(t) := \int_0^t e_i(r)\, dr$, $t \in [0,1]$, do not necessarily form a complete orthonormal system of $L^2(0,1)$. This is true only if

$$\langle \eta_i, \eta_j\rangle = \langle Qe_i, e_j\rangle = 0, \qquad i \neq j,$$

where Q is the operator defined in (5.3). On the other hand, the operator A is easily diagonalized: this corresponds to the equation

$$\begin{cases} -\dfrac{d^2 e_i}{dr^2} = \lambda_i e_i, \\ \\ e_i(0) = \dfrac{de_i}{dr}(1) = 0, \end{cases}$$

and some explicit computations show that

$$e_k(r) := \sqrt{2}\sin((2k-1)\pi r/2), \qquad r \in [0,1], \ k \in \mathbb{N}^*.$$

Then $e_k \in D(A)$ for all $k \in \mathbb{N}^*$ and $2Ae_k = -\frac{(2k-1)^2\pi^2}{4} e_k$, and this is a complete orthonormal system (a fact which requires a proof...). In particular, we obtain that the sequence $(\xi_k)_{k\in\mathbb{N}^*}$ is i.i.d. $\mathcal{N}(0,1)$, where

$$\xi_k := \frac{(2k-1)\pi}{2} \langle B, e_k \rangle$$

and the following random Fourier series converges in $L^2(\Omega; H)$

$$B = \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \xi_k \, e_k.$$

5.5. **The Brownian bridge.** Let $\beta_r := B_r - rB_1$, $r \in [0,1]$. Then $(\beta_r)_{r\in[0,1]}$ is a centered Gaussian process. Moreover the covariance between β and B_1 is $\mathbb{E}(\beta_r \, B_1) = 0$ for all $r \in [0,1]$, and by the main property of Gaussian vectors recalled above, $(\beta_r)_{r\in[0,1]}$ and B_1 are independent. Since we can write $B_r = \beta_r + rB_1$, with $(\beta_r)_{r\in[0,1]}$ and B_1 independent, then the law of $(B_r)_{r\in[0,1]}$ conditioned on $\{B_1 = y\}$ is equal to the law of $(\beta_r + ry)_{r\in[0,1]}$. In particular, $(\beta_r)_{r\in[0,1]}$ has the law of a BM conditioned on $\{B_1 = 0\}$.

For this reason, we expect that, in analogy with (5.1),

$$X = (\beta_r)_{r\in[0,1]} \sim \frac{1}{Z} \exp\left(-\frac{1}{2} \int_0^1 (\dot{x}_r)^2 \, dr \right) \mathbb{1}_{\{x_0 = x_1 = 0\}} \prod_{r\in[0,1]} dx_r. \tag{5.4}$$

We can interpret this formula as in the case of BM by computing the covariance operator of $(\beta_r)_{r\in[0,1]}$. We have

$$\mathbb{E}(\langle \beta, h\rangle\langle \beta, k\rangle) = \int_{[0,1]^2} (s \wedge r - sr) \, h_s \, h_r \, ds \, dr = \langle Qh, k\rangle, \qquad h, k \in H,$$

where we denote now

$$Q : H \mapsto H, \qquad (Qh)_r := \int_{[0,1]} (s \wedge r - sr) \, h_s \, ds, \quad r \in [0,1]. \tag{5.5}$$

And an easy computation shows that $f := Qh$ is the (unique) solution of the equation

$$\begin{cases} -\dfrac{d^2 f}{dr^2} = h, \\ \\ f(0) = f(1) = 0. \end{cases}$$

Therefore, Q is the inverse of the operator $(-2A)$, where

$$D(A) := \left\{ f \in H : \, f'' \in H, \, f(0) = f(1) = 0 \right\}, \qquad Af := \frac{1}{2} \frac{d^2 f}{dr^2}.$$

Notice also that, again by integrating by parts, for all $f \in D(A)$

$$\langle Af, f\rangle = -\frac{1}{2}\int_0^1 \dot{f}^2 dr + [f'f]_0^1 = -\frac{1}{2}\int_0^1 \dot{f}^2 dr$$

where the boundary terms vanish because of the boundary conditions in the definition of $D(A)$. The closure of $D(A)$ w.r.t. the norm $\int_0^1 \dot{f}^2 dr$ is equal to

$$D(\mathcal{E}) = \big\{ f \in H : \, f' \in H, \, f(0) = f(1) = 0 \big\}$$

and we can define the symmetric bilinear form

$$\mathcal{E}(f,g) := \int_0^1 \dot{f}\,\dot{g}\,dr, \qquad f, g \in D(\mathcal{E}).$$

Indeed, we can write

$$D(A) := \left\{ f = \int_0^{\cdot} g\,dr : \, g' \in H, \, \int_0^1 g_r\,dr = 0 \right\}, \qquad \mathcal{E}(f,f) = \|g\|_H^2.$$

Now, the space $\left\{ g : g' \in H, \, \int_0^1 g_r\,dr = 0 \right\}$ is dense in the space $\left\{ g \in H : \, \int_0^1 g_r\,dr = 0 \right\}$, and therefore the closure of $D(A)$ w.r.t. to the scalar product $\mathcal{E}$ is

$$\left\{ f = \int_0^{\cdot} g\,dr : \, g \in H, \, \int_0^1 g_r\,dr = 0 \right\} = D(\mathcal{E}).$$

In analogy with the case of BM, the space $D(\mathcal{E})$ is a Cameron-Martin space for the Brownian bridge (this analogy can be made much more precise).

5.6. **Fourier expansion of Brownian bridge.** We want now to obtain a Fourier expansion for β in terms of a complete orthonormal system and a sequence of iid standard normal variables. Again, this is

Again, it is enough to diagonalize the covariance operator Q defined in (5.5). This corresponds to the equation

$$\begin{cases} -\dfrac{d^2 e_i}{dr^2} = \lambda_i e_i, \\ \\ e_i(0) = e_i(1) = 0, \end{cases}$$

and some explicit computations show that

$$e_k(x) := \sqrt{2}\,\sin(k\pi x), \qquad x \in [0,1], \; k \in \mathbb{N}^*.$$

Then $e_k \in D(A)$ for all $k \in \mathbb{N}^*$ and $2Ae_k = -k^2\pi^2\,e_k$, and this is a complete orthonormal system (a fact which requires a proof...). In particular, we obtain that the sequence $(\xi_k)_{k\in\mathbb{N}^*}$ is i.i.d. $\mathcal{N}(0,1)$, where

$$\xi_k := k\pi\,\langle \beta, e_k\rangle$$

and the following random Fourier series converges in $L^2(\Omega; H)$

$$\beta = \sum_{k=1}^{\infty} \frac{1}{k\pi}\,\xi_k\,e_k.$$

See subsection 6.4 for related material.

6. The stochastic heat equation

We want to study the stochastic PDE

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \dot{W}, \\ \\ u(t,0) = u(t,1) = 0 \\ \\ u(0,x) = u_0(x), \qquad x \in [0,1] \end{cases} \tag{6.1}$$

where $W(t,x)$ is a Brownian sheet over $[0,+\infty[\times[0,1]$ and $\dot{W}(t,x)$ is therefore a space-time white-noise. We suppose that $u_0 \in L^2(0,1)$.

6.1. **The deterministic heat equation.** Let us start from the heat equation without noise:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2}, \\ \\ v(t,0) = v(t,1) = 0 \\ \\ v(0,x) = v_0(x), \qquad x \in [0,1] \end{cases} \tag{6.2}$$

where $v_0 \in L^2(0,1)$. We set for all $k \geq 1$:

$$e_k(x) := \sqrt{2}\,\sin(k\pi x), \qquad x \in [0,1]. \tag{6.3}$$

We recall the following result:

Exercise 6.1. $\{e_k\}_{k\geq 1}$ is a complete orthonormal basis of $L^2(0,1)$.

Notice that $\{e_k\}_{k\geq 1}$ is a complete basis of eigenvectors of the second derivative with homogeneous Dirichlet boundary conditions:

$$\frac{d^2}{dx^2} e_k = -(\pi k)^2 e_k, \qquad e_k(0) = e_k(1) = 0, \qquad k \geq 1.$$

Setting

$$D(A) = \left\{ f \in L^2(0,1) : \sum_{k\geq 1} k^4 \langle e_k, f\rangle^2_{L^2(0,1)} < +\infty \right\}.$$

$$Af = -\sum_{k\geq 1} \frac{(k\pi)^2}{2} \langle e_k, f\rangle\, e_k, \qquad f \in\ D(A),$$

we obtain a closed operator in $L^2(0,1)$ extending $\frac{1}{2}\frac{\partial^2}{\partial x^2}$ on $C^2_c(0,1)$. The solution of the heat equation (6.2) is therefore

$$v(t,x) = \sum_{k\geq 1} e^{-t\frac{(k\pi)^2}{2}} \langle e_k, v_0\rangle\, e_k(x), \qquad t > 0,\ x \in [0,1].$$

Since $|e_k(x)| \leq \sqrt{2}$ and $\sum_{k\geq 1} e^{-t\frac{(k\pi)^2}{2}} k^m < +\infty$ for all $m \in \mathbb{N}$, the above series converges uniformly on $[0,1]$ together with all its partial derivatives in t and x. One can write more compactly, using the semigroup notation,

$$v_t := v(t,\cdot) = e^{tA} v_0, \qquad t \geq 0.$$

6.2. **Fourier expansion of** (6.1)**.** Let us consider the scalar product of both terms of (6.1) and e_k. Setting $u_t^k := \langle u(t,\cdot), e_k\rangle$ we obtain

$$\begin{cases} du^k = -\frac{(k\pi)^2}{2}\, u^k\, dt + dW_t^k, \\ u_0^k = \langle u_0, e_k\rangle \end{cases}$$

where

$$W_t^k := \int_{[0,t]\times[0,1]} e_k(x)\, W(ds, dx).$$

Exercise 6.2. Prove that $(W_t^k, t \geq 0)_{k\geq 1}$ is an independent sequence of Brownian motions.

Setting $\lambda_k = (\pi k)^2/2$, we obtain that $(u_t^k, t \geq 0)_{k\geq 1}$ is an independent family of Ornstein-Uhlenbeck processes, i.e.

$$u_t^k = e^{-\lambda_k t} u_0^k + \int_0^t e^{-\lambda_k (t-s)}\, dW_s^k, \qquad t \geq 0, \tag{6.4}$$

or, equivalently,

$$u_t^k = u_0^k - \lambda_k \int_0^t u_s^k\, ds + W_t^k, \qquad t \geq 0, \tag{6.5}$$

An important remark is the following:

$$\sum_k \frac{1}{\lambda_k} = \sum_k \frac{2}{(\pi k)^2} < +\infty.$$

Since $u_t^k \sim \mathcal{N}\left(e^{-\lambda_k t} u_0^k, \frac{1}{2\lambda_k}(1 - e^{-2\lambda_k t})\right)$, then

$$\mathbb{E}\left(\left\| \sum_{k=n+1}^m u_t^k\, e_k \right\|^2\right) = \sum_{k=n+1}^m \left[e^{-2\lambda_k t} \left(u_0^k\right)^2 + \frac{1}{2\lambda_k}(1 - e^{-2\lambda_k t}) \right] \to 0$$

as $n, m \to +\infty$. Therefore the series

$$u_t := \sum_{k=1}^{+\infty} u_t^k\, e_k$$

converges in $L^2(\Omega; L^2(0,1))$ to a well-defined r.v. u_t taking values in $L^2(0,1)$. Formula (6.4) becomes

$$u_t = e^{tA} u_0 + \int_0^t e^{(t-s)A}\, dW_s, \qquad u_0 \in L^2(0,1),\ t \geq 0, \tag{6.6}$$

while formula (6.5) becomes

$$\langle u_t, h\rangle = \langle u_0, h\rangle + \frac{1}{2} \int_0^t \langle u_s, h''\rangle\, ds + \langle W_t, h\rangle, \qquad t \geq 0,\ h \in D(A), \tag{6.7}$$

which can be interpreted as a weak formulation of

$$du = \frac{1}{2}\, \frac{\partial^2 u}{\partial x^2}\, dt + dW.$$

6.3. **Path continuity.** Until now we have considered u_t as a $L^2(0,1)$-valued random variable. However, if $x \in [0,1]$ is fixed, then

$$u_n(t,x) := \sum_{k=1}^n u_t^k \, e_k(x) = \sum_{k=1}^n \langle u_t, e_k \rangle \, e_k(x) \in \mathbb{R}$$

is well defined for all $n \geq 1$, $t \geq 0$. Let us suppose that $u_0 = 0$, so that $u_0^k := 0$ for all $k \geq 1$. Then

$$\begin{aligned}\mathbb{E}\left(|u_n(t,x) - u_m(t,x)|^2\right) &= \sum_{k=n+1}^m \mathbb{E}\left(\left(u_t^k\right)^2\right) e_k^2(x) \leq \sum_{k=n+1}^m \int_0^t e^{-2\lambda_k(t-s)}\, ds \\ &= \sum_{k=n+1}^m \frac{1 - e^{-2\lambda_k t}}{2\lambda_k} \to 0\end{aligned}$$

as $n, m \to +\infty$. Therefore, there exists a well defined stochastic process $(u(t,x), t \geq 0, x \in [0,1])$, limit in $L^2(\mathbb{P})$ of $(u_n(t,x), t \geq 0, x \in [0,1])$ as $n \to \infty$.

Lemma 6.3. *For all $m \in \mathbb{N}$ there exists a constant $C_m < +\infty$ such that*

$$\mathbb{E}\left(|u(t,x) - u(s,y)|^{2m}\right) \leq C_m \left(|t-s|^{m/2} + |x-y|^m\right), \quad \forall\, t, s \geq 0, \quad x, y \in [0,1].$$

Proof. Notice that $(u(t,x) - u(s,y))$ is a real Gaussian variable with 0 mean; in order to estimate its moments, it is enough to compute the second one, i.e. it is enough to prove that for some constant C

$$\mathbb{E}\left(|u(t,x) - u(s,y)|^2\right) \leq C \left(|t-s|^{1/2} + |x-y|\right), \quad \forall\, t, s \geq 0, \quad x, y \in [0,1].$$

Since $(u(t,x) - u(s,y))$ is the limit in $L^2(\mathbb{P})$ of $(u_n(t,x) - u_n(s,y))$ as $n \to \infty$, then it is enough to estimate the variance of $(u_n(t,x) - u_n(s,y))$ uniformly in n. First we have

$$|u_n(t,x) - u_n(s,y)|^2 \leq 2\,|u_n(t,x) - u_n(s,x)|^2 + 2\,|u_n(s,x) - u_n(s,y)|^2.$$

Now:

$$\begin{aligned}\mathbb{E}\left(|u_n(s,x) - u_n(s,y)|^2\right) &= \mathbb{E}\left(\left|\sum_{k=1}^n u_s^k\,(e_k(x) - e_k(y))\right|^2\right) \\ &= \sum_{k=1}^n \frac{1 - e^{-2\lambda_k s}}{2\lambda_k}(e_k(x) - e_k(y))^2 \leq \sum_{k=1}^n \frac{1 \wedge (|x-y|\,k)^2}{k^2} \\ &\leq 1 \wedge |x-y| + \int_1^\infty \frac{1 \wedge (|x-y|\,k)^2}{k^2}\, dk \leq 3\,|x-y|.\end{aligned}$$

With similar computations:

$$\begin{aligned}\mathbb{E}\left(|u_n(t,x) - u_n(s,x)|^2\right) &= \sum_{k=1}^n e_k^2(x) \left[(1 - e^{-\lambda_k(t-s)})^2 \frac{1 - e^{-2\lambda_k s}}{2\lambda_k} + e^{-2\lambda_k s} \frac{1 - e^{-2\lambda_k (t-s)}}{2\lambda_k}\right] \\ &\leq 2 \sum_{k=1}^n \frac{1 \wedge (|t-s|\,k^2)}{k^2} \leq 2\left(1 \wedge |t-s| + \int_1^\infty \frac{1 \wedge (|t-s|\,k^2)}{k^2}\, dk\right) \\ &\leq 6\,\sqrt{|t-s|}.\end{aligned}$$

We have used the fact that

$$(1 - e^{-\lambda_k(t-s)})^2 \leq (1 \wedge [\lambda_k(t-s)])^2 \leq 1 \wedge [\lambda_k(t-s)].$$

In order to obtain the desired result, it is enough to pass to the limit in $n \to \infty$. $\square$

Proposition 6.4. *There exists an a.s. continuous stochastic process* $(u(t,x), t \geq 0, x \in [0,1])$ *such that for all* $t \geq 0$, $u(t,\cdot) = u_t$ *in* $L^2(0,1)$, *a.s.*

Proof. By the Kolmogorov criterion, we obtain that there exists an a.s. continuous modification of v, that we call again v, such that in particular for all $\varepsilon \in]0,1[$ and $T < +\infty$

$$\sup_{x,y\in[0,1],\, t,s\in[0,T]} \frac{|u(t,x)-u(s,y)|}{|t-s|^{\frac{1-\varepsilon}{4}} + |x-y|^{\frac{1-\varepsilon}{2}}} < +\infty, \qquad \text{a.s.}$$

$\square$

Finally, continuity of $u_0 = (u_0(x), x \in [0,1])$ implies continuity of $(e^{tA}u_0(x), t \geq 0, x \in [0,1])$, while if u_0 is merely in $L^2(0,1)$, then we have continuity of $(e^{tA}u_0(x), t > 0, x \in [0,1])$.

6.4. **The invariant measure.** Since u_t can be written as a sequence of independent O.U. processes, it is easy to extend properties from the single processes to u_t. For instance, the unique probability invariant measure of the sequence is necessarily $\otimes_{k=1}^{+\infty} \mu_{\lambda_k}$, which means that the only probability invariant measure of $(u_t)_{t\geq 0}$ is the distribution of

$$\beta := \sum_{k=1}^{+\infty} \frac{1}{\pi k}\, e_k\, Z_k \in L^2(0,1),$$

where $(Z_k)_{k\geq 1}$ is an i.i.d. sequence of $\mathcal{N}(0,1)$ variables. Notice that

$$\mathbb{E}\left(\beta_x\, \beta_y\right) = \sum_{k=1}^{+\infty} \frac{1}{(\pi k)^2}\, e_k(x)\, e_k(y), \qquad x,y \in [0,1].$$

Is it possible to compute explicitly this covariance function? Notice that for $h \in L^2(0,1)$ we have

$$f := (-2A)^{-1}h = \sum_{k=1}^{+\infty} \frac{1}{2\lambda_k} \langle h, e_k\rangle\, e_k = \sum_{k=1}^{+\infty} \frac{1}{(\pi k)^2} \langle h, e_k\rangle\, e_k.$$

Moreover, f is the (unique) solution of the equation

$$\begin{cases} -\dfrac{d^2 f}{dx^2} = h, \\[2ex] f(0) = f(1) = 0, \end{cases}$$

Exercise 6.5. Prove that

$$f(x) = \int_0^1 (x \wedge y - xy) h(y)\, dy, \qquad x \in [0,1].$$

Therefore

$$\mathbb{E}\left(\beta_x\, \beta_y\right) = \sum_{k=1}^{+\infty} \frac{1}{(\pi k)^2}\, e_k(x)\, e_k(y) = x \wedge y - xy, \qquad x,y \in [0,1],$$

which is the covariance function of a Brownian bridge on $[0,1]$: $\beta_x := B_x - xB_1$, where B is a BM.

6.5. **The Stochastic Convolution.** The process Z defined in (6.6) above is called the *stochastic convolution*. We recall that the function

$$g_t(x,y) := \sum_{k=1}^{\infty} e^{-t(\pi k)^2/2}\, e_k(x)\, e_k(y), \qquad t>0,\ x,y\in[0,1], \tag{6.8}$$

where e_k is as in (6.3), is C^∞ for $(t,x,y)\in\,]0,+\infty[\times[0,1]\times[0,1]$, and is called the *fundamental solution* of the heat equation on $[0,1]$ with Dirichlet boundary condition. Indeed, g satisfies

$$\begin{cases} \dfrac{\partial g}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 g}{\partial x^2}, \\[2ex] g_t(0,y) = g_t(1,y) = 0, \\[2ex] g_0(x,y) = \delta(x-y). \end{cases} \tag{6.9}$$

Indeed, for all $t>0$ the series in (6.8) converges uniformly with all partial derivatives w.r.t. x and by (6.3), for $t>0$

$$\begin{aligned} \frac{\partial g}{\partial t}(t,x) &= \frac{\partial}{\partial t}\sum_{k=1}^{\infty} e^{-t(\pi k)^2/2}\, e_k(x)\, e_k(y) = -\sum_{k=1}^{\infty}\frac{(\pi k)^2}{2}\, e^{-t(\pi k)^2/2}\, e_k(x)\, e_k(y) \\ &= \frac{1}{2}\frac{\partial^2}{\partial x^2}\sum_{k=1}^{\infty} e^{-t(\pi k)^2/2}\, e_k(x)\, e_k(y) = \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,x) \end{aligned}$$

and for $t=0$ and any $f,g\in L^2(0,1)$

$$\begin{aligned} &\int_{[0,1]^2} f(y)\, g(x)\, g_0(x,y)\, dx\, dy = \int_{[0,1]^2} f(y)\, g(x)\sum_{k=1}^{\infty} e_k(x)\, e_k(y)\, dx\, dy = \sum_{k=1}^{\infty}\langle g, e_k\rangle\,\langle f, e_k\rangle \\ &= \langle f, g\rangle = \int_{[0,1]} f(x)\, g(x)\, dx = \int_{[0,1]^2} f(y)\, g(x)\,\delta(x-y)\, dx\, dy. \end{aligned}$$

If we use the fundamental solution g of the heat equation defined in (6.8) and the space-time white noise representation $W(ds,dy)$ in terms of the Brownian sheet, then from (6.6) we obtain yet another expression for z,

$$z(t,x) = \int_0^1 g_t(x,y)\, u_0(y)\, dy + \int_0^t\int_0^1 g_{t-s}(x,y)\, W(ds,dy).$$

We have the following useful estimate on g

Lemma 6.6. *We have*

$$0\le g_t(x,y)\le G_t(x,y) := \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{|x-y|^2}{2t}\right), \qquad x,y\in[0,1],\ t>0.$$

Proof. Let $f\in C_c(0,1)$ be non-negative. Then by (6.9) the function

$$f_t(x) := \int_0^1 g_t(x,y)\, f(y)\, dy, \qquad t>0,\ x\in[0,1]$$

is a solution of

$$\begin{cases} \dfrac{\partial f}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x^2}, \\ f_t(0) = f_t(1) = 0, \\ f_0(x) = f(x) \end{cases} \tag{6.10}$$

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and let $\tau_x := \inf\{s > 0 : x + B_s \notin]0,1[\}$ for $x \in]0,1[$. Then Ito's formula applied to $(f_{t-s}(x + B_{s\wedge\tau_x}))_{s\leq t}$ yields that

$$f_t(x) = \mathbb{E}\left(f(x + B_t)\, \mathbb{1}_{\{s>\tau_x\}}\right).$$

Therefore we obtain

$$0 \leq f_t(x) \leq \mathbb{E}\left(f(x + B_t)\right) = \int_0^1 G_t(x, y)\, f(y)\, dy$$

and since $f \in C_c(0, 1)$ non-negative is generic, then we obtain

$$0 \leq g_t(x, y) \leq G_t(x, y), \qquad t > 0,\ x, y \in [0, 1].$$

□

7. Martingale measures

In this section we show how to construct stochastic integrals w.r.t. a multi-parameter white noise $W(dt, dx)$. We follow the presentation given in [12, Chapter Two].

Let $(E, \mathcal{B}, \eta)$ be a finite measure space and W a white noise over $(E \times [0, T], \eta \otimes dt)$. Let us define the filtration $\mathcal{F}_t := \sigma(W(\mathbb{1}_{A\times[0,s]}), s \in [0, t], A \in \mathcal{B})$.

By the properties of white noises, the process $t \mapsto W(\mathbb{1}_{A\times[0,t]}) =: M_t(A)$ is a $(\mathcal{F}_t)_t$-BM multiplied by $(\eta(A))^{1/2}$, and in particular a $(\mathcal{F}_t)_t$-martingale. Let $A, B \in \mathcal{B}$. Then we can write

$$M_t(A) \cdot M_t(B) = [M_t(A \cap B)]^2 + M_t(A\backslash B) \cdot M_t(A \cap B) + M_t(B\backslash A) \cdot M_t(A \cap B).$$

By the independence properties of Gaussian processes, $(M_t(A\backslash B), t \geq 0)$, $(M_t(A\cap B), t \geq 0)$ and $(M_t(B\backslash A), t \geq 0)$ are independent martingales. Therefore

$$M_t(A) \cdot M_t(B) = N_t + t\, \eta(A \cap B),$$

where $(N_t)_t$ is a martingale and we obtain that the covariance functional is

$$\langle M(A), M(B)\rangle_t = t\, \eta(A \cap B) = t \int_E \mathbb{1}_A(x)\, \mathbb{1}_B(x)\, \eta(dx).$$

We can and shall use the notation

$$M_t(A) = \int_{[0,T]\times E} \mathbb{1}_{[0,t]\times A}(s, x)\, W(ds, dx).$$

The aim is to extend this stochastic integral to more general integrands.

Lemma 7.1. *The process $(M_t(A), A \in \mathcal{B}, t \geq 0)$ is a martingale measure, i.e.*

1. $M_0(A) \equiv 0$
2. *for all $A \in \mathcal{B}$, $(M_t(A), t \geq 0)$ is a $(\mathcal{F}_t)_{t\geq 0}$-martingale.*

(3) *$M_t : \mathcal{B} \mapsto L^2(\mathbb{P})$ is a L^2-valued measure, i.e. for all $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\eta(\cup_n A_n) < +\infty$ we have*

$$\lim_{n\to+\infty} \mathbb{E}\left[\left(M_t\left(\cup_k A_k\right) - \sum_{k=1}^{n} M_t(A_k)\right)^2\right] = 0.$$

The third property can be interpreted as an equality in $L^2(\Omega)$

$$M_t\left(\cup_k A_k\right) = \sum_k M_t(A_k) \qquad \text{in } L^2(\Omega).$$

Notice that the series in the right hand side converges in $L^2(\Omega)$ but not necessarily almost surely.

Proof of Lemma 7.1. The first two properties are obvious from the definitions of W and $\mathcal{F}_t$ and have already been discusses above. To check the last property, let us recall that $(M_t(A_k))_{k\in\mathbb{N}}$ is an independent sequence, since it is a Gaussian family such that pairwise covariances vanish

$$\mathrm{Cov}(M_t(A_i)M_t(A_j)) = \mathbb{E}(M_t(A_i)M_t(A_j)) = \eta(A_i \cap A_j) = 0, \qquad i \neq j.$$

Then the sequence $(M_t(A_k))_{k\in\mathbb{N}}$ is orthogonal in $L^2(\mathbb{P})$ and a.s.

$$M_t\left(\cup_{k=1}^n A_k\right) = \sum_{k=1}^{n} M_t(A_k).$$

Since

$$\mathbb{E}\left[\left(M_t\left(\cup_k A_k\right) - M_t\left(\cup_{k=1}^n A_k\right)\right)^2\right] = \mathbb{E}\left[\left(M_t\left(\cup_{k\geq n+1} A_k\right)\right)^2\right] = \eta(\cup_{k\geq n+1} A_k)$$

then we obtain the claim as $n \to +\infty$. □

We define the covariance functional

$$\langle M(A), M(B)\rangle_t = t\,\eta(A \cap B). \tag{7.1}$$

Then we can also say that $(M_t(A), A \in \mathcal{B}, t \geq 0)$ is *orthogonal* in the sense that $A \cap B = \emptyset$ implies that $(M_t(A), t \geq 0)$ and $(M_t(B), t \geq 0)$ are orthogonal martingales, i.e. the product $(M_t(A)M_t(B), t \geq 0)$ is a martingale (the quadratic covariation vanishes identically).

We are going to construct stochastic integrals w.r.t. the martingale measure $(M_t(A), A \in \mathcal{B}, t \in [0,T])$. We introduce the class of *elementary* integrands:

$$f(x,s,\omega) = X(\omega)\, \mathbb{1}_{]a,b]}(s)\, \mathbb{1}_A(x), \qquad x \in E,\ s \in [0,T],\ \omega \in \Omega, \tag{7.2}$$

where $0 \leq a \leq b \leq T$, X is bounded and $\mathcal{F}_a$-measurable, $A \in \mathcal{B}$. A *simple* function is a sum of a finite number of elementary functions.

If f is an elementary function of the form (7.2) and M a martingale measure, we define a new martingale measure $f \bullet M$ by

$$f \bullet M_t(B) := X \cdot (M_{t\wedge b}(A \cap B) - M_{t\wedge a}(A \cap B)), \qquad t \in [0,T].$$

We define $f \bullet M_t(B)$ by linearity if f is simple.

Lemma 7.2. *For any simple function f, $(f \bullet M_t(B),\ t \geq 0, B \in \mathcal{B})$ is an orthogonal martingale measure with covariance functional*

$$\langle f \bullet M(A), f \bullet M(B)\rangle_t = \int_{[0,t]\times(A\cap B)} f^2(x,s,\cdot)\, ds\, \eta(dx). \tag{7.3}$$

Proof. The martingale measure properties follow from linearity. Let now two elementary functions

$$f_1(x,s,\omega) = X_1(\omega)\,\mathbb{1}_{]a_1,b_1]}(s)\,\mathbb{1}_{A_1}(x), \quad f_2(x,s,\omega) = X_2(\omega)\,\mathbb{1}_{]a_2,b_2]}(s)\,\mathbb{1}_{A_2}(x),$$

where $0 \le a_i \le b_i \le T$, X_i is bounded and $\mathcal{F}_{a_i}$-measurable, $A_i \in \mathcal{B}$. Let us set $N_t^i := M_t(A_i \cap B_i)$, with $B_i \in \mathcal{B}$, $i = 1, 2$. Then $(N_t^i)_{t\ge 0}$ is a classical $(\mathcal{F}_t)_{t\ge 0}$-martingale and

$$f_i \bullet M_t(B_i) = \int_0^t X_i \mathbb{1}_{]a_i,b_i]}(s)\, dN_s^i, \qquad t \ge 0,$$

where the right hand side is a classical stochastic integral. Then we have by (7.1)

$$\langle f_1 \bullet M(B_1), f_2 \bullet M(B_2)\rangle_t = \int_0^t X_1 X_2\,\mathbb{1}_{]a_1,b_1]\cap]a_2,b_2]}(s)\, d\langle N^1, N^2\rangle_s$$
$$= \int_0^t X_1 X_2\,\mathbb{1}_{]a_1,b_1]\cap]a_2,b_2]}(s)\, ds\, \eta(A_1 \cap B_1 \cap A_2 \cap B_2) = \int_{[0,t]\times(B_1\cap B_2)} f_1\, f_2\, ds\, d\eta.$$

By linearity, (7.3) follows easily. □

Now we define the predictable σ-algebra $\mathcal{P}$ as the σ-algebra generated by all elementary functions. We say that a function f is predictable if it is $\mathcal{P}$-measurable. We define a norm on all predictable functions

$$\|f\|_M := \left[\mathbb{E}\left(\int_{E\times[0,T]} |f(y,s)|^2\, \eta(dy)\, ds\right)\right]^{\frac{1}{2}}.$$

As in the classical case, we can prove that simple functions are dense in the space of predictable functions with $\|f\|_M < +\infty$. Notice that for all simple functions

$$\mathbb{E}\left((f \bullet M_t(B))^2\right) = \mathbb{E}\left(\int_{[0,t]\times B} f^2(x,s,\cdot)\, ds\, \eta(dx)\right) \le \|f\|_M^2. \tag{7.4}$$

Indeed, this follows by considering $A = B$ and taking expectation (7.3). Therefore, by density, we can define the process $(f \bullet M_t(B), t \ge 0, B \in \mathcal{B})$ for all $(\mathcal{F}_t)_{t\ge0}$-predictable f such that $\|f\|_M < +\infty$. Now we have the

Lemma 7.3. *For any predictable function f with $\|f\|_M < +\infty$, $(f \bullet M_t(B),\ t \ge 0, B \in \mathcal{B})$ is an orthogonal martingale measure with covariance functional given by* (7.3) *and such that* (7.4) *holds.*

Proof. Since $f \bullet M$ is defined by density, the martingale property follows since the L^2 limit of a sequence of martingales is still a martingale. Moreover, (7.4) holds by density.

To prove that $B \mapsto f \bullet M_t(B)$ is a L^2-values measure, let us consider $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}$ such that $A_i \cap A_j = \emptyset$ for $i \ne j$ and $\eta(\cup_n A_n) < +\infty$. Then by (7.4)

$$\mathbb{E}\left[\left(f \bullet M_t\left(\cup_k A_k\right) - \sum_{k=1}^n f \bullet M_t(A_k)\right)^2\right] = \mathbb{E}\left[\left(f \bullet M_t(\cup_{k\ge n+1} A_k)\right)^2\right]$$
$$= \int_{\cup_{k\ge n+1} A_k} \eta(dx)\, \mathbb{E}\left(\int_0^t f^2(x,s,\cdot)\, ds\right) \to 0$$

as $n \to +\infty$. □

We shall denote

$$\int_0^t \int_B f(s,y)\, W(ds,dy) := f \bullet M_t(B), \qquad B \in \mathcal{B},\ t \geq 0.$$

and in particular

$$\int_{[0,t]\times E} f(s,y)\, W(ds,dy) = \int_0^t \int_E f(s,y)\, W(ds,dy) := f \bullet M_t(E), \qquad t \geq 0.$$

8. Non-linear SPDEs

We want to study the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + f(u) + \sigma(u)\, \dot{W}, \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x), \qquad x \in [0,1] \end{cases} \tag{8.1}$$

where $W = (W(s,x), s \in [0,t], x \in [0,1])$ is a Brownian sheet and $f : \mathbb{R} \mapsto \mathbb{R}$, $\sigma : \mathbb{R} \mapsto \mathbb{R}$. If we set $\mathcal{F}_t := \sigma(W(s,x), s \in [0,t], x \in [0,1])$, then $u(t,\cdot)$ is assumed to be predictable. We make the standard assumption on the coefficients that there exists a constant C such that

$$|f(u) - f(v)| + |\sigma(u) - \sigma(v)| \leq C|u-v|, \qquad u, v \in \mathbb{R}. \tag{8.2}$$

A motivation for considering a nonlinear coefficient σ multiplying the space-time white noise comes from population genetics. Indeed, a natural model of branching Brownian motions has as scaling limit the following SPDE

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \sqrt{u}\, \dot{W}$$

also known as the Super-Brownian motion equation, see [4]. On the other hand, notice that the square root is not Lipschitz and, indeed, only uniqueness in law is known for this equation, while pathwise uniqueness is still an open problem. This class of equations makes sense also in dimension $d \geq 1$ as a martingale problem, but u is a positive measure without a density w.r.t. Lebesgue measure.

First we notice that, if $f \equiv 0$ and $\sigma \equiv 1$ then this equation is the stochastic heat equation (6.1). In analogy with that case, we can expect that the solution can not be regular enough to satisfy (8.1) in a pointwise sense. We first need to properly interpret (8.1).

8.1. **Mild solutions.** We retain the notations of section 6. If we multiply (8.1) by a test function $h \in D(A)$ and we integrate by parts, we obtain that

$$\begin{aligned} \langle u_t, h \rangle = \langle u_0, h \rangle &+ \int_0^t \langle u_s, Ah \rangle\, ds + \int_0^t \langle f(u_s), h \rangle\, ds \\ &+ \int_0^t \int_0^1 \sigma(u(s,y))\, h(y)\, W(ds,dy) \end{aligned}$$

If we choose $h = e_k$ then we obtain for $u^k_t := \langle u_t, e_k\rangle$

$$\begin{aligned} u^k_t =& u^k_0 - \frac{(k\pi)^2}{2} \int_0^t u^k_s \, ds + \int_0^t \langle f(u_s), e_k\rangle \, ds \\ &+ \int_0^t \int_0^1 \sigma(u(s,y)) \, e_k(y) \, W(ds, dy) \end{aligned}$$

and, by applying the Ito formula to $t \mapsto e^{t\frac{(k\pi)^2}{2}} u^k_t$, we obtain

$$\begin{aligned} u^k_t =& e^{-t\frac{(k\pi)^2}{2}} u^k_0 + \int_0^t e^{-(t-s)\frac{(k\pi)^2}{2}} \, \langle f(u_s), e_k\rangle \, ds \\ &+ \int_0^t \int_0^1 e^{-(t-s)\frac{(k\pi)^2}{2}} \, \sigma(u(s,y)) \, e_k(y) \, W(ds, dy). \end{aligned}$$

Multiplying each term by $e_k(x)$ and summing over k we find that (8.1) is equivalent to

$$\begin{aligned} u(t,x) =& \int_0^1 g_t(x,y) \, u_0(y) \, dy + \int_0^t \int_0^1 g_{t-s}(x,y) \, f(u(s,y)) \, dy \, ds \\ &+ \int_0^t \int_0^1 g_{t-s}(x,y) \, \sigma(u(s,y)) \, W(dy, ds) \end{aligned} \tag{8.3}$$

where g is the fundamental solution of the heat equation (6.10). Equation (8.3) is called the *mild formulation* of (8.1).

The stochastic convolution. Notice that the last term in (8.3) is a stochastic integral which requires the theory developed in section 7. Let us check that the assumptions are satisfied. By the previous section, the space-time white noise defines a martingale measure by

$$M_t(A) := \int_{[0,T]\times[0,1]} \mathbb{1}_{([0,t]\times A)}(s,y) \, W(ds, dy), \qquad t \geq 0,$$

and the condition $\|f\|_M < +\infty$ on a $(\mathcal{F}_t)-$predictable process is just

$$\int_{[0,T]\times[0,1]} \mathbb{E}\left(f^2(s,y)\right) ds \, dy < +\infty.$$

8.2. **Existence and uniquenenss with Lipschitz continuous coefficients.**

Proposition 8.1. *Assume* (8.2). *For all* $u_0 \in L^2(0,1)$ *there exists a unique predictable process* $(u(t,x), t \geq 0, x \in [0,1])$ *such that* u *solves* (8.3) *and*

$$\sup_{(t,x)\in[0,T]\times[0,1]} \mathbb{E}\left(u^2(t,x)\right) < +\infty, \qquad \forall T > 0.$$

Proof. We follow [12, Chapter Three]. Suppose u_1 and u_2 are two predictable solutions of (8.3). Then $U := u_1 - u_2$ satisfies

$$\begin{aligned} U(t,x) =& \int_0^t \int_0^1 g_{t-s}(x,y) \, \left(f(u_1(s,y)) - f(u_2(s,y))\right) \, dy \, ds \\ &+ \int_0^t \int_0^1 g_{t-s}(x,y) \, \left(\sigma(u_1(s,y)) - \sigma(u_2(s,y))\right) \, W(dy, ds). \end{aligned} \tag{8.4}$$

We define

$$F(t,x) := \mathbb{E}\left(U^2(t,x)\right), \qquad H(t) := \sup_{x\in[0,1]} F(t,x).$$

By assumption, H is bounded over $[0,T]$. From (8.4), for all $t\in[0,T]$

$$\begin{aligned}F(t,x) \leq & 2T\,\mathbb{E}\left(\int_0^t\int_0^1 g^2_{t-s}(x,y)\,|f(u_1(s,y))-f(u_2(s,y))|^2\,dy\,ds\right)+\\ &+2\,\mathbb{E}\left(\int_0^t\int_0^1 g^2_{t-s}(x,y)\,|\sigma(u_1(s,y))-\sigma(u_2(s,y))|^2\,dy\,ds\right)\\ \leq & K\int_0^t\int_0^1 g^2_{t-s}(x,y)\,F(s,y)\,dy\,ds.\end{aligned}$$

where $K:=C^2(2T+1)$. Thus

$$H(t)\leq K\int_0^t H(s)\int_0^1 g^2_{t-s}(x,y)\,dy\,ds\leq K\int_0^t H(s)\,\frac{ds}{\sqrt{2\pi(t-s)}},$$

since by Lemma 6.6

$$\int_0^1 g^2_{t-s}(x,y)\,dy\leq\int_0^1 G^2_{t-s}(x,y)\,dy=\frac{1}{2\pi(t-s)}\int_{\mathbb{R}} e^{-\frac{y^2}{t-s}}\,dy=\frac{1}{\sqrt{2\pi(t-s)}}.$$

By iterating we have

$$H(t)\leq K\int_0^t H(s)\int_0^1 g^2_{t-s}(x,y)\,dy\,ds\leq\frac{K^2}{2\pi}\int_0^t\int_0^s H(u)\,\frac{du\,ds}{\sqrt{(t-s)(s-u)}}.$$

Setting $v:=(s-u)/(t-u)$ we have

$$\int_u^t\frac{ds}{\sqrt{(t-s)(s-u)}}=\int_0^1\frac{dv}{\sqrt{v(1-v)}}\leq 2\int_0^{\frac12}\frac{dv}{\sqrt{v}}\leq 4.$$

Therefore, by Fubini

$$H(t)\leq\frac{2\,K^2}{\pi}\int_0^t H(s)\,ds.$$

By Gronwall's Lemma

$$H(t)\leq H(0)\,e^{\frac{2\,K^2}{\pi}\,t},$$

and since $H(0)=0$ we have $H\equiv 0$ and $u_1\equiv u_2$.

We prove now existence of a solution. We define

$$\begin{cases} u^0(t,x):=\displaystyle\int_0^1 g_t(x,y)\,u_0(y)\,dy\\ u^{n+1}(t,x):=u^0(t,x)+\displaystyle\int_0^t\int_0^1 g_{t-s}(x,y)\,\left(f(u^n(s,y))\,dy\,ds+\sigma(u^n(s,y))\,W(dy,ds)\right)\end{cases}$$

and

$$F_n(t,x):=\mathbb{E}\left(\left|u^{n+1}(t,x)-u^n(t,x)\right|^2\right),\qquad H_n(t):=\sup_{x\in[0,1]}F_n(t,x).$$

Then with the same computations as above

$$\begin{aligned}F_{n+1}(t,x)\leq & 2T\,\mathbb{E}\left(\int_0^t\int_0^1 g^2_{t-s}(x,y)\,|f(u^{n+1}(s,y))-f(u^n(s,y))|^2\,dy\,ds\right)+\\ &+2\,\mathbb{E}\left(\int_0^t\int_0^1 g^2_{t-s}(x,y)\,|\sigma(u^{n+1}(s,y))-\sigma(u^n(s,y))|^2\,dy\,ds\right)\\ \leq & K\int_0^t\int_0^1 g^2_{t-s}(x,y)\,F_n(s,y)\,dy\,ds\end{aligned}$$

and we obtain by iteration

$$H_{n+2}(t) \leq C \int_0^t H_n(s)\, ds$$

where C is a constant. By induction on $m \geq 1$, we find for $n = 0, 1$

$$H_{n+2m}(t) \leq \frac{C^m}{(m-1)!} \int_0^t H_n(s)\, (t-s)^m\, ds.$$

Therefore

$$\sum_{n=1}^{+\infty} \sqrt{H_n(t)} = \sum_{n=0}^{1} \sum_{m=1}^{+\infty} \sqrt{H_{n+2m}(t)} < +\infty.$$

If follows that (u^n) is a Cauchy sequence in $C([0,T] \times [0,1]; L^2(\Omega))$. Therefore $u^n \to u$ and it is easy to show that u is a solution of (8.3) by passing to the limit in the definition of u^{n+1} above. □

Remark 8.2. In the existence part of the proof we have been lazy, in that we have considered convergence in L^2. It is possible to prove that for $u_0 \in L^p(0,1)$, the sequence u^n converges in $C([0,T] \times [0,1]; L^p(\Omega))$, $p > 1$. One can also obtain the estimate

$$\mathbb{E}\left(|u(t,x) - u(s,y)|^p\right) \leq C \left(|x-y|^{\frac{1}{2} - \frac{1}{p}} + |t-s|^{\frac{1}{4} - \frac{2}{p}} \right)$$

and construct, by Kolmogorov's Theorem, a continuous modification of $(u(t,x), t \geq 0, x \in [0,1])$.

8.3. **Additive white noise.** If the noise coefficient σ is constant, e.g. $\sigma \equiv 1$, then existence and uniqueness of solutions are easier, since (8.3) becomes

$$\begin{aligned} u(t,x) = & \int_0^1 g_t(x,y)\, u_0(y)\, dy + \int_0^t \int_0^1 g_{t-s}(x,y)\, W(dy,ds) \\ & + \int_0^t \int_0^1 g_{t-s}(x,y)\, f(u(s,y))\, dy\, ds \end{aligned} \tag{8.5}$$

and the sum of the first two terms in the right hand side is the solution of the linear heat equation (6.1). Then one can simply define a map $\Gamma : C([0,T]; H) \mapsto C([0,T]; H)$

$$\begin{aligned} \Gamma(v)(t) := & \int_0^1 g_t(\cdot,y)\, u_0(y)\, dy + \int_0^t \int_0^1 g_{t-s}(\cdot,y)\, W(dy,ds) \\ & + \int_0^t \int_0^1 g_{t-s}(\cdot,y)\, f(v(s,y))\, dy\, ds \end{aligned}$$

and look for a fixed point. This can be done analogously to the proof of Theorem 2.1, by introducing $B := C([0,T]; H)$ with norm $\|f\|_B := \sup_{t \in [0,T]} e^{-Kt} \|f_t\|$, for some $K > 0$, and showing that Γ is a contraction in B. Notice that, as in the proof of Theorem 2.1, by considering the difference $\Gamma(u) - \Gamma(v)$ the stochastic parts cancel out and this simplifies considerably the proof.

We can in fact study a more difficult class of equations, namely

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial}{\partial x} [f(u)] + \dot{W}, \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x), \qquad x \in [0,1]. \end{cases} \tag{8.6}$$

Notice that, as for the linear equation (6.1), we do not expect u to be differentiable in x, and therefore the nonlinearity in (8.6) has to be properly interpreted. A classical example of equation (8.6) is when $f(u) = u^2$, a case known as *Burgers's equation.* This is a one-dimensional version of the more important (and much more difficult) stochastic Navier-Stokes equation.

Arguing as in the proof of (8.3), the standard way is to multiply by a test function and integrate by parts. We obtain for $h \in D(A)$

$$\langle u_t, h\rangle = \langle u_0, h\rangle + \int_0^t \langle u_s, Ah\rangle\, ds - \int_0^t \langle f(u_s), h'\rangle\, ds + \int_0^t \int_0^1 h(y)\, W(ds, dy)$$

and therefore, after choosing $h = e_k$ and then going back to the Fourier series, we obtain the mild formulation

$$\begin{aligned} u(t,x) = &\int_0^1 g_t(x,y)\, u_0(y)\, dy + \int_0^t \int_0^1 g_{t-s}(x,y)\, W(dy, ds) \\ &- \int_0^t \int_0^1 \frac{\partial}{\partial y} g_{t-s}(x,y)\, f(u(s,y))\, dy\, ds. \end{aligned} \tag{8.7}$$

Actually, the last term requires some care: indeed, it is not obvious that it is well-defined, due to the singularity of $\frac{\partial}{\partial y} g_{t-s}(x,y)$. However, notice that

$$\frac{\partial}{\partial y} g_t(x,y) = \sum_{k=1}^\infty e^{-t(\pi k)^2/2}\, e_k(x)\, k\, \eta_k(y), \qquad t > 0,\ x, y \in [0,1],$$

where $\eta_k(y) := \sqrt{2}\, \cos(\pi k y)$. Notice that $(\eta_k)_{k\geq 1}$ is again a complete orthonormal system in H. Then for all $h \in H$

$$\begin{aligned} \int_0^1 dx \left(\int_0^1 dy\, \frac{\partial}{\partial y} g_t(x,y)\, h(y) \right)^2 &= \left\| \sum_{k=1}^\infty e^{-t(\pi k)^2/2}\, e_k\, k\, \langle h, \eta_k\rangle \right\|^2 \\ = \sum_{k=1}^\infty k^2 e^{-t(\pi k)^2} \langle h, \eta_k\rangle^2 &\leq \sup_{k\geq 1} k^2 e^{-t(\pi k)^2}\, \|h\|^2 \leq \frac{C}{t}\, \|h\|^2, \end{aligned}$$

since

$$\sup_{k\geq 1} k^2 e^{-t(\pi k)^2} = \frac{1}{t\pi^2} \sup_{k\geq 1} t(\pi k)^2 e^{-t(\pi k)^2} \leq \frac{1}{t\pi^2} \sup_{r>0} r e^{-r} = \frac{C}{t}.$$

Thus

$$\left\| \int_0^1 \frac{\partial}{\partial y} g_{t-s}(\cdot, y)\, f(u(s,y))\, dy \right\|_H \leq \frac{C}{\sqrt{t-s}}\, \|f(u(s,\cdot))\|_H$$

and therefore if $u \in C([0,T]; H)$

$$\left\| \int_0^t \int_0^1 \frac{\partial}{\partial y} g_{t-s}(\cdot, y)\, f(u(s,y))\, dy\, ds \right\|_H \leq \int_0^t \frac{C}{\sqrt{t-s}}\, ds \sup_{r\in[0,t]} \|f(u(r,\cdot))\|_H < +\infty.$$

Therefore the last term of (8.7) is well defined for all t in H. With the same estimates, we prove now existence and uniqueness of solutions of (8.7) under a standard Lipschitz-continuity assumption on f. We define the map $\Gamma : C([0,T]; H) \mapsto C([0,T]; H)$

$$\begin{aligned} \Gamma(v)(t) := &\int_0^1 g_t(\cdot, y)\, u_0(y)\, dy + \int_0^t \int_0^1 g_{t-s}(\cdot, y)\, W(dy, ds) \\ &- \int_0^t \int_0^1 \frac{\partial}{\partial y} g_{t-s}(\cdot, y)\, f(v(s,y))\, dy\, ds. \end{aligned}$$

By the above estimate

$$\|\Gamma(v)(t)-\Gamma(u)(t)\|_H = \left\| \int_0^t \int_0^1 \frac{\partial}{\partial y} g_{t-s}(\cdot,y)\, \left(f(v(s,y))-f(u(s,y))\right) dy\, ds \right\|_H$$
$$\le \int_0^t \frac{C}{\sqrt{t-s}} \, \|f(v(s,\cdot))-f(u(s,\cdot))\|_H \, ds \le L \int_0^t \frac{Ce^{Ks}}{\sqrt{t-s}} \, e^{-Ks} \, \|v(s,\cdot)-u(s,\cdot)\|_H \, ds$$

since f is L-Lipschitz. Therefore by multiplying both sides by e^{-Kt} and taking the supremum over $t \in [0,T]$ we obtain

$$\|\Gamma(v)-\Gamma(u)\|_B \le L \int_0^T \frac{Ce^{-Ks}}{\sqrt{s}} \, ds \, \|v-u\|_B \, .$$

Therefore, for K sufficiently large Γ is a contraction in B and it has a unique fixed point.

9. Ergodic theory

Let E be a Polish space. We consider a family $(\Theta_t)_{t\in\mathbb{N}}$ of maps $\Theta_t : E \mapsto E$ such that

(1) $\Theta_0 = Id$
(2) $\Theta_s \circ \Theta_t = \Theta_{t+s}$ for all $s,t \in \mathbb{N}$
(3) the map Θ_t is measurable for all $t \in \mathbb{N}$.

Such a family is called a *dynamical system*. Denoting by $\mathcal{M}_1(E)$ the space of all probability measures on E, we define the set of all *invariant probability measures*

$$\mathcal{J}(\Theta) := \{\mathbf{P} \in \mathcal{M}_1(E) : \ \Theta_t^*\mathbf{P} = \mathbf{P}, \ \forall\, t \in \mathbb{N}\}$$

where $\Theta_t^*\mathbf{P}$ is the image measure of $\mathbf{P}$ under Θ_t. For all $\mathbf{P} \in \mathcal{J}(\Theta)$ we define the σ-algebra of all *invariant sets*

$$\mathcal{I}_{\mathbf{P}} := \{A \in \mathcal{B}(E) : \ \mathbb{1}_A \circ \Theta_t = \mathbb{1}_A \quad \mathbf{P}-\text{a.s.}\ \forall\, t \in \mathbb{N}\} \, .$$

Then we have the classical *Birkhoff Ergodic Theorem*

Theorem 9.1. *Let $\mathbf{P} \in \mathcal{J}(\Theta)$ and $F \in L^1(E,\mathbf{P})$. Then*

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} F(\Theta_n(\omega)) = \mathbf{E}(F \,|\, \mathcal{I}_{\mathbf{P}})(\omega), \qquad \text{for } \mathbf{P}-\text{a.e. } \omega \in E.$$

For a proof, see [10, section 2.2]. We say that $\mathbf{P} \in \mathcal{J}(\Theta)$ is *ergodic* if $\mathbf{P}(A) \in \{0,1\}$ for all $A \in \mathcal{I}_{\mathbf{P}}$. In this case Theorem 9.1 yields

Proposition 9.2. $\mathbf{P} \in \mathcal{J}(\Theta)$ *is ergodic iff for all* $F \in L^1(E,\mathbf{P})$

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\Theta_n(\omega)) = \mathbf{E}(f), \qquad \text{for } \mathbf{P}-\text{a.e. } \omega \in E.$$

9.1. **Applications to Markov processes.** Let $(X_t)_{t\ge0}$ be a continuous Markov process taking values in a Polish space T and set $E := T^{\mathbb{N}}$. Let $(P_t(x,A), x \in T, A \in \mathcal{B}(T))_{t\ge0}$ be the transition kernel of X, and denote by $(P_t)_{t\ge0}$ the transition semigroup

$$P_t f(x) := \int_T f(x)\, P_t(x,dy), \qquad \forall\, t \ge 0,\ x \in T,\ f \in C_b(T).$$

We suppose that $(P_t)_{t\ge0}$ is *Feller*, i.e. $P_t f \in C_b(T)$ for all $f \in C_b(T)$ and $t \ge 0$.

If the initial law of X, i.e. the law of X_0, is ν, then we denote by $\mathbf{P}_\nu$ the law of $(X_t)_{t\in\mathbb{N}}$ on E. If $\nu = \delta_x$, a Dirac mass concentrated at $x \in T$, then we denote $\mathbf{P}_{\delta_x} = \mathbf{P}_x$. Recall that by the Markov property

$$\mathbf{P}_\nu = \int_T \mathbf{P}_x \, \nu(dx), \qquad \nu \in \mathcal{M}_1(T). \tag{9.1}$$

Suppose that μ is an invariant probability measure for $(P_t)_{t\geq 0}$, namely

$$\int_T P_t f \, d\mu = \int_T f \, d\mu, \qquad \forall\, t \geq 0,\ f \in C_b(T),$$

i.e. $P_t^* \mu = \mu$ for all $t \geq 0$. If X_0 has law μ, then X is stationary, i.e. $(X_{t+s})_{t\geq 0}$ has the same distribution as $(X_t)_{t\geq 0}$. We define the *shift semigroup*

$$\Theta_n : E \mapsto E, \quad \Theta_n(\omega) := \omega(n + \cdot), \qquad n \in \mathbb{N}.$$

Then $(\Theta_n)_{n\in\mathbb{N}}$ is a dynamical system and $\mathbf{P}_\mu \in \mathcal{J}(\Theta)$. We denote by

$$\mathcal{J}(P) := \{\mu \in \mathcal{M}_1(T) :\ P_t^* \mu = \mu,\ \forall\, t \geq 0\}.$$

For all $\mu \in \mathcal{J}(P)$ we set

$$\mathcal{I}_\mu := \{A \in \mathcal{B}(T) :\ P_t \mathbb{1}_A = \mathbb{1}_A \quad \mu - \text{a.s.}\ \forall\, t \geq 0\}.$$

We say that $\mu \in \mathcal{J}(P)$ is *ergodic* if $\mathbf{P}_\mu$ is ergodic for the dynamical system $(\Theta_n)_{n\in\mathbb{N}}$. We need first a technical result, whose statement is intuitive, although the proof is not completely obvious. The interpretation is that a Θ-invariant set is necessarily identified by its projection at time 0.

Lemma 9.3. *Every invariant set $\mathcal{A} \in I_{\mathbf{P}_\mu}$ is equal, for some $A \in I_\mu$, to*

$$\{\omega \in E : \omega_n \in A,\ \forall\, n \in \mathbb{N}\} = A^{\mathbb{N}}$$

up to a $\mathbf{P}_\mu$-negligible set, i.e. $\mathbf{P}_\mu(\mathcal{A} \Delta A^{\mathbb{N}}) = 0$. In particular, $\mathbf{P}_\mu(\mathcal{A}) = \mathbf{P}_\mu(A^{\mathbb{N}}) = \mu(A)$.

Proof. For the proof of the existence of A, see [7, Proposition 5.2, Corollaries 5.3-5.5]. Let us prove the formula $\mathbf{P}_\mu(A^{\mathbb{N}}) = \mu(A)$. We claim that, for $A \in \mathcal{I}_\mu$

$$\mathbf{P}_x(\omega_1 \in A, \ldots, \omega_n \in A) = 1, \qquad \forall\, x \in A,\ n \geq 0.$$

We proceed by induction. For $n = 0$ the result being obvious, we suppose it holds for n. Then by the Markov property

$$\mathbf{P}_x(\omega_0 \in A, \ldots, \omega_{n+1} \in A) = \mathbb{1}_A(x)\, \mathbf{P}_x(\omega_1 \in A, \ldots, \omega_{n+1} \in A) = \mathbf{P}_x(\omega_0 \in A, \ldots, \omega_n \in A).$$

Therefore

$$\mathbf{P}_\mu(\omega_0 \in A, \ldots, \omega_n \in A) = \int_A \mu(dx)\, \mathbf{P}_x(\omega_0 \in A, \ldots, \omega_n \in A) = \mu(A)$$

and

$$\mathbf{P}_\mu(A^{\mathbb{N}}) = \lim_{n\to\infty} \mathbf{P}_\mu(\omega_1 \in A, \ldots, \omega_n \in A) = \lim_{n\to\infty} \mu(A) = \mu(A).$$

□

By Birkhoff's Theorem, we obtain

Proposition 9.4. *$\mu \in \mathcal{J}(P)$ is ergodic iff for all $f \in L^1(T, \mu)$*

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(X_n) = \int_T f \, d\mu, \quad \mathbf{P}_\mu\text{--a.s.}$$

Proof. Let $f \in L^1(T,\mu)$, and set $F(\omega) := f(\omega(0))$, $\omega \in E$. Then we have

$$\frac{1}{N}\sum_{n=0}^{N-1} f(X_n) = \frac{1}{N}\sum_{n=0}^{N-1} F\circ\Theta_n, \qquad \int_E F\,d\mathbf{P}_\mu = \int_T f\,d\mu.$$

If $\mathbf{P}_\mu$ is ergodic, then the desired result follows by

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(X_n) = \lim_{N\to+\infty}\frac{1}{N}\sum_{n=0}^{N-1} F\circ\Theta_n = \int_E F\,d\mathbf{P}_\mu = \int_T f\,d\mu, \quad \mathbf{P}_\mu\text{−a.s.}$$

by Birkhoff's Theorem. Conversely, suppose that the formula above holds for all $f \in L^1(T,\mu)$. Let $\mathcal{A} \in I_{\mathbf{P}_\mu}$ and let $A \in \mathcal{B}(T)$ as in Lemma 9.3. By applying the above formula to $f = \mathbb{1}_A$, we obtain by Birkhoff's Theorem that

$$\mathbf{P}_\mu(\mathcal{A}\,|\,I_{\mathbf{P}_\mu}) = \lim_{N\to+\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{1}_{\mathcal{A}}\circ\Theta_n = \lim_{N\to+\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{1}_{(X_n\in A)} = \mu(A),$$

$\mathbf{P}_\mu$-a.s. Since $\mu(A)$ is non-random, it means that $I_{\mathbf{P}_\mu}$ is trivial, i.e. $\mathbf{P}_\mu$ is ergodic. □

Corollary 9.5. *$\mu \in \mathcal{J}(P)$ is ergodic if and only if $\mu(A) \in \{0,1\}$ for all $A \in I_\mu$.*

Proof. If $\mu \in \mathcal{J}(P)$ is ergodic then we obtain by the previous result that

$$\mu(A) = \mathbf{P}_\mu(\mathcal{A}\,|\,I_{\mathbf{P}_\mu}) \Longrightarrow \mu(A) = \mathbf{P}_\mu(\mathcal{A}) \in \{0,1\}.$$

Conversely, if μ is not ergodic, then $\mathbf{P}_\mu$ is not ergodic, i.e. there exists $\mathcal{A} \in I_{\mathbf{P}_\mu}$ with $\mathbf{P}_\mu(\mathcal{A}) \notin \{0,1\}$. Let A as in Lemma 9.3; then $\mu(A) = \mathbf{P}_\mu(\mathcal{A}) \notin \{0,1\}$. □

The set of invariant measures $\mathcal{J}(P)$ has an interesting structure. Indeed it is convex and bounded in the weak-$*$ topology, namely

$$\sup_{\varphi\in C_b(T)\setminus\{0\}}\frac{\left|\int\varphi\,d\mu\right|}{\|\varphi\|_\infty} \le 1, \qquad \forall\,\mu\in\mathcal{J}(P).$$

We recall that $\mu,\nu \in \mathcal{M}_1(T)$ are said to be (mutually) singular if there exists $A \in \mathcal{B}(T)$ such that $\mu(A) = 1$ and $\nu(A) = 0$.

Proposition 9.6. *If $\mu,\nu \in \mathcal{J}(P)$ are ergodic and $\mu \neq \nu$ then they are mutually singular.*

Proof. Let $C \in \mathcal{B}(T)$ such that $\mu(C) \neq \nu(C)$ and define the event

$$A := \left\{x \in T : \lim_{N\to+\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{1}_C(X_n) = \mu(C), \quad \mathbf{P}_x - \text{a.s.}\right\}.$$

By ergodicity of $\mathbf{P}_\mu$ we have $\mathbf{P}_\mu(W) = 1$, where

$$W := \left\{\omega \in E : \lim_{N\to+\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{1}_C(\omega(n)) = \mu(C)\right\}.$$

By Fubini's Theorem and by (9.1) we have

$$1 = \mathbf{P}_\mu(W) = \int_T \mathbf{P}_x(W)\,\mu(dx)$$

so that $\mathbf{P}_x(W) = 1$ for μ-a.e. x, i.e. $\mu(A) = 1$. Analogously we have $\mathbf{P}_\nu(\tilde{W}) = 1$, where

$$\tilde{W} := \left\{\omega \in E : \lim_{N\to+\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{1}_C(\omega(n)) = \nu(C)\right\}$$

and $W \cap \tilde{W} = \emptyset$, so that $\mathbf{P}_\nu(W) = 0$. By Fubini's Theorem and by (9.1) we have

$$0 = \mathbf{P}_\nu(W) = \int_T \mathbf{P}_x(W)\, \nu(dx)$$

so that $\mathbf{P}_x(W) = 0$ for ν-a.e. x, i.e. $\nu(A) = 0$. □

We say that $\mu \in \mathcal{J}(P)$ is *extremal* if it can not be written as a non-trivial convex combination of $\mu_1, \mu_2 \in \mathcal{J}(P)$, i.e. if

$$\mu = (1-\alpha)\mu_1 + \alpha\mu_2, \quad \alpha \in (0,1) \Longrightarrow \mu_1 = \mu_2 = \mu.$$

Proposition 9.7. *$\mu \in \mathcal{J}(P)$ is ergodic iff μ is extremal.*

Proof. Let us first prove that if $\mu \in \mathcal{J}(P)$ is ergodic and $\nu \in \mathcal{J}(P)$ is absolutely continuous w.r.t. μ, then $\nu = \mu$. For all $A \in \mathcal{B}(T)$, by Birkhoff's Theorem

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{1}_A(X_n) = \mu(A), \quad \mathbf{P}_\mu\text{−a.s.}$$

Since ν is absolutely continuous w.r.t. μ, then $\mathbf{P}_\nu$ is absolutely continuous w.r.t. $\mathbf{P}_\mu$ and we have

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{1}_A(X_n) = \mu(A), \quad \mathbf{P}_\nu\text{−a.s.}$$

By dominated convergence

$$\lim_{N\to+\infty} \mathbf{E}_\nu \left[\frac{1}{N} \sum_{n=0}^{N-1} \mathbb{1}_A(X_n) \right] = \mu(A).$$

On the other hand

$$\mathbf{E}_\nu \left[\frac{1}{N} \sum_{n=0}^{N-1} \mathbb{1}_A(X_n) \right] = \frac{1}{N} \sum_{n=0}^{N-1} \nu(A) = \nu(A),$$

so that $\mu(A) = \nu(A)$. Let us now consider an ergodic $\mu \in \mathcal{J}(P)$ such that

$$\mu = (1-\alpha)\mu_1 + \alpha\mu_2, \quad \alpha \in (0,1), \ \mu_1, \mu_2 \in \mathcal{J}(P).$$

Then μ_1 is absolutely continuous w.r.t. μ and therefore $\mu_1 = \mu = \mu_2$.

Conversely, if μ is extremal but not ergodic, then there exists $C \in \mathcal{I}_\mu$ such that $\mu(C) \in]0,1[$. If we define $\mu_1 := \mu(\cdot \,|\, C)$, $\mu_2 := \mu(\cdot \,|\, T \backslash C)$, then $\mu_1, \mu_2 \in \mathcal{J}(P)$ and

$$\mu = \mu(C)\, \mu_1 + (1 - \mu(C))\, \mu_2,$$

a contradiction. □

As an immediate corollary we obtain the important

Theorem 9.8. *If $\mathcal{J}(P) = \{\mu\}$, i.e. there exists a unique invariant probability measure μ, then μ is ergodic.*

10. An existence result

Let us consider again a *Feller* transition semigroup $(P_t)_{t\geq 0}$, i.e. $P_t f \in C_b(T)$ for all $f \in C_b(T)$ and $t \geq 0$. We recall that a family $(\mu_i)_{i\in I} \subset \mathcal{M}_1(T)$ is *tight* if for all $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset T$ such that

$$\sup_{i\in I} \mu_i(K_\varepsilon) \geq 1 - \varepsilon.$$

Since T is Polish, Prohorov's Theorem states that tightness and pre-compactness in $\mathcal{M}_1(T)$ are equivalent. Then we have the classical

Theorem 10.1 (Krylov-Bogolioubov). *Let $(P_t)_{t\geq 0}$ be Feller and suppose that there exists $\mu_0 \in \mathcal{M}_1(T)$ such that $(P_t^*\mu_0)_{t\geq 0}$ is tight in T. Then there exists an invariant probability measure $\mu \in \mathcal{J}(P)$.*

Proof. Let $\mu_t \in \mathcal{M}_1(T)$ be defined by

$$\mu_t := \frac{1}{t}\int_0^t P_s^*\mu_0 \, ds, \qquad t > 0.$$

Tightness of $(P_t^*\mu_0)_{t\geq 0}$ easily implies tightness of $(\mu_t)_{t>0}$. Then, by Prohorov's Theorem there exists a sequence $t_n \to +\infty$ and $\mu_\infty \in \mathcal{M}_1(T)$ such that $\mu_{t_n} \rightharpoonup \mu_\infty$. Then for any test function $\varphi \in C_b(T)$ and $r \geq 0$ we have $P_r\varphi \in C_b(T)$ and

$$\begin{aligned} \langle P_r\varphi, \mu\rangle &= \lim_n \langle P_r\varphi, R_{t_n}\rangle = \lim_n \frac{1}{t_n}\int_0^{t_n} \langle \varphi, P_{r+s}^*\mu_0\rangle \, ds \\ &= \lim_n \left[\frac{1}{t_n}\int_0^{t_n} \langle \varphi, P_s^*\mu_0\rangle \, ds + \frac{1}{t_n}\int_{t_n}^{t_n+r} \langle \varphi, P_s^*\mu_0\rangle \, ds - \frac{1}{t_n}\int_0^{r} \langle \varphi, P_s^*\mu_0\rangle \, ds\right] = \langle \varphi, \mu\rangle. \end{aligned}$$

□

10.1. **Lyapunov functions.** A classical sufficient condition for existence of an invariant probability measure is the existence of a *Lyapunov function*, which means a function which tends to $+\infty$ at ∞ but remains bounded (in a sense to be made precise) along the dynamics.

Let us consider a continuous Markov process $(X_t)_{t\geq 0}$ in $\mathbb{R}^d$ with infinitesimal generator $(L, D(L))$, in the sense that for all $\varphi \in C_b^2(\mathbb{R}^d)$ we have

$$\frac{d}{dt} P_t\varphi = P_t L\varphi, \qquad t > 0.$$

A Lyapunov function is $V : \mathbb{R}^d \mapsto \mathbb{R}$ of class C^2 such that

1. $V(x) \to +\infty$ as $|x| \to +\infty$
2. $LV \leq C_1 - C_2 V$ for some $C_1, C_2 > 0$.

Then we have

Proposition 10.2. *If there exists a Lyapunov function then X has an invariant probability measure.*

Proof. Let us compute $V(X_t)$. By Itô's formula we have

$$\frac{d}{dt} P_t V(x) = P_t LV(x) \leq C_1 - C_2 P_t V(x),$$

so that

$$P_t V(x) \leq \frac{C_1}{C_2}(1 - e^{-C_2 t}) + V(x)\, e^{-C_2 t} \leq \frac{C_1}{C_2} + V(x) \leq C(1 + V(x)), \qquad t \geq 0.$$

Let now $K_\varepsilon := \{x \in \mathbb{R}^d : V(x) \le \varepsilon^{-1}\}$, $\varepsilon > 0$. Since $V(x) \to +\infty$ as $|x| \to +\infty$, then K_ε is compact. Moreover by the Markov inequality

$$\mathbb{P}_x(X_t \notin K_\varepsilon) = \mathbb{P}_x(V(X_t) > \varepsilon^{-1}) \le \varepsilon\,\mathbb{E}_x(V(X_t)) \le \varepsilon\, C(1 + V(x)),$$

so that the family $(P_t(x,\cdot))_{t\ge 0}$ is tight. By the Krylov-Bogolioubov Theorem, we conclude. □

A classical example is the solution of

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \qquad X_0 = x,$$

with b and σ Lipschitz and

$$\langle b(x), x\rangle \le -c|x|^2, \qquad \|\sigma| \le C,$$

for some positive constants c, C. Then $V(x) = \|x\|^2$ is a Lyapunov function, since it satisfies

$$LV(x) = \mathrm{Tr}[\sigma^*\sigma](x) + 2\langle b(x), x\rangle \le C_1(1 - V(x)),$$

for some positive constant C_1.

11. Uniqueness results

Let $(X_t)_{t\ge 0}$ be a continuous Markov process in a Polish space T. We call $(P_t)_{t\ge 0}$ its transition semigroup.

We say that $(P_t)_{t\ge 0}$ is

(1) t_0-regular if all transition probabilities $(P_{t_0}(x,\cdot), x \in T)$ are equivalent.
(2) *irreducible* if for all nonempty open $A \subset T$

$$P_t \mathbb{1}_A(x) > 0, \qquad \forall\, x \in T,\ t > 0.$$

(3) *Strong Feller* if for all bounded Borel $\varphi : T \mapsto \mathbb{R}$ we have $P_t\varphi \in C_b(T)$ for all $t > 0$.

For instance, the Brownian motion is t_0-regular for all $t_0 > 0$, irreducible and Strong Feller in $\mathbb{R}$.

Proposition 11.1. *Suppose that $(P_t)_{t\ge 0}$ is irreducible and Strong Feller. Then*

(1) *$(P_t)_{t\ge 0}$ is t_0-regular for all $t_0 > 0$.*
(2) *There exists at most one invariant probability measure.*
(3) *If μ is an invariant probability measure, then $P_t(x,\cdot)$ is equivalent to μ for all $t > 0$ and $x \in T$.*

Proof. Let us prove (1). Suppose that $P_t(x_0, A) > 0$ for $t > 0$. Since

$$P_t(x_0, A) = \int P_{t/2}(x_0, dy)\, P_{t/2}(y, A) > 0$$

then there must be some y s.t. $P_{t/2}(y, A) > 0$. Since $P_{t/2}(y, A) = P_{t/2}\mathbb{1}_A(y)$ and $P_{t/2}\mathbb{1}_A \in C_b(T)$ by the Strong Feller property, then there must be a non-empty open set O such that $P_{t/2}(z, A) \ge \delta > 0$ for all $z \in O$. Now, for any other x, we know by irreducibility that $P_{t/2}(x, O) > 0$. Therefore

$$P_t(x, A) = \int P_{t/2}(x, dz)\, P_{t/2}(z, A) \ge P_{t/2}(x, O)\cdot\delta > 0.$$

Therefore $P_{t/2}(x,\cdot) \ll P_{t/2}(x_0,\cdot)$.

Let us prove that any invariant probability measure is necessarily ergodic. Let $\mu \in \mathcal{J}(P)$ be an invariant probability measure and $\Gamma \in \mathcal{I}_\mu$ an invariant set, i.e. $P_t\mathbb{1}_\Gamma = \mathbb{1}_\Gamma$, μ-a.e.

such that $\mu(\Gamma) > 0$. We have to show that $\mu(\Gamma) = 1$. By invariance we have $P_t(x, \Gamma) = 1$ for μ-a.e. $x \in \Gamma$, and since $\mu(\Gamma) > 0$, then there is at least a $x_0 \in T$ such that $P_t(x_0, \Gamma) = 1$. Since all transition probabilies $(P_t(x, \cdot), x \in T)$ are equivalent, then $P_t(y, \Gamma) = 1$ for all $y \in T$. Therefore

$$\mu(\Gamma) = \int P_t(y, \Gamma)\, \mu(dy) = 1.$$

Let us now prove that μ is equivalent to $P_t(x, \cdot)$ for any $t > 0$ and $x \in T$. If $P_t(x, A) = 0$ then $P_t(y, A) = 0$ for any $y \in T$ and therefore

$$\mu(A) = \int P_t(y, A)\, \mu(dy) = 0.$$

Viceversa, if $\mu(A) = 0$, by the same formula, $P_t(y, A) = 0$ for μ-a.e. $y \in T$. Since all transition laws are equivalent, we have that $P_t(y, A) = 0$ for all $y \in T$.

Finally, suppose we have $\nu \in \mathcal{J}(P)$ and $\nu \neq \mu$. Then ν is ergodic and equivalent to $P_t(y, \cdot)$ for all y, therefore equivalent to μ. By Proposition 9.6, this is a contradiction. $\square$

11.1. **Uniqueness through coupling.** Let us consider the stochastic differential equation (SDE)

$$dX = F(X_t)\, dt + B\, dW_t$$

where $W_t : [0, T] \mapsto \mathbb{R}^d$ is a Brownian motion, $B : \mathbb{R}^d \mapsto \mathbb{R}^d$ is *any* linear map, and $F : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a Lipshitz continuous map satisfying

$$\langle F(x) - F(y), x - y\rangle \leq -c\|x - y\|^2, \qquad \forall\, x, y \in \mathbb{R}^d,$$

for some positive constant $c > 0$. We say that F is *dissipative*.

We want to prove that there is at most one invariant probability measure of $(X_t, t \geq 0)$. We prove actually more:

Theorem 11.2. *There exists a unique invariant probability measure μ of $(P_t)_{t\geq 0}$. Moreover, for any probability measure ν on $\mathbb{R}^d$*

$$\lim_{t\to+\infty} \int P_t f\, d\nu = \int f\, d\mu, \qquad \forall\ f \in C_b(\mathbb{R}^d).$$

Proof. Let us first notice that the function $V(x) := \|x\|^2$ is a Lyapunov function, since

$$\begin{aligned} LV(x) &= \mathrm{Tr}[B^*B] + 2\langle F(x), x\rangle = \mathrm{Tr}[B^*B] + 2\langle F(x) - F(0), x - 0\rangle + 2\langle F(0), x\rangle \\ &\leq \mathrm{Tr}[B^*B] - c\|x\|^2 + C\|x\| \leq \mathrm{Tr}[B^*B] + \frac{C^2}{2c} - \frac{c}{2}\,\|x\|^2 \end{aligned}$$

where we have applied the inequality $|2ab| \leq a^2 + b^2$ to $a = \|x\|\sqrt{c}$ and $b = C/\sqrt{c}$. Then an invariant probability measure μ exists by (10.2).

Let Y and Z be two random variables such that Y has law μ, Z has law ν and (Y, Z, W) are independent. Since μ is invariant for $(P_t)_{t\geq 0}$ then for all $t \geq 0$:

$$\int f\, d\mu = \int P_t f\, d\mu = \int \mathbb{E}[f(X_t(x))]\, d\mu(x) = \mathbb{E}[f(X_t(Y))].$$

In particular, $(X_t(Y), t \geq 0)$ is a *stationary* solution of

$$dX_t = -F(X_t)\, dt + B\, dW_t.$$

Let us define now $Y_t := X_t(Y)$, $Z_t := X_t(Z)$. Notice that $Y_t - Z_t$ satisfies

$$Y_t - Z_t = Y - Z - \int_0^t (F(Y_s) - F(Z_s))\, ds, \qquad t \geq 0.$$

Then we obtain by the dissipativity assumption on F that

$$\frac{d}{dt}\|Y_t - Z_t\|^2 = -\langle Y_t - Z_t, F(Y_t) - F(Z_t)\rangle \le -c\|Y_t - Z_t\|^2.$$

Then $\|Y_t - Z_t\|^2 \le e^{-ct}\|Y - Z\|^2$. Let now $f : \mathbb{R}^d \mapsto \mathbb{R}$ be any bounded Lipschitz-continuous function, in particular there exists $K \ge 0$ such that

$$|f(x) - f(y)| \le K\,(\|x - y\| \wedge 1), \qquad \forall\, x, y \in \mathbb{R}^d.$$

Then

$$\begin{aligned} |\mu(f) - \nu(P_t f)| &= |\mu(P_t f) - \nu(P_t f)| = |\mathbb{E}[f(X_t(Y)) - f(X_t(Z))]| \\ &\le K\,\mathbb{E}(\|Y_t - Z_t\| \wedge 1) \to 0 \end{aligned}$$

as $t \to +\infty$, by dominated convergence.

Suppose now that also ν is an invariant probability measure of $(P_t)_{t\ge 0}$. Then $\nu(P_t f) = \nu(f)$ and we obtain that $\int f\,d\mu = \int f\,d\nu$, for all bounded Lipschitz-continuous function f; by approximation and dominated convergence we have the same equality for all $f \in C_b(\mathbb{R}^d)$. This means that μ is the only invariant probability measure of $(P_t)_{t\ge 0}$. $\square$

The trick of considering two copies of the solutions with independent initial condition but with *the same noise* is often used and is an example of *coupling*, namely a pair of processes with specified marginal distributions.

11.2. **Dissipative gradient systems.** Throughout these notes, unless otherwise specified, $U : \mathbb{R}^d \mapsto \mathbb{R}$ denotes a C^3 function such that:

- there exists $\delta \in]0, 1]$ with $\delta I \le D^2 U \le \delta^{-1} I$,
- all derivatives of U of order up to the third are Lipschitz continuous.

By the first assumption

$$\liminf_{|x|\to+\infty} \frac{U(x)}{|x|^2} > 0, \tag{11.1}$$

so that, in particular, $Z := \int \exp(-U(z))\,dz < +\infty$. By the second assumption, $\nabla U : \mathbb{R}^d \mapsto \mathbb{R}^d$ is Lipschitz continuous. Then we consider the SDE

$$dX = -\frac{1}{2}\nabla U(X)\,dt + B\,dW_t, \qquad X_0 = x$$

Let us show that $-\nabla U$ is a dissipative non-linearity, i.e.

$$-\langle y - x, \nabla U(y) - \nabla U(x)\rangle \le -\delta\|y - x\|^2, \qquad \forall y, x \in \mathbb{R}^d. \tag{11.2}$$

Set $V(x) := U(x) - \delta|x|^2/2$, $x \in \mathbb{R}^d$. Notice that V is convex by the assumption $D^2 U \ge \delta \cdot Id$. Then for all $t \in [0, 1]$:

$$V(ty + (1-t)x) \le tV(y) + (1-t)V(x).$$

Subtracting $V(x)$ from both sides, dividing by t and letting $t \to 0+$, we obtain

$$\langle \nabla V(x), y - x\rangle \le V(y) - V(x)$$

and exchanging x with y

$$\langle \nabla V(y), x - y\rangle \le V(x) - V(y).$$

By summing the two inequalities one gets

$$-\langle \nabla V(y) - \nabla V(x), y - x\rangle \le 0,$$

and (11.2) is easily obtained. Therefore, Theorem 11.2 applies to this case.

12. Differentiability properties of the transition semigroup

Let $F:\mathbb{R}^d\mapsto\mathbb{R}^d$ a map of class C^2 such that

$$\|F(x)-F(y)\|\le K\|x-y\|,\qquad \forall\, x,y\in\mathbb{R}^d,$$
$$\langle F(x)-F(y),x-y\rangle\le\omega\|x-y\|^2,\qquad \forall\, x,y\in\mathbb{R}^d,$$

and defining for all $x\in\mathbb{R}^d$ the matrix $H(x)$ by

$$\langle H(x)h,k\rangle:=\lim_{t\to0}\frac{1}{t}\langle F(x+th)-F(x),k\rangle.$$

By the assumption on F, we have that $\|H(x)h\|\le K\|h\|$ and

$$\langle H(x)\cdot h,h\rangle\le\omega\|h\|^2.$$

We also assume that $H(\cdot)$ is Lipschitz-continuous. We do not assume in this section dissipativity of F, i.e. we not make any assumption on the sign of $\omega\in\mathbb{R}$. We want to study the differentiability properties of the unique solution of the SDE

$$X_t(x)=x+\int_0^t F(X_s(x))\,ds+B\,W_t,\qquad t\ge0. \tag{12.1}$$

w.r.t. the initial condition x. We set for all $\varphi\in C^2(\mathbb{R}^d)$

$$\mathcal{L}\varphi(x):=\frac{1}{2}\Delta\varphi(x)+\langle F(x),\nabla\varphi(x)\rangle,\qquad x\in\mathbb{R}^d.$$

12.1. **Smoothness estimates.** We want to prove that the semigroup $(P_t)_{t\ge0}$ enjoys some regularity properties. We start with differentiability of $X_t(x)$ w.r.t. the initial datum x.

Lemma 12.1. *Let $t\ge0$. The map $\mathbb{R}^d\ni x\mapsto X_t(x)$ is a.s. twice differentiable. Setting*

$$\eta^h_t(x):=\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\left(X_t(x+\varepsilon h)-X_t(x)\right)\in\mathbb{R}^d,\qquad h\in\mathbb{R}^d,$$
$$\zeta^{h,k}_t(x):=\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\left(\eta^h_t(x+\varepsilon k)-\eta^h_t(x)\right)\in\mathbb{R}^d,\qquad h,k\in\mathbb{R}^d,$$

then there exists a constant $C_t\ge0$ such that

$$|\eta^h_t(x)|\le e^{\omega t}|h|,\qquad |\zeta^{h,k}_t(x)|\le C_t\,|h|\,|k|,\qquad \forall\, t\ge0,\ x\in\mathbb{R}^d. \tag{12.2}$$

Proof. We want to prove differentiability of $X_t(x)$ w.r.t. x. We set $Y_t(x):=X_t(x)-B\,W_t$. Then

$$Y_t(x)=x+\int_0^t F(Y_s(x)+B\,W_s)\,ds,\qquad \forall\, t\ge0.$$

Then Y solves a standard ODE (with random coefficients). Since the function $x\mapsto F(x+W_s)$ is twice continuously differentiable, uniformly in $s\in[0,T]$, then by standard considerations we obtain that $x\mapsto Y_t(x)$ and therefore $x\mapsto X_t(x)$ are twice differentiable. Moreover, the first derivatives $\eta^h_t(x)$ solve the equation (we drop the dependence on x for simplicity)

$$\eta^h_t=h+\int_0^t H(X_s)\cdot\eta^h_s\,ds$$

(notice that H is a matrix in $\mathbb{R}^d$ and therefore $H(X_s)\cdot\eta^h_s\in\mathbb{R}^d$); the second derivatives $\zeta^{h,k}_t(x)$ solve the equation

$$\zeta^{h,k}_t=\int_0^t H(X_s)\cdot\zeta^{h,k}_s\,ds+\int_0^t\Theta^k_s\cdot\eta^h_s\,ds,$$

where Θ^k_s is the matrix in $\mathbb{R}^d$ defined by

$$(\Theta^k_s)_{ij} = \sum_{\ell=1}^d \frac{\partial^2 F_i}{\partial x_j \partial x_\ell}(X_s)\ (\eta^k_t)_\ell.$$

We obtain that

$$\frac{d}{dt}|\eta^h_t|^2 = 2\langle H(X_t)\cdot \eta^h_t, \eta^h_t\rangle \le 2\omega\,|\eta^h_t|^2 \implies |\eta^h_t|^2 = e^{2\omega t}|h|^2 \quad \forall\ t\ge 0.$$

Since H is assumed to be Lipschitz-continuous, then there exists a constant κ such that $\sup_x \|\Theta(x)\| \le \kappa$. We obtain that

$$\begin{aligned}\frac{d}{dt}|\zeta^{h,k}_t|^2 &= \langle H(X_t)\cdot \zeta^{h,k}_t, \zeta^{h,k}_t\rangle + \langle \Theta^k_t \cdot \eta^h_t, \zeta^{h,k}_t\rangle \\ &\le \omega\|\zeta^{h,k}_t\|^2 + e^{\omega t}\,\kappa\,|h|\,|k|\,|\zeta^{h,k}_t| \le \left(e^{\omega t}\kappa\,|h||k|\right)^2 + (\omega+1)|\zeta^{h,k}_t|^2\end{aligned}$$

so that $|\zeta^{h,k}_t|^2 \le C_t\,|h|^2|k|^2$. □

We denote by $C^2_b(\mathbb{R}^d)$ the space of all twice differentiable functions on $\mathbb{R}^d$ which are continuous and bounded with all first and second derivatives .

Theorem 12.2. *Let $f \in C^2_b(\mathbb{R}^d)$ and set $u(t,x) := P_t f(x)$, $t\ge 0$, $x\in\mathbb{R}^d$. Then*

(1) *u is continuosly differentiable in t and twice continuously differentiable in x*
(2) *for all $t \ge 0$, $u(t,\cdot)\in C^2_b(\mathbb{R}^d)$ and*

$$\sup_{x\in\mathbb{R}^d} |\nabla_x u(t,x)| \le e^{\omega t} \sup_{x\in\mathbb{R}^d}|\nabla f(x)| \tag{12.3}$$

(3) *for all $t\ge 0$, $x\in\mathbb{R}^d$*

$$\frac{\partial u}{\partial t}(t,x) = \mathcal{L}u(t,x). \tag{12.4}$$

Proof. By Ito's formula:

$$f(X_t(x)) = f(x) + \int_0^t \mathcal{L}f(X_s(x))\,ds + \int_0^t \langle \nabla f(X_s(x)), dW_s\rangle, \qquad t\ge 0.$$

Since ∇f is assumed to be bounded, by taking expectation we obtain

$$u(t,x) = f(x) + \int_0^t P_s\mathcal{L}f(x)\,ds, \qquad t\ge 0,$$

so that we obtain differentiability in time. Moreover $\frac{\partial u}{\partial t} = P_t\mathcal{L}f$ is jointly continuous in (t,x).

In order to prove differentiability of u w.r.t. x, we use differentiability of X w.r.t. x, proved in Lemma 12.1; we obtain

$$\langle \nabla_x u(t,x), h\rangle = \mathbb{E}(\langle \nabla f(X_t(x)), \eta^h_t(x)\rangle),$$

$$\langle D^2_x u(t,x)\cdot h, k\rangle = \mathbb{E}(\langle D^2 f(X_t(x))\cdot \eta^h_t(x), \eta^k_t(x)\rangle + \langle \nabla f(X_t(x)), \zeta^{h,k}_t(x)\rangle).$$

It follows, since $f\in C^2_b(\mathbb{R}^d)$ and by (12.2), that $\mathcal{L}u$ is jointly continuous in (t,x) and for some constant C_T and all $x\in\mathbb{R}^d$, $t\in[0,T]$:

$$|\nabla_x u(t,x)| \le e^{\omega t}\sup_{z\in\mathbb{R}^d}|\nabla f(z)|, \qquad |D^2_x u(t,x)| \le C_T, \qquad |\mathcal{L}u(t,x)| \le C_T(1+|x|).$$

In particular we have proved (12.3). We consider now, for fixed $T \geq 0$, the process $[0,T] \ni t \mapsto u(T-t, X_t(x))$. By Itô's formula:

$$u(T-t, X_t(x)) - u(T,x) = \int_0^t \left(-\frac{\partial u}{\partial t} + \mathcal{L}u \right) (T-s, X_s(x)) ds + \int_0^t \langle \nabla_x u(T-s, X_s(x)), dW_s \rangle.$$

Notice that for all $t \in [0,T]$

$$|X_t(x)| \leq |x| + \frac{1}{\delta} \int_0^t |X_s(x)| \, ds + \sup_{s \in [0,T]} |W_s|,$$

so that by Gronwall's Lemma

$$|X_t(x)| \leq \left(|x| + \sup_{s \in [0,T]} |W_s| \right) e^{T/\delta}, \qquad \forall\, t \in [0,T].$$

Since $W = (W^1, \ldots, W^d)$ with $\{W^i\}$ independent,

$$\sup_{s \in [0,T]} |W_s| \leq \sqrt{\sum_{i=1}^d \sup_{s \in [0,T]} |W_s^i|^2}$$

and $\sup_{s \in [0,T]} |W_s^i|$ has the distribution of $|W_T^i|$ by the reflection principle, then we obtain

$$\mathbb{E} \left(\sup_{t \in [0,T]} |X_t(x)|^k \right) < +\infty, \qquad \forall\, T \geq 0, \ k \in \mathbb{N}.$$

Since $|\nabla_x u|$ is uniformly bounded and

$$|\mathcal{L}u(T-s, X_s(x))| \leq C_T(1 + |X_s(x)|),$$

$$\left| \frac{\partial u}{\partial t}(T-s, X_s(x)) \right| \leq P_{T-s}|\mathcal{L}f|(X_s(x)) \leq C_T(1 + |X_s(x)|),$$

then we can take the expectation in the formula above

$$\mathbb{E}[u(T-t, X_t(x)) - u(T,x)] = \int_0^t \mathbb{E}\left[\left(-\frac{\partial u}{\partial t} + \mathcal{L}u \right) (T-s, X_s(x)) \right] ds, \qquad \forall\, t \in [0,T].$$

By the semigroup property $\mathbb{E}(u(T-t, X_t(x))) = P_t P_{T-t} f(x) = P_T f(x) = u(T,x)$. Hence

$$\int_0^t \mathbb{E}\left[\left(-\frac{\partial u}{\partial t} + \mathcal{L}u \right) (T-s, X_s(x)) \right] ds = 0, \qquad \forall\, t \in [0,T],$$

and taking the derivative at $t = 0$ we obtain (12.4). □

13. Explicit invariant measures for gradient systems

We consider now the case $F(x) = -\frac{1}{2} \nabla U(x)$, where U satisfies the assumptions of section 11.2, and $B = Id$. Therefore we study the unique solution of the equation

$$X_t(x) = x - \frac{1}{2} \int_0^t \nabla U(X_s(x)) \, ds + W_t, \qquad t \geq 0.$$

We denote

$$\mu(dx) := \frac{1}{Z} \exp(-U(x)) \, dx, \qquad Z := \int \exp(-U(z)) \, dz \tag{13.1}$$

so that μ is a Borel probability measure on $\mathbb{R}^d$. The key point is the following integration by parts formula.

Lemma 13.1. *Let $f \in C^1(\mathbb{R}^d)$ such that $|\nabla f(x)| \le c(1+|x|)$ for all $x \in \mathbb{R}^d$. Then for all $h \in \mathbb{R}^d$:*

$$\int \langle \nabla f, h \rangle \, e^{-U} \, dx = \int f \, \langle \nabla U, h \rangle \, e^{-U} \, dx. \tag{13.2}$$

If $f, g \in C_b^2(\mathbb{R}^d)$, then $\mathcal{L} f \in L^2(\mu)$ and

$$\frac{1}{2} \int \langle \nabla f, \nabla g \rangle \, d\mu = - \int g \cdot \mathcal{L} f \, d\mu. \tag{13.3}$$

In particular, for $g \equiv 1$ we obtain for all $f \in C_b^2(\mathbb{R}^d)$

$$\int \mathcal{L} f \, d\mu = 0. \tag{13.4}$$

Proof. The first formula follows from the classical Gauss-Green formula, first applied to functions in $C^1(\mathbb{R}^d)$ with compact support and then using dominated convergence. If $f \in C_b^2(\mathbb{R}^d)$, then the estimate $\mathcal{L} f \in L^2(\mu)$ follows from the assumptions on U and the second formula is an application of the first one:

$$\begin{aligned}
&\int g \cdot \Delta f \, e^{-U} \, dx = \sum_{i=1}^d \int g \, \langle \nabla(\langle \nabla f, e_i \rangle), e_i \rangle \, e^{-U} \, dx \\
&= \sum_{i=1}^d \int \langle \nabla (g \, \langle \nabla f, e_i \rangle), e_i \rangle \, e^{-U} \, dx - \sum_{i=1}^d \int \langle \nabla g, e_i \rangle \, \langle \nabla f, e_i \rangle \, e^{-U} \, dx \\
&= \sum_{i=1}^d \int g \, \langle \nabla f, e_i \rangle \, \langle \nabla U, e_i \rangle \, e^{-U} \, dx - \sum_{i=1}^d \int \langle \nabla g, e_i \rangle \, \langle \nabla f, e_i \rangle \, e^{-U} \, dx \\
&= \int \langle \nabla f, \nabla U \rangle \, e^{-U} \, dx - \int \langle \nabla f, \nabla g \rangle \, e^{-U} \, dx.
\end{aligned}$$

□

Theorem 13.2. *The measure $\mu(dx)$ is invariant for X.*

Proof. Let $f \in C_b^2(\mathbb{R}^d)$ and set $u(t,x) := P_t f(x)$, $t \ge 0$, $x \in \mathbb{R}^d$. Then by Theorem 12.2, for all $t \in [0,T]$ and $x \in \mathbb{R}^d$:

$$\left| \frac{\partial u}{\partial t}(t,x) \right| = |\mathcal{L} u(t,x)| \le C_T(1+|x|), \qquad \int C_T(1+|x|) \, \mu(dx) < +\infty,$$

where we use the assumption that $D^2 U(x) \ge \delta I$ with $\delta > 0$. Then by the theorem of differentiation under the integral sign and by (13.4):

$$\frac{d}{dt} \int u(t,x) \, \mu(dx) = \int \mathcal{L} u(t,x) \, \mu(dx) = 0.$$

Then $\int P_t f d\mu = \int f d\mu$ for all $f \in C_b^2(\mathbb{R}^d)$ and by approximation and dominated convergence for all $f \in C_b(\mathbb{R}^d)$. □

We shall see in Theorem 15.15 below that μ is in fact even *reversible*. This is due to the symmetry in u, v of the left hand side of (13.3).

13.1. **An example: Random Interfaces.** Let $d \geq 1$ and consider the lattice $\mathbb{Z}^d$. Given two points $x, y \in \mathbb{Z}^d$, we say that $x \sim y$ iff $|x - y| = 1$.

We fix a finite subset $\Lambda \subset \mathbb{Z}^d$ and a C^2 even function $V : \mathbb{R} \mapsto \mathbb{R}$ with $0 < c_- \leq V'' \leq c_+ < +\infty$ and set for all $\phi \in \mathbb{R}^\Lambda$

$$U(\phi) := \frac{1}{2} \sum_{x,y \in \Lambda,\ x \sim y} V(\phi(x) - \phi(y)).$$

Then U is clearly convex on $\mathbb{R}^\Lambda$ and its gradient is

$$\nabla U(\phi) = \left(\frac{\partial U(\phi)}{\partial \phi(x)},\ x \in \Lambda \right) = \left(\sum_{y \in \Lambda,\ x \sim y} (V'(\phi(x)) - V'(\phi(y))),\ x \in \Lambda \right).$$

Therefore by Theorem 13.2 we obtain that the unique solution of

$$d\phi(x) = \frac{1}{2} \sum_{y \in \Lambda,\ x \sim y} (V'(\phi(y)) - V'(\phi(x)))\, dt + dw_t(x),$$

where $(w_t(x), t \geq 0, x \in \Lambda)$ is a family of independent Brownian motions, has invariant probability measure

$$\mu(d\phi) = \frac{1}{Z} \exp\left(-U(\phi)\right)\, d\phi.$$

This is a Gibbs measure on a finite lattice. The limits as $\Lambda \to \mathbb{Z}^d$ of the dynamics and the invariant measure are a non trivial and interesting problem.

14. The Bismut-Elworthy formula and the strong Feller property

We want to prove an important property of the transition semigroup of the process X, solution of (12.1), namely that for all Borel and bounded $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$:

$$|P_t\varphi(x) - P_t\varphi(y)| \leq C\, e^{\omega t}\, \frac{\sup|\varphi|}{\sqrt{t}}\, |x - y|, \qquad \forall\ x, y \in \mathbb{R}^d,\ t > 0. \tag{14.1}$$

In particular, if φ is only Borel and bounded, then for positive t the function $P_t\varphi$ is Lipschitz-continuous. This is known as the *strong Feller property*; recall that the *Feller property* is given by $P_t(C_b) \mapsto C_b$ for all $t \geq 0$, see the discussion before Proposition 2.3.

Notice that the above estimate is "constant-free": on the right-hand side no constant depends on the dimension or on the specific choice of U (bar the convexity assumption).

We prove in fact a stronger result, namely the *Bismut-Elworthy formula*

$$\langle \nabla P_T\varphi(x), h \rangle = \frac{1}{T}\, \mathbb{E}\left[\varphi(X_T(x)) \int_0^T \langle \eta^h_s(x), dW_s \rangle \right]. \tag{14.2}$$

Let first $\varphi \in C^2_b(\mathbb{R}^d)$ and set as usual $u(t, x) := P_t\varphi(x)$. We know, by applying the Itô formula to $t \mapsto u(T - t, X_t(x))$, that a.s.

$$\varphi(X_T(x)) = u(T, x) + \int_0^T \langle \nabla_x u(T - s, X_s(x)), dW_s \rangle. \tag{14.3}$$

By the estimates of Lemma 12.1, $|\eta^h_t| \leq e^{\omega t}|h|$. In particular, by the Itô isometry we have

$$\mathbb{E}\left[\left(\int_0^T \langle \eta^h_s(x), dW_s \rangle \right)^2 \right] = \int_0^T \mathbb{E}\left[|\eta^h_s(x)|^2 \right] ds \leq C\, e^{\omega T}\, T < +\infty.$$

We multiply both sides of (14.3) by $\int_0^T \langle \eta_s^h(x), dW_s\rangle$ and we take expectation. On the right hand side, we obtain

$$\mathbb{E}\left[u(T,x)\ \int_0^T \langle \eta_s^h(x), dW_s\rangle\right] = u(T,x)\ \mathbb{E}\left[\int_0^T \langle \eta_s^h(x), dW_s\rangle\right] = 0,$$

and

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \langle \nabla_x u(T-s, X_s(x)), dW_s\rangle \int_0^T \langle \eta_s^h(x), dW_s\rangle\right] \\
&= \mathbb{E}\left[\int_0^T \langle \nabla_x u(T-s, X_s(x)), \eta_s^h(x)\rangle\, ds\right] \\
&= \int_0^T \langle \nabla_x \mathbb{E}\left[u(T-s, X_s(\cdot))\right], h\rangle\, ds = \int_0^T \langle \nabla_x u(T,x), h\rangle\, ds = T\langle \nabla_x u(T,x), h\rangle.
\end{aligned}$$

Then we have obtained (14.2). We prove now (14.1). By Cauchy-Schwarz and the Itô isometry

$$|\nabla P_T\varphi(x)|^2 \leq \frac{1}{T^2}\,\mathbb{E}\left[\varphi^2(X_T(x))\right]\ \mathbb{E}\left[\left(\int_0^T \langle \eta_s^h(x), dW_s\rangle\right)^2\right] \leq C\, e^{\omega T}\,\frac{\sup|\varphi|^2}{T^2}\, T,$$

and we conclude with standard arguments.

Since by the chain rule

$$\langle \nabla P_T\varphi(x), h\rangle = \mathbb{E}\left[\langle \nabla\varphi(X_t(x)), \eta_t^h(x)\rangle\right],$$

then for $\varphi \in C_b^1(\mathbb{R}^d)$ we obtain from formula (14.2)

$$\mathbb{E}\left[\langle \nabla\varphi(X_t(x)), \eta_t^h(x)\rangle\right] = \frac{1}{t}\,\mathbb{E}\left[\varphi(X_t(x))\ \int_0^t \langle \eta_s^h(x), dW_s\rangle\right].$$

Notice that this is an integration by parts formula on the law of X: indeed, in the left hand side we have an integral which contains the gradient of φ, while the right hand side contains only φ.

Also (14.2) can be considered as an integration by parts formula: an infinitesimal variation of the initial point x of the trajectory of X is transferred to the whole trajectory up to time t and results in a variation of the law of $X_t(x)$ only through a density. In other words, the law of $X_t(x+\varepsilon h)$ is approximately given by the law of $X_t(x)$ multiplied by a density:

$$\mathbb{E}\left[\varphi(X_t(x+\varepsilon h))\right] \approx \mathbb{E}\left[\varphi(X_t(x))\,\exp\left(\varepsilon\cdot\frac{1}{t}\int_0^t \langle \eta_s^h(x), dW_s\rangle\right)\right]$$

as $\varepsilon \to 0$.

14.1. **The monotone case.** An important remark is that for $\omega \leq 0$ the estimate (14.1) becomes uniform in the dimension d of the space. This is very important for applications to infinite-dimensional problems, in particular SPDEs.

15. The Dirichlet form

15.1. **Definition and examples.**

Definition 15.1. Let $(T, \mathcal{B}, \gamma)$ a σ-finite measure space. If $D \subset L^2(\gamma)$ is a dense linear space and $\mathcal{E} : D \times D \mapsto \mathbb{R}$ a symmetric bilinear function such that $\mathcal{E}(u,u) \geq 0$ for all $u \in D$, then we call $(\mathcal{E}, D)$ a *non-negative symmetric bilinear form*, or simply a *form*. We define the scalar product on D

$$\mathcal{E}_1(u,v) := \int u\, v\, d\gamma + \mathcal{E}(u,v), \qquad u,v \in D.$$

We say that

(1) $(\mathcal{E}, D)$ is *closed* in $L^2(\gamma)$ if D is complete w.r.t. $\mathcal{E}_1$, i.e. if for any sequence $(u_n) \subset D$ which is Cauchy w.r.t. $\mathcal{E}_1$ there exists $u \in D$ such that $\mathcal{E}_1(u_n - u, u_n - u) \to 0$.

(2) $(\mathcal{E}, D)$ is a *closable form* in $L^2(\gamma)$ if, for any $(u_n) \subset D$ which is Cauchy w.r.t. $\mathcal{E}_1$ and converges to 0 in $L^2(\gamma)$, we have that u_n converges to 0 w.r.t. $\mathcal{E}_1$. In other words, closability means that if $\|u_n\|_{L^2(\gamma)} \to 0$ and $\mathcal{E}(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to +\infty$, then $\mathcal{E}(u_n, u_n) \to 0$.

Lemma 15.2. *If $(\mathcal{E}, D)$ is a closable form in $L^2(\gamma)$, then there exists a unique closed form $(\overline{\mathcal{E}}, \overline{D})$ in $L^2(\gamma)$ such that*

(1) *$D \subset \overline{D}$ and D is dense in $\overline{D}$ w.r.t. $\overline{\mathcal{E}}_1$*

(2) *$\overline{\mathcal{E}}(u,v) = \mathcal{E}(u,v)$, for all $u, v \in D$.*

$(\overline{\mathcal{E}}, \overline{D})$ is called the closure of $(\mathcal{E}, D)$ in $L^2(\gamma)$ and it is customary to denote it by $(\mathcal{E}, D(\mathcal{E}))$.

The proof of this Lemma is essentially contained in the following

Remark 15.3. Notice that, for any form $(\mathcal{E}, D)$, the space $(D, \mathcal{E}_1)$ has an abstract completion $(\overline{D}, \overline{\mathcal{E}}_1)$. A form is closable if and only if $\overline{D}$ has an *injection* in $L^2(\gamma)$ which extends continuously the canonical immersion $i : D \mapsto L^2(\gamma)$ given by the identity map. Lack of closability means that there exists a sequence $(u_n) \subset D$ such that $u_n \to 0$ in $L^2(\gamma)$ and $u_n \to v \in \overline{D} \backslash \{0\}$ w.r.t. $\overline{\mathcal{E}}_1$.

Our main example is the following: we set

$$(T, \mathcal{B}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)), \qquad D := C_c^2(\mathbb{R}^d), \qquad \mathcal{E}(u,v) := \frac{1}{2} \int \langle \nabla u, \nabla v \rangle \, d\gamma,$$

where $C_c^2(\mathbb{R}^d)$ is the space of functions in $C^2(\mathbb{R}^d)$ with compact support and γ is a Borel measure on $\mathbb{R}^d$. When the form $(\mathcal{E}, D)$ of this example is closable, then we call its closure $(\mathcal{E}, D(\mathcal{E}))$ a *Dirichlet form*.

Example 15.4. Consider the case of γ equal to the Lebesgue measure on $\mathbb{R}$. Then we have

$$\mathcal{E}_1(u,u) = \|u\|_{L^2}^2 + \|u'\|_{L^2}^2 =: \|u\|_{H^1}^2, \qquad D := C_c^1(\mathbb{R}).$$

Let $(f_n) \subset D$ be a Cauchy sequence for $\|\cdot\|_{H^1}$. Then (f_n) and (f_n') are Cauchy in L^2; since L^2 is complete, there exist f and g in L^2 such that $f_n \to f$ and $f_n' \to g$ in L^2. In order to say that $g = f'$ in a weak sense, we have to prove that g only depends on f and not on the particular sequence (f_n); in other words, if any other sequence $\hat{f}_n \subset D$ converges to f in L^2 and $\hat{f}_n'$ converges to h in L^2, then h must be equal to g. By taking the difference $f_n - \hat{f}_n$, this is equivalent to say that for any sequence $u_n \subset D$ converging to 0 in L^2 such that u_n' converges to w in L^2 we must have $w = 0$.

Notice that closability of $(\mathcal{E}, D)$ is equivalent to closability of the linear operator $\nabla : C_c^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ in the norm of $L^2(\mathbb{R})$. Moreover, the space $D(\mathcal{E})$ is the classical Sobolev space $W^{1,2}(\mathbb{R}) = H^1(\mathbb{R})$ of all functions in $L^2(\mathbb{R})$ such that the (distributional) first derivative belongs to $L^2(\mathbb{R})$.

In the following, we shall be mainly interested in the case of γ equal to the probability measure μ defined by (13.1), and in the connection between $\mathcal{E}$ and the process X solution of (12.1).

The proof of closability of $(\mathcal{E}, D)$ is a not very exciting but necessary technical step. A useful criterion is the following

Lemma 15.5. *Let $(\mathcal{E}, D)$ be a non-negative symmetric bilinear form. If for all $(u_n) \subset D$ such that $u_n \to 0$ in $L^2(\gamma)$ we have $\mathcal{E}(u_n, v) \to 0$ for all $v \in D$, then $(\mathcal{E}, D)$ is closable.*

Proof. Let $(u_n) \subset D$ which is Cauchy w.r.t. $\mathcal{E}_1$ and converges to 0 in $L^2(\gamma)$. Then by bilinearity

$$\mathcal{E}_1(u_n, u_n) = \mathcal{E}_1(u_n - u_m, u_n - u_m) + \mathcal{E}_1(u_n - u_m, u_m) + \mathcal{E}_1(u_m, u_n).$$

For fixed n, by assumption we have $\lim_m \mathcal{E}_1(u_m, u_n) = 0$. Moreover since (u_n) is Cauchy w.r.t. $\mathcal{E}_1$, then $\sup_m \mathcal{E}_1(u_m, u_m) =: c < +\infty$ and for any $\varepsilon > 0$ we have

$$\mathcal{E}_1(u_n - u_m, u_n - u_m) \leq \varepsilon$$

for all $n, m \geq N_\varepsilon$. Since

$$\mathcal{E}_1(u_n - u_m, u_m) \leq \sqrt{\mathcal{E}_1(u_n - u_m, u_n - u_m)\, \mathcal{E}_1(u_m, u_m)} \leq \sqrt{c\, \mathcal{E}_1(u_n - u_m, u_n - u_m)},$$

then for all $n \geq N_\varepsilon$

$$\limsup_{m\to\infty} \mathcal{E}_1(u_n - u_m, u_m) \leq \limsup_{m\to\infty} \sqrt{c\, \mathcal{E}_1(u_n - u_m, u_n - u_m)} \leq \sqrt{c\,\varepsilon}.$$

Then for all $n \geq N_\varepsilon$

$$\mathcal{E}_1(u_n, u_n) = \limsup_{m\to\infty}\, \mathcal{E}_1(u_n, u_n) \leq \varepsilon + \sqrt{c\,\varepsilon}.$$

□

Proposition 15.6. *The form*

$$D := C_c^2(\mathbb{R}^d), \qquad \mathcal{E}(u, v) := \frac{1}{2} \int \langle \nabla u, \nabla v \rangle \, d\mu,$$

is closable in $L^2(\mu)$. Moreover $C_b^1(\mathbb{R}^d) \subset D(\mathcal{E})$.

Proof. For all $u, v \in C_c^2(\mathbb{R}^d)$ we can apply the integration by parts formula (13.3) and obtain

$$\mathcal{E}(u, v) = -\int u \cdot \mathcal{L}v \, d\mu.$$

Notice that $\mathcal{L}v \in C_c(\mathbb{R}^d)$ for $v \in C_c^2(\mathbb{R}^d)$, and in particular $\mathcal{L}v \in L^2(\mu)$.

Let now any $(u_n) \subset C_c^2(\mathbb{R}^d)$ such that $u_n \to 0$ in $L^2(\mu)$. Then for any $v \in C_c^2(\mathbb{R}^d)$:

$$\lim_{n\to\infty} \mathcal{E}(u_n, v) = -\lim_{n\to\infty} \int u_n \cdot \mathcal{L}v \, d\mu = 0$$

and we can apply Lemma 15.5.

In order to prove that $C_b^1(\mathbb{R}^d) \subset D(\mathcal{E})$, we have to show that $C_b^1(\mathbb{R}^d)$ is contained in the closure of $C_c^2(\mathbb{R}^d)$ w.r.t. $\mathcal{E}_1$. But this is easily obtained by approximating functions in $C_b^1(\mathbb{R}^d)$ with functions in $C_c^1(\mathbb{R}^d)$ and then functions in $C_c^1(\mathbb{R}^d)$ with functions in $C_c^2(\mathbb{R}^d)$. □

Remark 15.7. This is in fact only a *very* particular example of Dirichlet form.

15.2. **Generator and resolvent.** We fix throughout this section a closed form $(\mathcal{E}, D(\mathcal{E}))$ in $L^2(\gamma)$. We define for $\lambda > 0$:

$$\mathcal{E}_\lambda(u,v) := \lambda \int u\, v\, d\gamma + \mathcal{E}(u,v), \qquad u, v \in D(\mathcal{E}).$$

Proposition 15.8. *For all $\lambda > 0$ and $f \in L^2(\gamma)$, there exists a unique $v \in D(\mathcal{E})$ such that*

$$\mathcal{E}_\lambda(v,g) = \int f\, g\, d\gamma, \qquad \forall\, g \in D(\mathcal{E}).$$

We denote $v = R_\lambda f$. Moreover

1. *For all $\lambda > 0$ and $f \in L^2(\gamma)$, $\lambda \|R_\lambda f\|_{L^2(\gamma)} \le \|f\|_{L^2(\gamma)}$*
2. *The bounded operator $R_\lambda : L^2(\gamma) \mapsto L^2(\gamma)$ is symmetric and injective in $L^2(\gamma)$.*
3. *For all $\alpha, \beta > 0$:*

$$R_\alpha - R_\beta = -(\alpha - \beta) R_\alpha R_\beta = -(\alpha-\beta) R_\beta R_\alpha. \tag{15.1}$$

The family of operators $(R_\lambda)_{\lambda > 0}$ is called the Resolvent family *associated with $(\mathcal{E}, D(\mathcal{E}))$.*

Proof. Notice that $\mathcal{E}_\lambda$ defines a scalar product on $D(\mathcal{E})$, equivalent to $\mathcal{E}_1$. The linear form $D(\mathcal{E}) \ni g \mapsto \int f\, g\, d\gamma \in \mathbb{R}$ is continuous w.r.t. $\mathcal{E}_\lambda$, since

$$\left| \int f\, g\, d\gamma \right| \le \|f\|_{L^2(\gamma)} \|g\|_{L^2(\gamma)} \le \frac{\|f\|_{L^2(\gamma)}}{\lambda}\, \mathcal{E}_\lambda(g,g).$$

Therefore there exists a unique $v \in D(\mathcal{E})$ such that

$$\mathcal{E}_\lambda(v,g) = \int f\, g\, d\gamma, \qquad \forall\, g \in D(\mathcal{E}),$$

which is the desired result. Let us now choose $g = R_\lambda f$. Then

$$\lambda \|R_\lambda f\|^2_{L^2(\gamma)} \le \mathcal{E}_\lambda(R_\lambda f, R_\lambda f) = \int f\, R_\lambda f\, d\gamma \le \|f\|_{L^2(\gamma)} \|R_\lambda f\|_{L^2(\gamma)},$$

which yields $\lambda \|R_\lambda f\|_{L^2(\gamma)} \le \|f\|_{L^2(\gamma)}$, since $R_\lambda f \in D(\mathcal{E}) \subset L^2(\gamma)$. Let now $f, g \in L^2(\gamma)$. Then

$$\int g\, R_\lambda f\, d\gamma = \mathcal{E}_\lambda(R_\lambda g, R_\lambda f) = \int f\, R_\lambda g\, d\gamma,$$

which gives simmetry of R_λ in $L^2(\gamma)$. If $R_\lambda f = R_\lambda g$, then

$$\int f\, h\, d\gamma = \mathcal{E}_\lambda(R_\lambda f, h) = \mathcal{E}_\lambda(R_\lambda g, h) = \int g\, h\, d\gamma, \qquad \forall\, h \in D(\mathcal{E}),$$

and since $D(\mathcal{E})$ is dense in $L^2(\gamma)$, then $f = g$ and R_λ is injective.

In order to prove (15.1), consider now $f \in L^2(\gamma)$ and $g \in D(\mathcal{E})$. Then

$$\begin{aligned} \mathcal{E}_\alpha(R_\alpha f - R_\beta f, g) &= \mathcal{E}_\alpha(R_\alpha f, g) - \mathcal{E}_\beta(R_\beta f, g) - (\alpha - \beta) \int g\, R_\beta f\, d\gamma \\ &= -(\alpha-\beta) \int g\, R_\beta f\, d\gamma = -(\alpha - \beta)\, \mathcal{E}_\alpha(R_\alpha R_\beta f, g). \end{aligned}$$

Since this is true for any $g \in D(\mathcal{E})$ and $\mathcal{E}_\alpha$ is a scalar product on $D(\mathcal{E})$, equivalent to $\mathcal{E}_1$, then we have the first equality of (15.1); the second equality is obtained by exchanging α and β. □

Proposition 15.9. *There exists a unique operator $L : D(L) \subset L^2(\gamma) \mapsto L^2(\gamma)$ such that, for all $\lambda > 0$, $R_\lambda(L^2(\gamma)) = D(L)$ and*

$$(\lambda - L)\, R_\lambda f = f, \quad \forall\, f \in L^2(\gamma), \qquad R_\lambda\, (\lambda - L) f = f, \quad \forall\, f \in D(L). \tag{15.2}$$

We write $R_\lambda = (\lambda - L)^{-1}$. Moreover $(L, D(L))$ is self-adjoint in $L^2(\gamma)$, $D(L) \subset D(\mathcal{E})$ and

$$\mathcal{E}(u, v) = -\int u\, Lv\, d\gamma = -\int v\, Lu\, d\gamma, \qquad \forall\, u, v \in D(L). \tag{15.3}$$

The operator $(L, D(L))$ is called the generator *of $(\mathcal{E}, D(\mathcal{E}))$, while $(R_\lambda)_{\lambda>0}$ is called the* Resolvent family *of $(L, D(L))$.*

Proof. By (15.1) we see that $R_\alpha(L^2(\gamma)) = R_\beta(L^2(\gamma))$, for all $\alpha, \beta > 0$, since

$$R_\alpha = R_\beta \left[I - (\alpha - \beta) R_\alpha\right].$$

Then we define $D(L) := R_\alpha(L^2(\gamma))$ for any $\alpha > 0$. If $v \in D(L)$, since R_α is injective and $v \in R_\alpha(L^2(\gamma))$, then there exists a unique $f \in L^2(\gamma)$ s.t. $R_\alpha f = v$; at the same time, there exists a unique $g \in L^2(\gamma)$ s.t. $R_\beta g = v$. Notice that

$$R_\beta \left[(\alpha v - f) - (\beta v - g)\right] = (\alpha - \beta) R_\beta R_\alpha f - R_\beta f + R_\alpha f$$

which is equal to 0 by (15.1). Since R_β is injective, then $\alpha v - f = \beta v - g$, i.e. $\alpha - R_\alpha^{-1} = \beta - R_\beta^{-1}$ on $D(L)$. Then we define

$$D(L) := R_\alpha(L^2(\gamma)), \quad L := \alpha - R_\alpha^{-1}, \qquad \alpha > 0,$$

and it is easy to see that L satisfies (15.2).

Recall that the image of R_λ is in $D(\mathcal{E})$ by definition. For any $u, v \in D(L)$:

$$\mathcal{E}(u, v) = -\int u\, v\, d\gamma + \int v\, R_1^{-1} u\, d\gamma = -\int v\, Lu\, d\gamma$$

and, by exchanging u and v, we obtain both equalities in (15.3). □

We have the easy

Proposition 15.10. *The domain $D(L)$ of L is given by*

$$D(L) = \{f \in D(\mathcal{E}) : \text{the map } D(\mathcal{E}) \ni g \mapsto \mathcal{E}(f, g) \text{ is continuous w.r.t. } \mathcal{E}_1\}.$$

15.3. **The process associated with $\mathcal{E}$.**

Definition 15.11. Let γ be a Borel measure on $\mathbb{R}^d$ and $(\mathcal{E}, D(\mathcal{E}))$ a Dirichlet form in $L^2(\gamma)$. A semigroup $(P_t)_{t\geq 0}$ in $L^2(\gamma)$, such that $[0, \infty) \ni t \mapsto P_t f \in L^2(\gamma)$ is continuous for all $f \in L^2(\gamma)$, is associated with $(\mathcal{E}, D(\mathcal{E}))$ if there exists $\lambda_0 > 0$ s.t.

$$R_\lambda f = \int_0^\infty e^{-\lambda t}\, P_t f\, dt, \qquad \forall\ \lambda > \lambda_0.$$

In particular, the resolvent family is the Laplace transform in time of the semigroup. A Markov process X in $\mathbb{R}^d$ is associated with $(\mathcal{E}, D(\mathcal{E}))$ if

$$\mathbb{E}[f(X_t(x))] = P_t f(x), \quad \mu\text{–a.e. } x, \qquad \forall\ f \in C_b(\mathbb{R}^d), \quad t \geq 0.$$

Remark 15.12. Recall that

$$\int_0^\infty e^{-\alpha t}\, dt = \frac{1}{\alpha}, \qquad \forall\ \alpha > 0.$$

Since, in the setting of the previous definition,

$$\int_0^\infty e^{-\lambda t}\, P_t f\, dt = R_\lambda f = (\lambda - L)^{-1}, \qquad \forall\, \lambda > 0,$$

then it is customary to write that $P_t = e^{tL}$. This assertion can be made precise, for instance by means of spectral theory.

Remark 15.13. If a process X is associated with $(\mathcal{E}, D(\mathcal{E}))$ in $L^2(\gamma)$, then the transition semigroup (P_t) is symmetric in $L^2(\gamma)$. Indeed, R_λ is symmetric and this property transfers to (P_t) by the injectivity of the Laplace transform.

Suppose now that moreover the constant functions belong to $D(\mathcal{E})$; in this case the constant function equal to 1 is in $L^2(\gamma)$, which is possible if and only if γ is a finite measure. Then $\gamma(\cdot)/\gamma(\mathbb{R}^d)$ is an invariant reversible finite measure for X. Indeed we have for all $f \in L^2(\gamma)$

$$\mathcal{E}_\lambda(R_\lambda f, 1) = \int \lambda\, R_\lambda f\, d\gamma = \int f\, d\gamma, \qquad \forall\, \lambda > 0,$$

and again by injectivity of the Laplace transform we have $\int P_t f\, d\gamma = \int f\, d\gamma$, $t \geq 0$.

15.4. **Monotone gradient systems.** We go back to the example of the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$, closure in $L^2(\mu)$ of the form

$$D := C_c^2(\mathbb{R}^d), \qquad \mathcal{E}(u, v) := \frac{1}{2} \int \langle \nabla u, \nabla v \rangle\, d\mu,$$

see Proposition 15.6 above. Let X be the solution of (12.1). Then we have the

Proposition 15.14. *For all $\lambda > 0$, $f \in C_b(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, set:*

$$F_\lambda f(x) = \int_0^\infty e^{-\lambda t}\, \mathbb{E}[f(X_t(x))]\, dt. \tag{15.4}$$

Then $F_\lambda f = R_\lambda f$, μ-a.e. for all $\lambda > \omega$. In particular, X is associated with $(\mathcal{E}, D(\mathcal{E}))$.

Proof. It is enough to consider $f \in C_b^2(\mathbb{R}^d)$. By Theorem 12.2, we have $F_\lambda \in C_b^1(\mathbb{R}^d) \subset D(\mathcal{E})$ and for all $g \in D(\mathcal{E})$

$$\frac{d}{dt} \int u(t,x)\, g(x)\, \mu(dx) = \int \mathcal{L}u(t,x)\, g(x)\, \mu(dx) = -\int \langle \nabla_x u(t,x), \nabla g(x) \rangle\, \mu(dx).$$

We multiply by $e^{-\lambda t}$ and we integrate over $[0, \infty)$; we obtain in the left hand side

$$\begin{aligned} \int_0^\infty e^{-\lambda t} \left[\frac{d}{dt} \int u\, g\, d\mu \right] dt &= -\int f\, g\, d\mu + \lambda \int_0^\infty e^{-\lambda t} \left[\int u\, g\, d\mu \right] dt \\ &= \int (\lambda F_\lambda f - f)\, g\, d\mu; \end{aligned}$$

in the right hand side we have

$$-\int_0^\infty e^{-\lambda t} \left[\int \langle \nabla_x u, \nabla g \rangle\, d\mu \right] dt = -\mathcal{E}(F_\lambda f, g).$$

Since $F_\lambda \in D(\mathcal{E})$ and $\mathcal{E}_\lambda(F_\lambda f, g) = \int f\, g\, d\mu$ for all $g \in D(\mathcal{E})$, then $F_\lambda f = R_\lambda f$ in $L^2(\gamma)$. $\square$

Therefore we obtain the strengthening of Theorem 13.2

Theorem 15.15. *The measure $\mu(dx)$ is reversible for X.*

16. Examples

16.1. **The standard Brownian motion.** Let now $X_t(x) = x + B_t$ and $\gamma = \mathcal{L}_d$, Lebesgue measure in $\mathbb{R}^d$. Define

$$D := C_c^2(\mathbb{R}^d), \qquad \mathcal{E}(u,v) := \frac{1}{2}\int \langle \nabla u, \nabla v\rangle \, d\mathcal{L}_d.$$

Also in this case we have an integration by parts formula

$$\frac{1}{2}\int \langle \nabla u, \nabla v\rangle \, d\mathcal{L}_d = -\frac{1}{2}\int u\, \Delta v \, d\mathcal{L}_d, \qquad \forall\ u, v \in D.$$

Moreover it is well known that the function $u(t,x) = \mathbb{E}[\varphi(x+B_t)]$, $t \geq 0$, $x \in \mathbb{R}^d$, $\varphi \in C_b(\mathbb{R}^d)$, satisfies

$$u(t,x) = \varphi(x) + \int_0^t \frac{1}{2}\Delta u(s,x)\, ds, \qquad \forall\ t \geq 0,\ x \in \mathbb{R}^d.$$

We can therefore repeat most of the above considerations, obtaining that $(\mathcal{E}, D)$ is closable and B is associated with the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. Notice that $D(\mathcal{E}) = H^1(\mathbb{R}^d)$ and $\mathcal{E}$ is also known as the Dirichlet integral, from which the name Dirichlet form comes.

Clearly $\mathcal{L}_d$ is not a finite measure and $1 \notin L^2(\mathcal{L}_d)$. However, $\mathcal{L}_d$ is still a (not finite) invariant measure for B; in fact, it is even reversible, in the sense that the transition semigroup is symmetric in $L^2(\mathcal{L}_d)$. Indeed, the law of $x + B_t$ is $\mathcal{N}(x, t\cdot I)$ and for all $f \in C_c(\mathbb{R}^d)$

$$\begin{aligned}
&\int dx \int \mathcal{N}(x, t\cdot I)(dy)\ f(y) = \int dx \int \mathcal{N}(0, t\cdot I)(dy)\ f(y+x) \\
&= \int \mathcal{N}(0, t\cdot I)(dy)\ \int dx\, f(y+x) = \int \mathcal{N}(0, t\cdot I)(dy)\ \int dx\, f(x) = \int dx\, f(x),
\end{aligned}$$

and for all $f, g \in C_c(\mathbb{R}^d)$:

$$\begin{aligned}
&\int dx\, g(x) \int \mathcal{N}(x, t\cdot I)(dy)\ f(y) = \int \mathcal{N}(0, t\cdot I)(dy)\ \int dx\ g(x)\, f(y+x) \\
&= \int \mathcal{N}(0, t\cdot I)(dy)\ \int dx\ g(x-y)\, f(x) = \int \mathcal{N}(0, t\cdot I)(dy)\ \int dx\ g(x+y)\, f(x) \\
&= \int dx\, f(x) \int \mathcal{N}(x, t\cdot I)(dy)\ g(y).
\end{aligned}$$

16.2. **Ornstein-Uhlenbeck processes.** Let A be a symmetric matrix in $\mathbb{R}^d$, with strictly negative eigenvalues $(-\lambda_i)_{i=1,\dots,d}$. Set $U(x) := -\langle Ax, x\rangle$, $x \in \mathbb{R}^d$. Then $D^2U = -2A$ and $2\min\{\lambda_1, \dots, \lambda_d\}I \leq D^2U \leq 2\max\{\lambda_1, \dots, \lambda_d\}I$. By the above results, the Ornstein-Uhlenbeck process, unique solution of

$$X_t(x) = x + \int_0^t AX_s(x)\, ds + W_t, \qquad t \geq 0,\ x \in \mathbb{R}^d,$$

is the process associated with the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$, closure of the form

$$\frac{1}{2}\int \langle \nabla u, \nabla v\rangle \, d\mu, \qquad u, v \in C_b^2(\mathbb{R}^d)$$

in $L^2(\mu)$, where

$$\mu(dx) = \frac{1}{Z}\exp\left(\langle Ax, x\rangle\right) dx = \mathcal{N}(0, (-2A)^{-1})(dx). \tag{16.1}$$

Therefore μ is the unique invariant probability measure of X and it is moreover reversible.

Notice also that A can be diagonalized, i.e. there exists a matrix U such that $U^*U = UU^* = I$ and U^*AU =diag$(-\lambda_1, \dots, -\lambda_d)$. Setting $\hat{X} := U^*X$, $\hat{x} := U^*x$, $\hat{W} := U^*W$, we have that $\hat{W}$ has the same law as W and

$$\hat{X}_t(\hat{x}) = \hat{x} + \int_0^t \mathrm{diag}(-\lambda_1, \dots, -\lambda_d)\hat{X}_s(\hat{x})\, ds + \hat{W}_t, \qquad t \geq 0,\ \hat{x} \in \mathbb{R}^d.$$

In particular, setting $\hat{x}^i_t := \langle \hat{X}_t, e_i\rangle$ and $\hat{w}^i_t := \langle \hat{W}_t, e_i\rangle$, where (e_i) is a basis of $\mathbb{R}^d$ such that $U^*AUe_i = -\lambda_i e_i$, we obtain

$$\hat{x}^i_t = \hat{x}^i - \lambda_i \int_0^t \hat{x}^i_s\, ds + \hat{w}^i_t, \qquad t \geq 0,\ i = 1, \dots, d.$$

Since the $(\hat{w}^i)$ are independent, so are the $(\hat{x}^i)$. Therefore, X can be constructed as a linear function of a vector of independent one-dimensional O.-U. processes, each with invariant measure $\mathcal{N}(0, (2\lambda_i)^{-1})$.

16.3. **BM killed at a boundary.** We consider for simplicity the case of $d = 1$, more precisely of BM killed at its first exit from $]0, 1[$. We set for all $k \geq 1$:

$$e_k(x) := \sqrt{2}\, \sin(k\pi x), \qquad x \in [0, 1].$$

We recall that $\{e_k\}_{k\geq 1}$ is a complete orthonormal basis of $L^2(0, 1)$ (exercice: prove it!). Notice that $\{e_k\}_{k\geq 1}$ is a complete basis of eigenvectors of the second derivative with homogeneous Dirichlet boundary conditions:

$$\frac{d^2}{dx^2}\, e_k = -(\pi k)^2 e_k, \qquad e_k(0) = e_k(1) = 0, \qquad k \geq 1.$$

Notice that each $\varphi \in C^2_c(0, 1)$ is naturally extended to a function φ in $C^2_c(\mathbb{R})$ by setting $\varphi_{|(0,1)^c} := 0$. Then both the canonical scalar product on $(0, 1)$ and the form

$$D := C^2_c(0, 1), \qquad \mathcal{E}_{\mathrm{Dir}}(u, v) := \frac{1}{2} \int_0^1 u'\, v'\, dx,$$

are restrictions to D of, respectively, the canonical scalar product on $\mathbb{R}$ and the Dirichlet form

$$D(\mathcal{E}) := H^1(\mathbb{R}), \qquad \mathcal{E}(u, v) := \frac{1}{2} \int_0^1 u'\, v'\, dx.$$

Therefore $(\mathcal{E}_{\mathrm{Dir}}, D)$ is closable and the domain of the closure is the closure of D in $H^1(\mathbb{R})$, i.e. the classical Sobolev space $H^1_0(0, 1)$.

We start from the integration by parts formula

$$\frac{1}{2} \int_0^1 u'\, v'\, dx = -\frac{1}{2} \int_0^1 u''\, v\, dx, \qquad \forall\ u, v \in C^2_c(0, 1). \tag{16.2}$$

By Proposition 15.10 we obtain that

$$C^2_c(0, 1) \subset D(L_{\mathrm{Dir}}), \qquad L_{\mathrm{Dir}} f = \frac{1}{2}\, u'', \quad \forall\ f \in C^2_c(0, 1).$$

Moreover we find that for all $f, g \in C^2_c(0, 1)$

$$\mathcal{E}_{\mathrm{Dir}}(f, g) = \sum_{k\geq 1} \frac{(k\pi)^2}{2} \langle e_k, f\rangle\, \langle e_k, g\rangle,$$

$$L_{\rm Dir} f = -\sum_{k\geq 1} \frac{(k\pi)^2}{2} \langle e_k, f\rangle \, e_k, \qquad \text{in } L^2(0,1).$$

We obtain the identifications of the domains of the form $(\mathcal{E}_{\rm Dir}, H^1_0(0,1))$ and of the generator $(L_{\rm Dir}, D(L_{\rm Dir}))$

$$H^1_0(0,1) = \left\{ f \in L^2(0,1) : \sum_{k\geq 1} k^2 \langle e_k, f\rangle^2_{L^2(0,1)} < +\infty \right\},$$

$$D(L_{\rm Dir}) = \left\{ f \in L^2(0,1) : \sum_{k\geq 1} k^4 \langle e_k, f\rangle^2_{L^2(0,1)} < +\infty \right\}.$$

and moreover

$$L_{\rm Dir} u = -\sum_{k\geq 1} \frac{(k\pi)^2}{2} \langle e_k, u\rangle_{L^2(0,1)} \, e_k, \qquad \text{in } L^2(0,1), \ \forall \, u \in D(L_{\rm Dir}),$$

$$\mathcal{E}_{\rm Dir}(f,g) = \sum_{k\geq 1} \frac{(k\pi)^2}{2} \langle e_k, f\rangle_{L^2(0,1)} \, \langle e_k, g\rangle_{L^2(0,1)}, \qquad \forall \, f,g \in D(\mathcal{E}_{\rm Dir}).$$

Therefore

$$(\lambda - L_{\rm Dir})^{-1} f = \sum_{k\geq 1} \frac{1}{\lambda - \frac{(k\pi)^2}{2}} \, \langle e_k, f\rangle_{L^2(0,1)} \, e_k, \qquad \text{in } L^2(0,1), \ \forall \, f \in L^2(0,1).$$

Setting for all $f \in L^2(0,1)$

$$P_t f := \sum_{k\geq 1} e^{-(k\pi)^2 t/2} \langle e_k, f\rangle_{L^2(0,1)} \, e_k, \qquad t \geq 0.$$

we obtain that $P_t = e^{tL}$. If $f \in C^2_c(0,1)$ then we set $u(t,x) := P_t f(x)$, $t \geq 0$, $x \in [0,1]$. Then u satisfies the heat equation with homogeneous Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2} \dfrac{\partial^2 u}{\partial x^2}(t,x), \quad t > 0, \ x \in (0,1) \\[2ex] u(t,0) = u(t,1) = 0, \quad t \geq 0 \\[2ex] u(0,x) = f(x), \quad x \in [0,1] \end{cases}$$

Setting $\eta_k(x) := \sqrt{2}\,\cos(k\pi x)$, $x \in [0,1]$, we find that

$$\frac{\partial u}{\partial x}(t,x) = \sum_{k\geq 1} e^{-(k\pi)^2 t/2} \langle e_k, f\rangle_{L^2(0,1)} \, k\pi \, \eta_k(x) = -\sum_{k\geq 1} e^{-(k\pi)^2 t/2} \, \frac{1}{k\pi} \, \langle e_k, f''\rangle_{L^2(0,1)} \, \eta_k(x)$$

and therefore we have the uniform bound

$$\left| \frac{\partial u}{\partial x}(t,x) \right|^2 \leq \sum_{k\geq 1} \frac{1}{(k\pi)^2} \ \sum_{j\geq 1} \langle e_j, f''\rangle^2_{L^2(0,1)} < +\infty.$$

Let now $x \in (0,1)$ and B a standard BM in $\mathbb{R}$. Set $\tau := \inf\{s > 0 : x + B_s \notin (0,1)\}$. We apply the Itô formula to $t \mapsto u(T - t \wedge \tau, x + B_{t\wedge\tau})$ for $t \le T$ and we find

$$u(T - t \wedge \tau, x + B_{t\wedge\tau}) - u(T,x) = \int_0^{t\wedge\tau} \left(-\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \right) (T - s \wedge \tau, x + B_{s\wedge\tau})\, ds + \int_0^t \frac{\partial u}{\partial x}(T - s\wedge\tau, x + B_{s\wedge\tau})\, dB_s$$

By the uniform bound above on $\frac{\partial u}{\partial x}$, we can take expectation and find

$$\mathbb{E}[u(T - t \wedge \tau, x + B_{t\wedge\tau})] = u(T,x), \qquad \forall t \in [0,T].$$

In particular, if $t \to T$:

$$u(T,x) = \mathbb{E}[u(T - T\wedge\tau, x + B_{T\wedge\tau})] = \mathbb{E}[u(T-\tau, x + B_\tau)\, 1_{(\tau \le T)}] + \mathbb{E}[u(0, x + B_T)\, 1_{(\tau > T)}].$$

Since $x + B_\tau \in \{0,1\}$ a.s., by the homogeneous Dirichlet boundary condition we find $u(T-\tau, x+B_\tau) = 0$ a.s. on the event $\{\tau \le T\}$. Then we obtain

$$u(T,x) = \mathbb{E}[f(x + B_T)\, 1_{(\tau > T)}].$$

Notice that the indicator of the event $\{\tau > T\}$ means that a trajectory disappears from the expectation at the first visit of $x + B$ to $\{0,1\}$; it is customary to say that the process is *killed* at the boundary.

Notice that the constant functions belong to $L^2(0,1)$ but not to $H^1_0(0,1)$. Indeed, since for all $f \in C^2_c(0,1)$:

$$f(x) = \int_0^x f'(y)\, dy, \qquad x \in [0,1],$$

then convergence in $H^1_0(0,1)$ implies uniform convergence over $[0,1]$, and therefore $H^1_0(0,1)$ is included in the space of continuous functions which vanish at $\{0,1\}$. The Lebesgue measure is not an invariant measure, and in fact there is no invariant measure, since the process dies out as $t \to +\infty$:

$$|P_t f(x)|^2 \le \sum_{k\ge1} e^{-(k\pi)^2 t} \sum_{j\ge1} \langle e_j, f\rangle^2_{L^2(0,1)} \to 0$$

by dominated convergence.

Finally, by Proposition 15.10 and the integration by parts formula (16.2), we could obtain that

$$D(L_{\mathrm{Dir}}) = \{f \in H^2(0,1) : f(0) = f(1) = 0\} = H^2(0,1) \cap H^1_0(0,1).$$

The condition $f(0) = f(1) = 0$ is called a *homogeneous Dirichlet boundary condition.*

16.4. **Reflecting BM.** We consider now the set $[0,\infty[$, endowed with the Lebesgue measure, and the form

$$D := C^1_c([0,\infty[), \qquad \mathcal{E}_{\mathrm{Neu}}(u,v) := \frac{1}{2}\int_0^\infty \langle \nabla u, \nabla v\rangle\, dx.$$

Notice that D has no "boundary condition" at 0. We want to prove closability in $L^2([0,\infty[)$. We notice that, as in the previous subsection, we can "embed" the form $(\mathcal{E}, D_{\mathrm{Neu}})$ in another closed form; indeed, we set

$$D_{\mathrm{even}} := \{f \in C_c(\mathbb{R}),\ f(-x) = f(x) \quad \forall\, x \in \mathbb{R}, \quad f_{|[0,\infty[} \in C^1_c([0,\infty[)\},$$

and

$$\mathcal{E}(u,v) := \frac{1}{2}\int_{\mathbb{R}} \langle \nabla u, \nabla v\rangle \, dx$$

noting that u' is continuous and bounded on $\mathbb{R}\backslash\{0\}$, with a jump at 0 if $\lim_{x\to 0+} u'(x) \neq 0$. If we endow $\mathbb{R}$ with the Lebesgue measure, then there is an isomorphism between $(\mathcal{E}_{\mathrm{Neu},1}, D)$ and $(\mathcal{E}_1, D_{\mathrm{even}})$. Since we know that $(\mathcal{E}_1, D_{\mathrm{even}})$ is closable by the example of the standard Brownian Motion, then $(\mathcal{E}_{\mathrm{Neu}}, D)$ is also closable. We obtain a Dirichlet form $(\mathcal{E}_{\mathrm{Neu}}, D(\mathcal{E}_{\mathrm{Neu}}))$.

We notice the important integration by parts formula:

$$\int_0^\infty \varphi'(x)\, dx = -\varphi(0), \qquad \forall\, \varphi \in C^1_c([0,\infty[),$$

which yields

$$\mathcal{E}_{\mathrm{Neu}}(u,v) = -\frac{1}{2}\int_0^\infty u''\, v\, dx - \frac{1}{2}\, u'(0)\, v(0), \qquad \forall\, u, v \in C^1_c([0,\infty[).$$

One can prove from this formula and Proposition 15.10 that

$$D(L_{\mathrm{Neu}}) = \{u \in H^2([0,\infty[) : u'(0) = 0\}, \qquad L_{\mathrm{Neu}} u = \frac{1}{2}\, u''.$$

The condition $u'(0) = 0$ is called a *homogeneous Neumann boundary condition.*

Set now $C_{b,\mathrm{even}}(\mathbb{R}) := \{f \in C_b(\mathbb{R}) : \ f(-x) = f(x) \quad \forall\, x \in \mathbb{R}\}$ and $H^1_{\mathrm{even}}(\mathbb{R}) := \{f \in H^1(\mathbb{R}) : \ f(-x) = f(x) \quad \forall\, x \in \mathbb{R}\}$ and notice that the definition makes sense, since $H^1(\mathbb{R}) \subset C(\mathbb{R})$, and therefore $H^1_{\mathrm{even}}(\mathbb{R}) = H^1(\mathbb{R}) \cap C_{b,\mathrm{even}}(\mathbb{R})$. It is easy to see that the semigroup of the Brownian Motion leaves $C_{b,\mathrm{even}}(\mathbb{R})$ invariant:

$$\begin{aligned} P^{BM}_t f(-x) &= \int \mathcal{N}(-x,t)(dy)\ f(y) = \int \mathcal{N}(0,t)(dy)\ f(y-x) \\ &= \int \mathcal{N}(0,t)(dy)\ f(-y+x) = \int \mathcal{N}(0,t)(dy)\ f(y+x) \\ &= \int \mathcal{N}(x,t)(dy)\ f(y) = P^{BM}_t f(x), \end{aligned}$$

for all $x \in \mathbb{R}$ and $f \in C_{b,\mathrm{even}}(\mathbb{R})$. In particular, the resolvent family of BM leaves invariant both $H^1(\mathbb{R})$ and $C_{b,\mathrm{even}}(\mathbb{R})$ and therefore their intersection $H^1_{\mathrm{even}}(\mathbb{R})$. We recall that for all $f \in L^2(\mathbb{R})$

$$\mathcal{E}_\lambda(R^{BM}_\lambda f, g) = \int_{\mathbb{R}} f\, g\, dx, \qquad \forall\, g \in H^1(\mathbb{R}),\ \lambda > 0.$$

We obtain, by restriction to $[0,\infty[$ and to $g \in C^1_{c,\mathrm{even}}(\mathbb{R})$, since

$$\int_{\mathbb{R}} u\, v\, dx = 2\int_0^\infty u\, v\, dx, \qquad \forall\, u, v \in C_c(\mathbb{R}) \cap C_{b,\mathrm{even}}(\mathbb{R}),$$

that

$$\mathcal{E}_{\mathrm{Neu},\lambda}(R^{BM}_\lambda f, g) = \int_0^\infty f\, g\, dx, \qquad \forall\, g \in C^1_c([0,\infty[),\ \lambda > 0.$$

In particular, by injectivity of the Laplace tranform, the semigroup (P^{Neu}_t) associated with $(\mathcal{E}_{\mathrm{Neu}}, D(\mathcal{E}_{\mathrm{Neu}}))$ has the representation

$$P^{\mathrm{Neu}}_t f(x) = \mathbb{E}[f(|x + B_t|)], \qquad \forall\, x \in [0,\infty[,\ t \geq 0,\ f \in C_c(\mathbb{R}).$$

In particular, $P_t^{\mathrm{Neu}} f$ is the restriction to $[0,\infty[$ of $P_t^{BM}\hat f$, where $\hat f(x) := f(|x|)$, $x \in \mathbb{R}$. Since $P_t^{BM}\hat f$ is smooth and even, we obtain that its derivative w.r.t. x at 0 is 0, i.e. $P_t^{\mathrm{Neu}} f$ satisfies the homogeneous Neumann boundary condition.

We have now the classical

Lemma 16.1 (Skorohod)**.** *For any continuous function $a : [0,\infty[\mapsto \mathbb{R}$ such that $a(0) \ge 0$ there exists a unique pair (x,ℓ) of continuous functions from $[0,\infty[$ to $[0,\infty[$ such that*

1. *ℓ is monotone non-decreasing and $\ell(0) = 0$,*
2. *$\int_0^\infty x_t \, d\ell_t = 0$,*
3. *$x_t = a_t + \ell_t$, $t \ge 0$.*

Moreover, we have the explicit representation

$$\ell_t = \sup_{s\le t}\,(a_s)^-, \qquad x_t = a_t + \sup_{s\le t}\,(a_s)^-, \qquad t \ge 0.$$

We apply this lemma to $a_t := x + B_t$, where $x \ge 0$ and B is a standard BM. Then we obtain that there exists a unique pair of continuous processes $(X_t(x), L_t)_{t\ge 0}$ such that $X \ge 0$, $L_0 = 0$ and L is monotone non-decreasing, and

$$X_t(x) = x + B_t + L_t, \qquad \int_0^\infty X_t \, dL_t = 0.$$

This condition is equivalent to saying that the measure dL_t supported by the set $\{t : X_t(x) = 0\}$.

The process $X_t(x)$ is called the *reflecting BM at* 0. Indeed, as long as $X_t > 0$ we have $dL_t = 0$ and therefore $dX_t = dB_t$; however, when $X_t(x) = 0$, then dL_t can be non-zero and gives a kick to the process, keeping it positive. The condition $\int_0^\infty X_t \, dL_t = 0$ means that the reflection term dL_t acts only when the process hits the obstacle 0.

We now define for any $f \in C_b^2([0,\infty[)$: $u(t,x) := P_t^{\mathrm{Neu}} f(x)$. We know that u is smooth and satisfies the homogeneous Neumann boundary condition. We consider as usual the process $[0,T] \ni t \mapsto u(T-t, X_t(x))$ and we obtain by the Itô formula for semimartingales

$$\begin{aligned} u(T-t, X_t(x)) - u(T,x) = &\int_0^t \left(-\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\right)(T-s, X_s(x))\, ds \\ &+ \int_0^t \frac{\partial u}{\partial x}(T-s, X_s(x))\, dB_s + \int_0^t \frac{\partial u}{\partial x}(T-s, X_s(x))\, dL_s. \end{aligned}$$

By the above considerations

$$\int_0^t \frac{\partial u}{\partial x}(T-s, X_s(x))\, dL_s = \int_0^t \frac{\partial u}{\partial x}(T-s, 0)\, dL_s = 0.$$

We obtain, taking expectation and letting $t = T$

$$u(T,x) = \mathbb{E}[f(X_t(x))].$$

Concerning the invariant measure of X, in this case we have that the constant function 1 does not belong to $L^2(0,\infty)$. However, the representation of the semigroup of X in terms of the semigroup of BM shows that the Lebesgue measure on $[0,\infty[$ is an invariant and reversible measure for X. It is interesting to notice that we have the representation

$$1_{[0,\infty[}(x)\, dx = e^{-U(x)}\, dx, \qquad U(x) := \begin{cases} 0, & x \ge 0, \\ +\infty, & x < 0 \end{cases}$$

The function U is discontinuous but convex.

17. Back to SPDEs

Let us recall some results from the previous sections. We have seen that the solution of the stochastic heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \dot{W}, \\ \\ u(t,0) = u(t,1) = 0 \\ \\ u(0,x) = u_0(x), \qquad x \in [0,1] \end{cases} \tag{17.1}$$

$\dot{W}(t,x)$ is a space-time white-noise, can be written as

$$u_t := \sum_{k=1}^{+\infty} u_t^k \, e_k$$

where

$$u_t^k = u_0^k - \frac{(k\pi)^2}{2} \int_0^t u_s^k \, ds + W_t^k, \qquad t \geq 0, \tag{17.2}$$

$$e_k(x) := \sqrt{2}\, \sin(k\pi x), \qquad x \in [0,1].$$

The process $(u(t,x), t \geq 0)$ is stationary if and only if $(u_0(x), x \in [0,1])$ has the distribution of a Brownian bridge. Moreover $(u_t^k, t \geq 0)_{k \geq 1}$ is an independent family of O-U processes, with respective parameters $(\frac{(k\pi)^2}{2})_{k\geq 1}$. Remark that (17.2) is a monotone gradient system in $\mathbb{R}$ with (convex) potential

$$U(x) = \frac{(k\pi)^2}{2}\, x^2, \qquad x \in \mathbb{R}.$$

Therefore the results of Proposition 15.14 and Theorem 15.15 in this particular case coincide with Propositions 3.1 and 3.3 and those of section 3.9.

Invariant measures of products of independent Markov processes are clearly products measures. What about Dirichlet forms? In this case we clearly have that $(u_t^k, t \geq 0)_{k=1,\dots,N}$ is a monotone gradient system in $\mathbb{R}^N$ with convex potential

$$U_N(x) = \sum_{k=1}^{N} \frac{(k\pi)^2}{2}\, x_k^2, \qquad x = (x_1, \dots, x_N) \in \mathbb{R}^N.$$

The associated invariant measure is

$$\mu_N(dx) = \frac{1}{Z_N} \exp\left(- \sum_{k=1}^{N} \frac{(k\pi)^2}{2}\, x_k^2 \right) dx_1 \cdots dx_N$$

and the Dirichlet Form is the closure of

$$\mathcal{E}^N(u,v) = \frac{1}{2} \int_{\mathbb{R}^N} \langle \nabla u, \nabla v \rangle_{\mathbb{R}^N} \, d\mu_N, \qquad u, v \in C_b^1(\mathbb{R}^N).$$

Since $(e_k)_{k\geq 1}$ is a complete orthonormal system of $H := L^2(0,1)$, then the map

$$\mathbb{R}^N \ni x = (x_1, \dots, x_N) \mapsto \sum_{k=1}^{+\infty} x_k \, e_k \in H$$

is an isometry with a linear subspace $H_N \subset H$. In particular, we can identify $\mathcal{E}^N$ with

$$\mathcal{E}^N(u,v) = \frac{1}{2}\int_{\mathbb{R}^N} \langle \nabla u, \nabla v\rangle_H \, d\mu_N, \qquad u,v \in C^1_b(H_N).$$

If we pass to the limit, it is possible to prove that

Proposition 17.1. *The solution of* (17.1) *is the Markov process in* $L^2(0,1)$ *associated with the Dirichlet form closure of*

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{H} \langle \nabla u, \nabla v\rangle_H \, d\mu, \qquad u,v \in C^1_b(H).$$

Notice that $C^1_b(H)$ is the space of all Fréchet differentiable $f : H \mapsto \mathbb{R}$ which are bounded continuous and have a bounded continuous Fréchet differential $\nabla f : H \mapsto \mathbb{R}$.

By the result of Exercise 6.5, we have that the invariant measure μ is the law of a Brownian bridge $(\beta_x)_{x\in[0,1]}$; moreover the covariance function

$$\mathbb{E}(\beta_x\,\beta_y) = x \wedge y - xy, \qquad x,y \in [0,1]$$

is associated with the elliptic equation

$$\begin{cases} -\dfrac{d^2 f}{dx^2} = h, \\ \\ f(0) = f(1) = 0, \end{cases}$$

since we have $f(x) = \int_0^1 (x\wedge y - xy)\, h(y)\, dy$. Recall now the explicit formula (16.1) for the density of a Gaussian measure in $\mathbb{R}^d$ with mean 0 and covariance $Q = (-2A)^{-1}$. In our case, we have just proven that $Qh = f$ and therefore

$$-2Af = -\frac{d^2 f}{dx^2}, \qquad \langle Af, f\rangle = -\frac{1}{2}\int_0^1 (f')^2 dx, \qquad f(0) = f(1) = 0.$$

This yields the formal expression for the law of the invariant measure of (17.1)

$$\mu(df) = \frac{1}{2}\exp(-U(f))\, df, \qquad U(f) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 (f')^2 dx, & \text{if } f(0) = f(1) = 0, \\ \\ +\infty & \text{otherwise} \end{cases}$$

It is easy to see that

$$\langle \nabla U(f), \alpha\rangle = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(U(f+\varepsilon\alpha) - U(f)) = \int_0^1 f'\alpha'\, dx = -\int_0^1 f''\alpha\, dx = \langle -f'', \alpha\rangle$$

i.e. $\nabla U(f) = -f''$ and equation (17.1) can be written as a gradient system in $L^2(0,1)$

$$du = -\frac{1}{2}\nabla U(u)\, dt + dW.$$

17.1. **Reaction-diffusion equations.** It is possible to go further and consider a nonlinear equation of reaction-diffusion type:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} - F'(u) + \dot{W}, \\ \\ u(t,0) = u(t,1) = 0 \\ \\ u(0,x) = u_0(x), \qquad x \in [0,1] \end{cases} \tag{17.3}$$

with $F : \mathbb{R} \mapsto \mathbb{R}$ of class C^1 and (for simplicity) convex, positive with at most quadratic growth. We define

$$G(f) := 2\int_0^1 F(f(x))\,dx, \qquad f \in L^2(0,1).$$

Then

$$\langle \nabla G(f), \alpha\rangle = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(G(f+\varepsilon\alpha) - G(f)) = 2\int_0^1 F'(f)\,\alpha'\,dx = 2\langle F'(f), \alpha\rangle$$

i.e. $\nabla G(f) = 2F'(f)$ and equation (17.3) can also be written as a gradient system

$$du = -\frac{1}{2}\nabla V(u)\,dt + dW, \qquad V := U + G.$$

Moreover, we have an expression for the invariant measure

$$\nu(df) = \frac{1}{Z}\exp(-V(f))\,df = \frac{1}{\int e^{-G} d\mu}\, e^{-G(f)}\,\mu(df),$$

where μ is the law of the Brownian bridge β. One can prove that ν is the unique probability invariant measure of the solution u of (17.3) and that the solution is reversible w.r.t. to ν.

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Laboratoire de Probabilités et Modèles Aléatoires (CNRS U.M.R. 7599), Université Paris 6 – Pierre et Marie Curie, U.F.R. Mathématiques, Case 188, 4 place Jussieu, 75252 Paris cedex 05, France

E-mail address: lorenzo.zambotti@upmc.fr