Branching Brownian Motion

and the spinal decomposition

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Chapter I

Branching Brownian motion

We introduce branching Brownian motion and its discrete-time analogue, the branching random walk. The elementary but useful many-to-one formula is derived. The additive martingale and the derivative martingale are introduced. The chapter ends with the second-order asymptotics of the minimal position.

1. Branching Brownian motion

Branching Brownian motion is a simple spatial branching process defined as follows. At time $t = 0$, a single particle (also called an "individual") starts at the origin, and moves as a standard one-dimensional Brownian motion, whose life-time is an exponential random variable of parameter 1. When the particle dies, it produces two new particles (we say that the original particle "splits into two"), moving as independent Brownian motions, each having a mean 1 exponential random life-time. These particles are subject to the same splitting rule. And the system goes on indefinitely. See Figure 1 below.

Let

$$f(x) := \mathbf{1}_{\{x\geq 0\}} \, ,$$

and let $X_1, \, X_2, \, \cdots, \, X_{N(t)}(t)$ denote the positions of the particles in the system at time t. Write

$$u(t, \, x) := \mathbf{E}\Big(\prod_{i=1}^{N(t)} f(x + X_i(t)) \Big) \, .$$

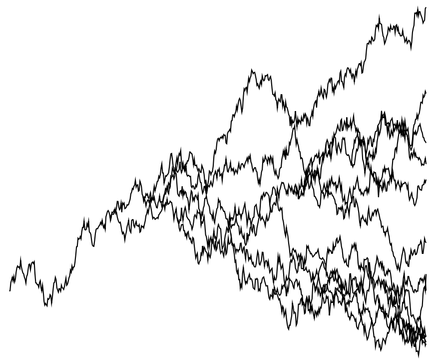

Figure 1. Branching Brownian motion

By discussing on the value of the life-time of the initial ancestor, we see that

$$\begin{aligned} u(t,\, x) &= \mathrm{e}^{-t}\,\mathbf{E}[f(x+B(t))] + \int_0^t \mathrm{e}^{-s}\,\mathrm{d}s\,\mathbf{E}[u^2(t-s,\, x+B(s)] \\ &= \mathrm{e}^{-t}\,\mathbf{E}[f(x+B(t))] + \mathrm{e}^{-t}\int_0^t \mathrm{e}^{r}\,\mathrm{d}s\,\mathbf{E}[u^2(r,\, x+B(s)]\,, \qquad (r := t-s) \end{aligned}$$

where $(B(s),\, s \ge 0)$ denotes standard Brownian motion. We then arrive at the so-called F-KPP equation (Fisher [61] who was interested in the evolution of a biological population, Kolmogorov, Petrovskii and Piskunov [78])

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u^2 - u\,. \tag{1.1}$$

This holds for a large class of measurable functions f. The special form of f we have taken here is of particular interest, since in this case,

$$u(t,\, x) = \mathbf{P}\Big(\min_{1\le i\le N(t)} X_i(t) \ge -x\Big) = \mathbf{P}\Big(\max_{1\le i\le N(t)} X_i(t) \le x\Big)\,,$$

which is the distribution function of the maximal position of branching Brownian motion at time t.

The F-KPP equation is known for its travelling wave solutions: let $m(t)$ denote the median of u, i.e., $u(t,\, m(t)) = \frac{1}{2}$, then

$$\lim_{t\to\infty} u(t,\, x+m(t)) = w(x)\,,$$

uniformly in $x \in \mathbb{R}$, and w is a wave solution of the F-KPP equation (1.1) at speed $2^{1/2}$, meaning that $w(x - 2^{1/2}t)$ solves (1.1), or, equivalently,

$$\frac{1}{2}\,w'' + 2^{1/2}w' + w^2 - w = 0\,.$$

It is proved by Kolmogorov, Petrovskii and Piskunov [78] that $\lim_{t\to\infty} \frac{m(t)}{t} = 2^{1/2}$, and by Bramson ([36] and [37]) that

$$m(t) = 2^{1/2} t - \frac{3}{2^{3/2}} \log t + C + o(1) , \qquad t \to \infty , \tag{1.2}$$

for some constant C.

For further use, we mention a probabilistic interpretation of the travelling wave solution w. By Lalley and Sellke [80], w can be written as

$$w(x) = \mathbf{E}\Big(\mathrm{e}^{-C_1 \, D_\infty \, \mathrm{e}^{-2^{1/2} x}} \Big) , \tag{1.3}$$

where $C_1 > 0$ is a constant, and $D_\infty > 0$ is a random variable whose distribution depends on the branching mechanism (in our description, it is binary branching). The idea of this interpretation is also present in the work of McKean [90]. We will come back to this point in Section 6.

The connection, observed by McKean [90], between the branching system and the F-KPP differential equation makes the study of branching Brownian motion particularly appealing.[1] As such, branching Brownian motion can be used to obtain — or explain — results for the F-KPP equation. For purely probabilistic approaches in the study of travelling wave solutions to the F-KPP equation, see Neveu [104], Harris [65], Kyprianou [79]. More recently, physicists have been much interested in the effect of noise on wave propagation. We are going to discuss on this feature in more details in Chapter II.

We study branching Brownian motion as a purely probabilistic object. Moreover, the Gaussian displacement of particles in the system does not play any essential role, which leads us to study the more general model of branching random walks.

2. Branching random walks

Consider a one-dimensional discrete-time branching random walk. At the beginning, there is a single particle located at the origin. Its children, who form the first generation, are positioned according to the distribution of a certain random vector $\mathscr{L} := (\xi_1, \cdots, \xi_N)$.[2]

[1] Another historic reference is a series of three papers by Ikeda, Nagasawa and Watanabe [71], who were interested in a general theory connecting probability with differential equations.

[2] As a matter of fact, the dimension of $\mathscr{L}$ can be random (so $\mathscr{L}$ is a point process, describing a random scattering of points in $\mathbb{R}$), and can be possibly infinite. Also, unlike branching Brownian motion, it is possible that $\mathbf{P}(\xi_i = \xi_j, \text{ for some } i \neq j)$ can be positive.

Each of the particles in the first generation gives birth to new particles that are positioned (compared to their birth places) according to an independent copy of $\mathscr{L}$; they form the second generation. The system goes on according to the same mechanism. We assume that for any n, each particle at generation n produces new particles independently of each other and of everything up to the n-th generation. See Figure 2 below.

In the special case that $N > 1$ is a fixed integer, and that $\xi_1, \cdots, \xi_N$ are i.i.d. random variables, we say that the branching random walk has i.i.d. displacements. It was the case with branching Brownian motion. However, we will see that in the problems which are of interest to us, the dependence structure between ξ_i will seldom cause any serious trouble.

We denote by $(V(x), |x| = n)$ the positions of the particles in the n-th generation, $|x|$ standing for the generation of the individual x. We always assume that $\mathbf{E}(N) > 1$; i.e., the branching is supercritical. [However, it is possible that $\mathbf{E}(N) = \infty$.] As such, the system survives with positive probability.

3. Examples

We give here some examples of branching random walks, and more general hierarchical fields.

In the literature, the branching random walk bears various names, all leading to equivalent or similar structure. Let us make a short list.

Example 3.1. (Mandelbrot's multiplicative cascades). Mandelbrot's multiplicative cascades are introduced by Mandelbrot [98], and studied by Kahane [73] and Peyrière [107], in an attempt of understanding the intermittency phenomenon in Kolmogorov's turbulence

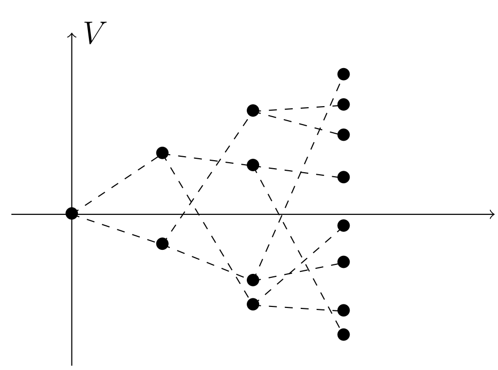

Figure 2. A branching random walk and its first three generations

theory. It can be formulated, for example, in terms of a stochastically self-similar measure on a compact interval. In fact, the standard Cantor set consists in dividing, at each step, a compact interval into three identical sub-intervals and removing the middle one. Instead of splitting an interval into identical sub-intervals, we can do it according to a given three-dimensional distributions, and the resulting lengths of sub-intervals form a Mandelbrot's multiplicative cascade. If we look at the logathrithm of the lengths, we have a branching random walk.

Mandelbrot's multiplicative cascades also bear other names, such as random recursive constructions (Mauldin and Williams [99]). A key ingredient is to study fixed points of the so-called **smoothing transforms** (Durrett and Liggett [55], Alsmeyer [9], Alsmeyer, Biggins and Meiners [10]). We will briefly come back to this point in Section 6. For surveys on these topics, see Liu [81], Biggins and Kyprianou [27]. □

Example 3.2. (Gaussian free fields and log-correlated Gaussian fields). The two-dimensional discrete Gaussian free field possesses a complicated structure of extreme values, but it turns out possible to compare it with that of the branching random walk. By comparison to analogue results for branching random walks, many deep results have been recently established for Gaussian free fields and more general logarithmically correlated Gaussian fields (Bolthausen, Deuschel and Giacomin [31], Madaule [89], Biskup and Louidor [30], Ding, Roy and Zeitouni [51]). In parallel, in the continuous-time setting, following Kahane's pioneer work in [74], the study of **Gaussian multiplicative chaos** has witnessed importance recent progress (Duplantier, Rhodes, Sheffield and Vargas [53], Garban, Rhodes and Vargas [63], Rhodes and Vargas [108]).

Via Dynkin's isomorphism theorem, local times of Markov processes are closely connected to (the square of) some Gaussian processes. As such, new lights have been recently shed on the **cover time** of the two-dimensional torus by simple random walk (Ding [50], Belius and Kistler [16]). □

Example 3.3. (Directed polymers on trees). In [49], Derrida and Spohn introduced directed polymers on trees, as a hierarchical extension of Derrida's REM (Random Energy Model) for spin glasses. In this setting, the energy of a polymer, being the sum of i.i.d. random variables assigned on each edge of the tree, is exactly a branching random walk with i.i.d. displacements. The continuous-time setting has also been studied in the literature (Bovier and Kurkova [35]).

Directed polymers on trees also provide an interesting example of random environment for random walks. The tree-valued random walk in random environment is an extension of Lyons' biased random walk on trees ([82], [83]), in the sense that the random walk is randomly biased. Chapter III will be devoted to this model. □

The list of interesting examples can be very long. Let us add just a couple more: Arguin [11] has given a series of lectures on his work in progress on characteristic polynomials of unitary matrices, and on the Riemann zeta function on the critical line, whereas Aïdékon [5] has successfully applied branching random walk techniques to CLE (Conformal Loop Ensembles).

4. The basic assumption

In order to obtain universality results, we need to exclude a few "pathological cases". Throughout, we assume[3]

$$(*) \qquad \mathbf{E}\Big(\sum_{x:\,|x|=1} \mathrm{e}^{-V(x)} \Big) = 1, \qquad \mathbf{E}\Big(\sum_{x:\,|x|=1} V(x)\, \mathrm{e}^{-V(x)} \Big) = 0,$$

and we call it Assumption $(*)$. In terms of the point process $\mathscr{L} := (\xi_1, \cdots, \xi_N)$, Assumption $(*)$ means $\mathbf{E}(\sum_{i=1}^N \mathrm{e}^{-\xi_i}) = 1$ and $\mathbf{E}(\sum_{i=1}^N \xi_i\, \mathrm{e}^{-\xi_i}) = 0$.

In general, given a branching random walk $(V(x))$, we should be able to find $a > 0$ and $b \in \mathbb{R}$ such that the linear transformation

$$\widehat{V}(x) := aV(x) + b|x| \, ,$$

which gives another branching random walk, satisfies Assumption $(*)$. However, the existence of the pair (a, b) is not automatic. There are examples of branching random walks for which the existence of (a, b) fails. Loosely speaking, the existence of (a, b) fails if and only if the law of $\inf_i \xi_i$ is bounded from below and $\mathbf{E}(\sum_i \mathbf{1}_{\{\xi_i = \mathrm{supp}_{\min}\}}) \geq 1$, with $\mathrm{supp}_{\min}$ denoting the minimum of the support of the law of $\inf_i \xi_i$ (i.e., the essential infimum of $\inf_i \xi_i$). In particular, for a branching random walk with i.i.d. Gaussian displacements, the pair (a, b) exist.

For an elementary but complete discussion on the existence of (a, b), see the arXiv version of Jaffuel [72], or Bérard and Gouéré [20].

[3] We implicitly assume in the second part that $\mathbf{E}(\sum_{x:\,|x|=1} |V(x)| \mathrm{e}^{-tV(x)}) < \infty$.

Exercise 4.1. Let $(V(x))$ be a branching random walk with i.i.d. displacements and satisfy Assumption $(*)$. Then $\mathbf{E}(\xi_1) > 0$. As such, along each branch, the random walk has a positive drift.

Special case: if the i.i.d. displacements are Gaussian, it must be $\mathscr{N}(2\log m, 2\log m)$, where $m := \mathbf{E}(N)$ is the mean number of branches. □

5. The many-to-one formula

Assume $\mathbf{E}(\sum_{x:\,|x|=1} \mathrm{e}^{-V(x)}) = 1$, which is the first part of Assumption $(*)$.

Let $S_0 := 0$ and let $(S_n - S_{n-1},\ n \ge 1)$ be a sequence of i.i.d. real-valued random variables such that for any measurable function $h : \mathbb{R} \to [0, \infty)$,

$$\mathbf{E}[h(S_1)] = \mathbf{E}\Big(\sum_{x:\,|x|=1} \mathrm{e}^{-V(x)} h(V(x)) \Big) . \tag{5.1}$$

The law of S_1 is well-defined due to our assumption $\mathbf{E}(\sum_{x:\,|x|=1} \mathrm{e}^{-V(x)}) = 1$.

Theorem 5.1. (The many-to-one formula) *Under Assumption $(*)$, for any $n \ge 1$ and any measurable function $g : \mathbb{R}^n \to [0, \infty)$, we have*[4]

$$\mathbf{E}\Big[\sum_{x:\,|x|=n} g(V(x_1), \cdots, V(x_n)) \Big] = \mathbf{E}\Big[\mathrm{e}^{S_n} g(S_1, \cdots, S_n) \Big],$$

where x_i is the ancestor of x at generation i, with $x_n := x$.

Proof. We prove by induction in n. For $n = 1$, this is the definition of the distribution of S_1. Assume the identity proved for n. Then, for $n + 1$, we condition on the branching random walk in the first generation; by the branching property, this yields

$$\begin{aligned} &\mathbf{E}\Big[\sum_{|x|=n+1} g(V(x_1), \cdots, V(x_{n+1})) \Big] \\ = \ &(\mathbf{E} \otimes \widetilde{\mathbf{E}}) \Big[\sum_{|y|=1} \sum_{|\widetilde{z}|=n} g(V(y), V(y) + \widetilde{V}(\widetilde{z}_1), \cdots, V(y) + \widetilde{V}(\widetilde{z}_n)) \Big], \end{aligned}$$

where $\widetilde{\mathbf{E}}$ is expectation with respect to the branching random walk $(\widetilde{V}(\widetilde{z}))$ which is independent of $(V(y), |y| = 1)$. By induction hypothesis, for any $u \in \mathbb{R}$,

$$\widetilde{\mathbf{E}}\Big(\sum_{|\widetilde{z}|=n} g(u + \widetilde{V}(\widetilde{z}_1), \cdots, u + \widetilde{V}(\widetilde{z}_n)) \Big) = \widetilde{\mathbf{E}}\Big(\mathrm{e}^{\widetilde{S}_n} g(u, u + \widetilde{S}_1, \cdots, u + \widetilde{S}_n) \Big),$$

[4] For notational simplification, we often write $\sum_{|x|=n}(\cdots)$ instead of $\sum_{x:\,|x|=n}(\cdots)$.

with the random walk $(\widetilde{S}_j,\ j \ge 1)$ independent of $(V(y), |y| = 1)$, and distributed as $(S_j,\ j \ge 1)$ under $\mathbf{P}$. Since

$$\mathbf{E}\Big[\sum_{|y|=1} h(V(y))\Big] = \mathbf{E}\Big[\mathrm{e}^{S_1} h(S_1)\Big],$$

it remains to note that $(\mathbf{E} \otimes \widetilde{\mathbf{E}})[\mathrm{e}^{S_1+\widetilde{S}_n} g(S_1, S_1 + \widetilde{S}_1, \cdots, S_1 + \widetilde{S}_n)]$ is nothing else but $\mathbf{E}[\mathrm{e}^{S_{n+1}} g(S_1, S_2, \cdots, S_{n+1})]$. This implies the desired identity for all $n \ge 1$. □

Remark 5.2. (i) Under Assumption $(*)$, we have $\mathbf{E}(S_1) = 0$, which means that $(S_n, n \ge 0)$ is a mean-zero non-degenerate[5] random walk. In particular, if the branching random walk has i.i.d. displacements, the walk along each branch is strictly larger than the drift of the new random walk (S_n). As a matter of fact, behind the innocent-looking new random walk (S_n) is a change-of-probabilities setting, which we will study in more details in Chapter IV.

(ii) The many-to-one formula is ready for use in the computation of the first moment, but requires some additional work for the computation of higher-order moments. For a "many-to-few" version, see Harris and Roberts [67]. □

6. A pair of martingales

Under Assumption $(*)$, there are a pair of martingales naturally associated with the branching random walk:

$$\begin{aligned} M_n &:= \sum_{|x|=n} \mathrm{e}^{-V(x)}, \\ D_n &:= \sum_{|x|=n} V(x)\mathrm{e}^{-V(x)}, \end{aligned}$$

with respect to their natural filtrations. In the literature, (M_n) is referred to as an additive martingale, whereas (D_n) is called a derivative martingale.

Since (M_n) is a non-negative martingale, it converges a.s. to a finite random variable; the convergence does not hold in L^1, as (M_n) is not uniformly integrable:

Theorem 6.1. *Under Assumption $(*)$, we have*[6]

$$M_n \to 0, \qquad \text{a.s.}$$

[5] That is, not identically zero.

[6] This is, in fact, a special case of a more general result called the Biggins martingale convergence theorem (Biggins [25], Lyons [84]).

Proof. Postponed to Chapter IV. □

We need to be careful with "a.s." (or the forthcoming "in probability"). If the point process $\mathscr{L}$ is empty with positive probability, the system dies out at finite time with positive probability. So "a.s." really means almost surely on the system's non-extinction.

A straightforward consequence of Theorem 6.1 is

$$\min_{|x|=n} V(x) \to \infty, \qquad \text{a.s.} \tag{6.1}$$

So, under Assumption $(*)$, even though the branching random walk can take negative values at some sites, it becomes very large when the generation becomes large. Looking at the derivative martingale $D_n := \sum_{|x|=n} V(x)\mathrm{e}^{-V(x)}$; it is of no surprise that it also converges a.s.:[7]

Theorem 6.2. (Biggins and Kyprianou [26]) *If* $\mathbf{E}[\sum_{|x|=1} V(x)^2\,\mathrm{e}^{-V(x)}] < \infty$, *then* $(D_n,\, n \geq 0)$ *converges a.s. to a non-negative limit, denoted by* D_∞.

We do not prove Theorem 6.2, though we will have enough mathematical tools in Chapter IV. Attention: even though D_∞ exists, nothing guarantees that it is (strictly) positive, because the positivity of D_∞ requires some additional integrability condition. Biggins and Kyprianou [26] and Aïdékon [2] provided sufficient conditions for the positivity of D_∞; recently, a necessary and sufficient condition is established by Chen [47].

The positive random variable D_∞ in (1.3) is the continuous-time analogue for branching Brownian motion, and is proved to be positive by Lalley and Sellke [80]. As a matter of fact, McKean [90] used M_∞ (the limit of the additive martingale) in the expression (1.3) for w (the uniqueness in law, up to a constant multiple, of the fixed point being known; see the survey by Biggins and Kyprianou [27]). In view of Theorem 6.1, it is unfortunate that McKean used the vanishing solution. It took about a decade to see the error fixed, by Lalley and Sellke [80].[8]

Although we do not study the derivative martingale in depth in these notes,[9] it plays a crucial role when one looks for refined properties of the branching random walk. For

[7] The sufficient condition in [26] is slightly weaker than in the statement of Theorem 6.2.

[8] Actually, McKean [90] was not *that* wrong: it is possible ([8]) to prove that $\frac{n^{1/2} M_n}{D_n}$ converges in probability to a positive constant. So the approach used by McKean [90] is all right, as long as the additive martingale is multiplied by $n^{1/2}$.

[9] And even completely neglect another fundamental martingale: the multiplicative martingale.

example, it comes as of no surprise that it was exploited by Duplantier, Rhodes, Sheffield and Vargas [53] in the study of critical Gaussian multiplicative chaos.

It is obvious that both $M_\infty = 0$ and D_∞ are fixed points (in distribution) of the following smoothing transform:

$$Z \stackrel{\text{(law)}}{=} \sum_{|x|=1} \mathrm{e}^{-V(x)} Z(x)\,,$$

where, conditioning on $(V(x),\,|x|=1)$, $Z(x)$ are independent copies of Z. This is a simple example to illustrate the importance of fixed points of smoothing transform.

7. Extreme positions

We work under Assumption $(*)$, and are interested in the minimal position $\min_{|x|=n} V(x)$. We have already seen in (6.1) that

$$\min_{|x|=n} V(x) \to \infty, \qquad \text{a.s.}$$

The question is at which speed $\min_{|x|=n} V(x)$ goes to infinity. A general result called the law of large numbers for branching random walks (Biggins [24], Kingman [77], Hammersley [64]) applied under Assumption $(*)$ yields that

$$\frac{1}{n} \min_{|x|=n} V(x) \to 0, \qquad \text{a.s.} \tag{7.1}$$

Here is the answer to our question, and is a weak analogue of Bramson's estimate (1.2):

Theorem 7.1. *Under Assumption $(*)$ and suitable integrability condition, we have*

$$\frac{1}{\log n} \min_{|x|=n} V(x) \to \frac{3}{2}\,, \qquad \textit{in probability.}$$

Proof. We only prove the lower bound (and outline the proof of the upper bound), namely, for any $\varepsilon > 0$,

$$\mathbf{P}\Big(\min_{|x|=n} V(x) \le (\frac{3}{2} - \varepsilon) \log n \Big) \to 0, \qquad n \to \infty.$$

Let $K > 0$ and $0 < a < \frac{3}{2}$. Let

$$Z_n := \sum_{|x|=n} \mathbf{1}_{\{V(x) \le a \log n,\ V(x_i) \ge -K,\ \forall 1 \le i \le n\}}.$$

By the many-to-one formula (Theorem 5.1), we have,

$$\begin{aligned}\mathbf{E}(Z_n) &= \mathbf{E}\Big\{\mathrm{e}^{S_n}\,\mathbf{1}_{\{S_n\le a\log n,\, S_i\ge -K,\, \forall 1\le i\le n\}}\Big\}\\ &\le n^a\,\mathbf{P}\Big\{S_n\le a\log n,\ S_i\ge -K,\ \forall 1\le i\le n\Big\}.\end{aligned}$$

For n such that $a\log n\ge 1$, we have $\mathbf{P}\{S_n\le a\log n,\ \underline{S}_n\ge -K\}\le \frac{1}{n^{(3/2)+o(1)}}$. See Figure 3 below.

Since $a<\frac32$, it follows that $\lim_{n\to\infty}\mathbf{E}(Z_n)=0$. A fortiori, $Z_n\to 0$ in probability.

We already know that $\min_{|x|=n}V(x)\to\infty$ a.s. This yields the desired lower bound: $\mathbf{P}\{\min_{|x|=n}V(x)\ge(\frac32+\varepsilon)\log n\}\to 0$ for all $\varepsilon>0$.

We provide a sketch for the upper bound: for all $\varepsilon>0$,

$$\mathbf{P}\Big(\min_{|x|=n}V(x)\le(\frac32+\varepsilon)\log n\Big)\to 1,\qquad n\to\infty.$$

We are tempted to imitate the argument used in the proof of the lower bound by taking $Z_n:=\sum_{|x|=n}\mathbf{1}_{\{V(x)\le(\frac32+\varepsilon)\log n,\, V(x_i)\ge -K,\,\forall 1\le i\le n\}}$, using the Cauchy–Schwarz inequality

$$\mathbf{P}(Z_n\ge 1)\ge\frac{[\mathbf{E}(Z_n)]^2}{\mathbf{E}(Z_n^2)}\,.$$

and bounding $\mathbf{E}(Z_n)$ from below while bounding $\mathbf{E}(Z_n^2)$ from above.

Unfortunately, $\mathbf{E}(Z_n^2)$ is very large in this case. So we slightly reduce the size of the event in Z_n by considering (writing $a_n:=(\frac32+\varepsilon)\log n$)

$$Y_n:=\sum_{|x|=n}\mathbf{1}_{\{V(x)\le a_n,\, V(x_i)\ge\frac{a_n}{n}i-K,\,\forall 1\le i\le n\}}\,,$$

where $K>0$ is a large but fixed constant. The first moment $\mathbf{E}(Y_n)$ can be estimated as before. For the second moment $\mathbf{E}(Y_n^2)$, we argue that

$$\begin{aligned}\mathbf{E}(Y_n^2) &= \mathbf{E}\Big\{\sum_{|x|=n}\sum_{|y|=n}\mathbf{1}_{\{V(x)\le a_n,\, V(y)\le a_n,\, V(x_i)\ge\frac{a_n}{n}i-K,\, V(y_i)\ge\frac{a_n}{n}i-K,\,\forall 1\le i\le n\}}\Big\}\\ &= \mathbf{E}(Y_n)+\mathbf{E}\Big\{\sum_{j=0}^{n-1}\sum_{|z|=j}\mathbf{1}_{\{V(z_i)\ge\frac{a_n}{n}i-K,\,\forall 1\le i\le j\}}\times\\ &\qquad\times\sum_{(x_{j+1},\,y_{j+1})}\sum_{(x,\,y)}\mathbf{1}_{\{V(x_k),\,V(y_k)\ge\frac{a_n}{n}k-K,\,\forall j<k\le n,\, V(x)\le a_n,\, V(y)\le a_n\}}\Big\}\,,\end{aligned}$$

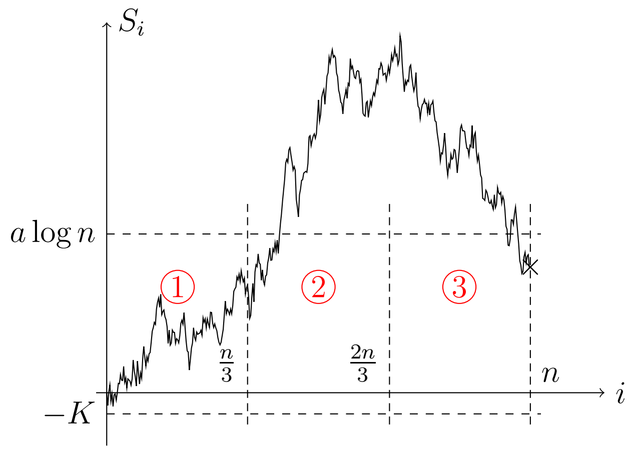

Figure 3. Computing $\mathbf{P}\{S_n \le a \log n,\ \underline{S}_n \ge -K\}$

where, the double sum $\sum_{(x_{j+1},\, y_{j+1})}$ is over pairs $(x_{j+1},\, y_{j+1})$ of distinct children of z, whereas $\sum_{(x,\, y)}$ is over pairs $(x,\, y)$ with $|x| = |y| = n$ such that[10] $x \ge x_{j+1}$ and $y \ge y_{j+1}$.

We apply the Markov property at generation $j+1$ and use the many-to-one formula to deal with the (conditional) expectation of $\sum_{(x,\, y)}$. The (conditional) expectation of $\sum_{(x_{j+1},\, y_{j+1})}$ is taken care of by an appropriate assumption of integrability for the point process $\mathscr{L} := (\xi_1, \cdots, \xi_N)$. Finally, the expectation of $\sum_{|z|=j}$, for any j, is treated by another application of the many-to-one formula. After some tedious computations, we arrive at:

$$\mathbf{E}(Y_n^2) \le C\, [\mathbf{E}(Y_n)]^2 ,$$

for some constant $C > 1$ and all sufficiently large n (say $n \ge n_0$). So

$$\mathbf{P}\Big(\min_{|x|=n} V(x) \le (\frac{3}{2} + \varepsilon) \log n \Big) \ge \frac{1}{C}, \qquad \forall n \ge n_0 .$$

Our goal is to say that $C > 1$ can be chosen as close to 1 as possible. This cannot be achieved by making the computation as precise as possible, but can be easily done by means of the tree structure. Indeed, fix $\eta > 0$, and let $k = k(\eta)$ be sufficiently large such that $(1 - \frac{1}{C})^{N^k} < \eta$.[11] Then

$$\begin{aligned} & \mathbf{P}\Big\{ \min_{|x|=n+k} V(x) > \max_{|y|=k} V(y) + (\frac{3}{2} + \varepsilon) \log n \Big\} \\ \le \quad & \Big[\mathbf{P}\Big\{ \min_{|x|=n} V(x) > (\frac{3}{2} + \varepsilon) \log n \Big\} \Big]^{N^k} , \end{aligned}$$

[10] By $x \ge y$, we mean either $x = y$, or y is an ancestor of x.

[11] By doing so, we assume implicitly that N is not random. In the general case (i.e., when it is random), we can apply a similar argument by taking a random k.

which is bounded by $(1-\frac{1}{C})^{N^k} < \eta$. This yields the desired upper bound in Theorem 7.1.□

The reason for which we have not given full details of the proof of Theorem 7.1 is that the theorem is weak: much more is true.

Theorem 7.2. (Aïdékon [2]). *Under Assumption* $(*)$ *and suitable integrability condition, if the distribution of* $\mathscr{L}$ *is non lattice,*[12] *then*

$$\min_{|x|=n} V(x) - \frac{3}{2}\log n$$

converges weakly to a Gumbel distribution shifted at the random position $\log(c\, D_\infty)$*, where* $c \in (0,\, \infty)$ *is a constant, and* $D_\infty > 0$ *is the almost sure limit of the derivative martingale.*

This deep result, like the corresponding result of Lalley and Sellke [80] for branching Brownian motion, shows the important role played by the derivative martingale in the asymptotics of the minimal position. See also the recent work of Bramson, Ding, and Zeitouni [38].

We now look at the sample path of the branching random walk leading to the minimal position[13] at time n. Intuitively, it would behave like a Brownian motion on $[0,\, n]$, starting at 0 and ending around $\frac{3}{2}\log n$, and staying above the line $i \mapsto \frac{\frac{3}{2}\log n}{n} i$ for $0 \le i \le n$. If we normalise this sample path with the same scaling as Brownian motion, then we would expect it to behave asymptotically like a normalised Brownian excursion. This is rigorously proved by Chen [46].

More precisely, let $|x^*_{n,n}| = n$ be such that $V(x^*_{n,n}) = \min_{|x|=n} V(x)$, and for $0 \le i \le n$, let $x^*_{n,i}$ be the ancestor of $x^*_{n,n}$ in the i-th generation. Let $\sigma^2 := \mathbf{E}(\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)})$.

Theorem 7.3. (Chen [46]). *Under Assumption* $(*)$ *and suitable integrability condition,*

$$\Big(\frac{V(x^*_{n,\lfloor nt\rfloor})}{(\sigma^2\, n)^{1/2}},\, t \in [0,\, t]\Big)$$

converges weakly in $C([0,\, 1],\, \mathbb{R})$ *to the normalised Brownian excursion.*[14]

---

[12] That is, no lattice supports $\mathscr{L}$.

[13] If there are several minima, one can choose any one at random according to the uniform distribution.

[14] A normalised Brownian excursion can be formally defined as a standard Brownian bridge conditioned to be non-negative. Rigorously, if $(B(t),\, t \ge 0)$ is a standard Brownian motion, writing $\mathfrak{G} := \sup\{t \le 1 : B(t) = 0\}$ and $\mathfrak{D} := \inf\{t \ge 1 : B(t) = 0\}$, then $(\frac{|B(\mathfrak{G}+(\mathfrak{D}-\mathfrak{G})t)|}{(\mathfrak{D}-\mathfrak{G})^{1/2}},\, t \in [0,\, 1])$ is a normalised Brownian excursion.

For any vertex x with $|x| \geq 1$, let us write

$$\overline{V}(x) := \max_{1 \leq i \leq |x|} V(x_i)\,, \tag{7.2}$$

which stands for the maximum value of the branching random walk along the path connecting the root and x. How small can $\overline{V}(x)$ when $|x| \to \infty$? If we take x to be the (or a) vertex on which the branching random walk reaches the minimum value at generation n, then we have seen in the previous paragraph that $\overline{V}(x)$ has the order of magnitude $n^{1/2}$. Can we do better?

The answer is yes.

Theorem 7.4. (Fang and Zeitouni [57]). *Under Assumption* $(*)$ *and suitable integrability condition,*

$$\lim_{n\to\infty} \frac{1}{n^{1/3}} \min_{|x|=n} \overline{V}(x) = \Big(\frac{3\pi^2\sigma^2}{2}\Big)^{1/3}, \qquad \text{a.s.}$$

Theorem 7.4, which will be useful in Chapter III, can be proved by means of the many-to-one formula.

There has been an important number of recent results on extreme values in the branching random walks. See for example Arguin, Bovier and Kistler [12], [13], [14] and [15], Roberts [109], Aïdékon et al. [6] for branching Brownian motion; Addario-Berry and Reed [1], Hu and Shi [68], Madaule [88] for branching random walks, together with the references therein. For a "spatial" version of convergence of the extremal process, see Bovier and Hartung [34]. For extensions to models with a time-inhomogeneous branching mechanism, see Fang and Zeitouni [58]–[59], Maillard and Zeitouni [93], Mallein [94]–[95], Bovier and Hartung [32]–[33].

Chapter II

Branching random walks with selection

We study two models of branching random walks with selection, both proposed and studied by Derrida and his coauthors. In the first model, an absorbing barrier is present with a slope that is slightly greater than the asymptotic speed of the minimal position in the branching random walk without selection. We study the asymptotic behaviour of the survival probability when the difference between the slope of the absorbing barrier and the speed of the minimal position tends to 0. In the second model, the number of individuals in each generation is fixed to be $N \geq 1$; only the N individuals with the smallest spatial values in each generation survive. Let v_N denote the asymptotic speed of the system; we study the asymptotecs of v_N when N goes to infinity.

1. Branching random walks with an absorbing barrier

Branching processes were introduced by Galton and Watson in the study of survival probability for families in Great Britain. In the supercritical case of the Galton–Watson branching process, when the system survives, the number of individuals in the population grows exponentially fast, a phenomenon that is not quite realistic in biology. From this point of view, it sounds natural to impose a criterion of *selection*, according to which only some individuals in the population are allowed to survive, while others are eliminated from the system, as well as their descendants.

In this section, we consider branching random walks in the presence of an absorbing barrier: any individual lying above the barrier gets erased. Although the study of branching diffusions with absorption goes back to Sevast'yanov [110] and Watanabe [111], it is the work of Kesten [76] on branching Brownian motion with an absorbing barrier that is the most relevant to the topic in this chapter. For recent progress on and refined properties of branching Brownian motion, see Berestycki, Berestyki and Schweinsberg [22] and [23], Aïdékon and Harris [7]. For the corresponding study on the one-sided F-KPP equation, see Harris, Harris and Kyprianou [66].

Let $(V(x))$ denote a branching random walk. Throughout the section, we work under Assumption $(*)$:

$$(*) \qquad \mathbf{E}\Big(\sum_{|x|=1} \mathrm{e}^{-V(x)}\Big) = 1, \qquad \mathbf{E}\Big(\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}\Big) = 0,$$

We recall that this implies (see (7.1) in Chapter I) that $\frac{1}{n}\min_{|x|=n} V(x) \to 0$ a.s.

Let $\varrho(\varepsilon)$ denote the survival probability of the system with an absorbing barrier of slope ε, that kills all individuals whose position is above or on the barrier. According to Biggins, Lubachevsky, Shwartz and Weiss [28], $\varrho(\varepsilon) > 0$ if and only if $\varepsilon > 0$. What can we say about $\varrho(\varepsilon)$ when $\varepsilon \downarrow 0$? This was a question raised by Pemantle [106].

Theorem 1.1. *Under Assumption $(*)$ and suitable integrability condition,*

$$\varrho(\varepsilon) = \exp\Big(- (1+o(1)) \frac{\pi\sigma}{(2\varepsilon)^{1/2}} \Big), \qquad \varepsilon \downarrow 0,$$

where $\sigma^2 := \mathbf{E}(\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)})$.

The proof of Theorem 1.1 relies on a second-moment argument by applying the many-to-one formula (Theorem 5.1 in Chapter I). For more details, see [62], and also [20] for a new proof, and some additional precision on the $o(1)$ expression. For branching Brownian motion, (much) more is known, see Berestycki, Berestyki and Schweinsberg [22], Aïdékon and Harris [7].

Theorem 1.1 also plays a crucial role in the study of branching random walks with competition, described in the next section.

2. Branching random walks with competition

Starting from 1990s, physicists have been interested in the slowdown phenomenon in the wave propagation of the F-KPP equation (Breuer, Huber and Petruccione [39]). Instead of the standard F-KPP equation[1]

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u(1-u)\,,$$

with initial condition $u(0,\,x) = \mathbf{1}_{\{x<0\}}$. Brunet and Derrida [40] introduced the cut-off version of the F-KPP equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u(1-u)\,\mathbf{1}_{\{u\ge \frac{1}{N}\}}\,,$$

and discovered that the solution to the equation with cut-off has a wave speed that is slower than the standard speed by a difference of order $(\log N)^{-2}$ when N is large.[2]

Later on, Brunet and Derrida [41] introduced a related F-KPP equation with white noise:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u(1-u) + \Big(\frac{u(1-u)}{N}\Big)^{1/2}\dot{W}\,,$$

where $\dot{W}$ is the standard space-time white noise. Once again, they found that the solution to the noised F-KPP equation has a wave speed that is delayed, compared to the standard speed, by a quantity of order $(\log N)^{-2}$ when N is large. This has been mathematically proved by Mueller, Mytnik and Quastel [101] and [102].

On the other hand, the following so-called N-BRW was introduced by Brunet, Derrida, Mueller and Munier ([42], [43] and [44]): in the branching random walk $(V(x))$, at each generation, only the N individuals having the smallest spatial values survive. The positions of the individuals in the resulting N-BRW are denoted by $(V^N(x))$. Since N is fixed, it is not hard to check that[3]

$$v_N := \lim_{n\to\infty}\frac{1}{n}\max_{|x|=n} V^N(x) = \lim_{n\to\infty}\frac{1}{n}\min_{|x|=n} V^N(x)\,,$$

exists a.s., and is deterministic. Several predictions are made by these authors (see [43] in particular), for example, concerning the velocity v_N:

$$v_N = \frac{\pi^2\sigma^2}{2(\log N)^2}\Big(1 - \frac{(6+o(1))\log\log N}{\log N}\Big)\,, \qquad N\to\infty\,, \tag{2.1}$$

[1] We have replaced u by $1-u$ (thus considering the tail distribution, instead of the distribution function, of the maximum of branching Brownian motion) in the F-KPP equation (1.1) of Chapter I.

[2] The notation is unfortunate, because N in this chapter has nothing to do with the random variable N representing the number of children for individuals in the system.

[3] In order to avoid trivial discussions, we assume that there are no leaves in the branching random walks, i.e., with probability one, every particle produces at least one child.

where, as before, $\sigma^2 := \mathbf{E}(\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)})$. [Of course, what is really interesting in the conjectured precision $\frac{(6+o(1)) \log\log N}{\log N}$ on the right-hand side is the universality of the main term.] All these predictions remain open, including a very interesting one concerning the genealogy of the particles in a suitable scale that would converge to the Bolthausen–Sznitman coalescent, though there is strong evidence that they are true in view of the recent progress made by Berestycki, Berestycki and Schweinsberg [23].

However, the following has been remarkably proved by Bérard and Gouéré [19] by means of a rigorous argument:

Theorem 2.1. *Under Assumption* $(*)$ *and suitable integrability condition,*[4]

$$v_N \sim \frac{\pi^2 \sigma^2}{2(\log N)^2}, \qquad N \to \infty .$$

where $\sigma^2 := \mathbf{E}(\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)})$.

Other rigorous results concerning the N-BRW (or the analogue for branching Brownian motion) have been obtained by Durrett and Remenik [56], Maillard [92], Bérard and Maillard [21], Mallein [96]–[97].

The proof of Theorem 2.1 is technical, requiring several delicate couplings between the N-BRW and the usual branching random walk in an appropriate scale. We describe a heuristic argument to see why v^N should behave asymptotically like $\frac{\pi^2\sigma^2}{2(\log N)^2}$.

The basic idea is that the following two properties are alike:

(a) A branching random walk, with an absorbing barrier of slope ε and starting with N particles at the origin, survives;

(b) An N-BRW moves at speed $\le \varepsilon$.

In (a), the survival probability is $1-(1-\varrho(\varepsilon))^N$, which suggests that v_N would behave like $\varepsilon = \varepsilon(N)$ where ε is defined by

$$\varrho(\varepsilon) \approx \frac{1}{N} .$$

Solving the equation by means of Theorem 1.1, we obtain:

$$\varepsilon \sim \frac{\pi^2\sigma^2}{2(\log N)^2} ,$$

which gives Theorem 2.1.

[4] Notation: By $a_N \sim b_N$, $N \to \infty$, we mean $\lim_{N\to\infty} \frac{a_N}{b_N} = 1$.

Bérard and Gouéré [19] have succeeded in making the heuristic argument rigorous. Unfortunately, it is believed that the heuristic will fail to lead to what is conjectured in (2.1). In other words, deeper understanding of the N-BRW will be required for a proof of (2.1).

Chapter III

Biased random walks on trees

So far, we have studied branching Brownian motion and branching random walks more or less indifferently. This chapter is devoted to an application of branching random walks, namely, randomly biased random walks on trees, and it concerns branching random walks only: no version for branching Brownian motion is involved. Randomly biased random walks on trees have been introduced by Lyons and Pemantle [85], extending the model of deterministically biased random walks on trees studied in depth by Lyons [82]–[83].

1. A simple example

Before introducing the general model, let us start with a simple example.

Example 1.1. Consider a rooted regular binary tree, and add a parent $\overleftarrow{\varnothing}$ to the root $\varnothing$.[1] The resulting tree is a planted tree. We give a random colour to each of the vertices of the tree; a vertex is coloured red with probability p_{red}, and blue with probability p_{blue}, with $p_{\mathrm{red}} > 0$ and $p_{\mathrm{blue}} > 0$ such that $p_{\mathrm{red}} + p_{\mathrm{blue}} = 1$.

A random walker performs a discrete-time random walk on the tree, starting from the root $\varnothing$. At each step, the walk stays at a vertex for a unit of time, then moves to one of the neighbours (either the parent, or one of the two children). The transition probabilities are $a^{\uparrow}_{\mathrm{red}}$ (moving to the parent), $a^{(1)}_{\mathrm{red}}$ and $a^{(2)}_{\mathrm{red}}$ (moving to either of the children) if the site where the walker stays currently is red, or $a^{\uparrow}_{\mathrm{blue}}$, $a^{(1)}_{\mathrm{blue}}$ and $a^{(2)}_{\mathrm{blue}}$ if the site is blue. As such, $a^{\uparrow}_{\mathrm{red}}$, $a^{(1)}_{\mathrm{red}}$, $a^{(2)}_{\mathrm{red}}$, $a^{\uparrow}_{\mathrm{blue}}$, $a^{(1)}_{\mathrm{blue}}$ and $a^{(2)}_{\mathrm{blue}}$ are positive numbers such that

$$a^{\uparrow}_{\mathrm{red}} + a^{(1)}_{\mathrm{red}} + a^{(2)}_{\mathrm{red}} = 1 = a^{\uparrow}_{\mathrm{blue}} + a^{(1)}_{\mathrm{blue}} + a^{(2)}_{\mathrm{blue}} .$$

[1]The root $\varnothing$ is a vertex of the tree, but $\overleftarrow{\varnothing}$ is not considered as a vertex of the tree.

We assume that $\overleftarrow{\varnothing}$ is reflecting: each time the walk is at $\overleftarrow{\varnothing}$, it automatically comes back to $\varnothing$ in the next step. □

The usual questions arise naturally: Is the random walker recurrent or transient? What can be said about its position after n steps? What is the maximal displacement in the first n steps?

2. The maximal displacement

Let $\mathbb{T}$ be a planted regular N-ary tree. For any $x \in \mathbb{T}$, let $\overleftarrow{x}$ denote the parent of x (recalling that $\overleftarrow{\varnothing}$ is not considered as a vertex of $\mathbb{T}$), and $x^{(1)}, \cdots, x^{(N)}$ the children of x. Let $(\omega(x), x \in \mathbb{T})$ be a family of i.i.d. random vectors, with $\omega(x) = (\omega(x, y), \ y \in \{\overleftarrow{x}\} \cup \{x^{(1)}, \cdots, x^{(N)}\})$. We assume that $\omega(\varnothing, y) > 0$ $\mathbf{P}$-a.s., for $y \in \{\overleftarrow{\varnothing}\} \cup \{\varnothing^{(1)}, \cdots, \varnothing^{(N)}\}$, and that $\omega(\varnothing, \overleftarrow{\varnothing}) + \sum_{i=1}^N \omega(\varnothing, \varnothing^{(i)}) = 1$. In Example 1.1, $\omega(\varnothing)$ (or any $\omega(x)$, for $x \in \mathbb{T}$) takes two possible values, with probability p_{red} and p_{blue}, respectively.

For each given ω (which, in Example 1.1, means that all the colours are known), let $(X_n, n \geq 0)$ be a Markov chain with $X_0 = \varnothing$ with transition probabilities

$$\begin{aligned} P_\omega(X_{n+1} = \overleftarrow{x} \,|\, X_n = x) &= \omega(x, \overleftarrow{x}), \\ P_\omega(X_{n+1} = x^{(i)} \,|\, X_n = x) &= \omega(x, x^{(i)}), \quad 1 \leq i \leq N, \end{aligned}$$

and $P_\omega(X_{n+1} = y \,|\, X_n = x) = 0$ if $y \notin \{\overleftarrow{x}\} \cup \{x^{(1)}, \cdots, x^{(N)}\}$.

Let us first do some elementary computations. Assume that the walk is recurrent. For any vertex $x \in \mathbb{T}$, we define

$$T_x := \inf\{i \geq 0 : \ X_i = x\},$$

the first hitting time at x, and also

$$T_\varnothing^+ := \inf\{i \geq 1 : \ X_i = \varnothing\},$$

the first *return* time to the root.

Let $x \in \mathbb{T}$ with $|x| = n \geq 1$. For any $0 \leq k \leq n$, write

$$a_k := P_\omega\{T_x < T_\varnothing \,|\, X_0 = x_k\}.$$

Then $a_0 = 0$, $a_n = 1$, and for $1 \leq k < n$,

$$a_k = \frac{\omega(x_k, x_{k+1})}{\omega(x_k, x_{k+1}) + \omega(x_k, x_{k-1})} a_{k+1} + \frac{\omega(x_k, x_{k-1})}{\omega(x_k, x_{k+1}) + \omega(x_k, x_{k-1})} a_{k-1},$$

which means

$$a_{k+1} - a_k = \frac{\omega(x_k,\, x_{k-1})}{\omega(x_k,\, x_{k+1})}\,(a_k - a_{k-1})\,.$$

Iterating the procedure, we obtain, for $1 \le k < n$,

$$a_{k+1} - a_k = \Big(\prod_{j=1}^{k} \frac{\omega(x_j,\, x_{j-1})}{\omega(x_j,\, x_{j+1})}\Big)(a_1 - a_0) = \Big(\prod_{j=1}^{k} \frac{\omega(x_j,\, x_{j-1})}{\omega(x_j,\, x_{j+1})}\Big) a_1\,,$$

due to the fact that $a_0 = 0$. Summing on both sides over $k \in [0,\, n-1] \cap \mathbb{Z}$: on the left-hand side, we have $\sum_{k=0}^{n-1}(a_{k+1} - a_k) = a_n - a_0 = 1$, so

$$1 = a_1 \sum_{k=0}^{n-1} \prod_{j=1}^{k} \frac{\omega(x_j,\, x_{j-1})}{\omega(x_j,\, x_{j+1})}\,,$$

with the notation $\prod_{j=1}^{0} := 1$. This yields

$$P_\omega\{T_x < T_\varnothing \,|\, X_0 = x_1\} = a_1 = \frac{1}{\sum_{k=0}^{n-1} \prod_{j=1}^{k} \frac{\omega(x_j, x_{j-1})}{\omega(x_j, x_{j+1})}}\,.$$

If the walk starts at $X_0 = \varnothing$, we obtain:

$$P_\omega\{T_x < T_\varnothing^+\} = \omega(\varnothing,\, x_1)\, P_\omega\{T_x < T_\varnothing \,|\, X_0 = x_1\} = \frac{\omega(\varnothing,\, x_1)}{\sum_{k=0}^{n-1} \prod_{j=1}^{k} \frac{\omega(x_j, x_{j-1})}{\omega(x_j, x_{j+1})}}\,.$$

Writing

$$V(y) = \sum_{i=0}^{|y|-1} \log \frac{\omega(y_i,\, y_{i-1})}{\omega(y_i,\, y_{i+1})}, \qquad y \in \mathbb{T}\backslash\{\varnothing\}\,, \qquad (y_{-1} := \overleftarrow{\varnothing})$$

we immediately see that $(V(y),\, y \in \mathbb{T}\backslash\{\varnothing\})$ is a branching random walk in the sense of the previous chapters!

By definition, $\prod_{j=1}^{k} \frac{\omega(x_j, x_{j-1})}{\omega(x_j, x_{j+1})} = \mathrm{e}^{V(x_{k+1}) - V(x_1)}$, so that

$$P_\omega\{T_x < T_\varnothing^+\} = \frac{\omega(\varnothing,\, x_1)}{\sum_{k=0}^{n-1} \mathrm{e}^{V(x_{k+1}) - V(x_1)}} = \frac{\omega(\varnothing,\, x_1)\, \mathrm{e}^{V(x_1)}}{\sum_{i=1}^{n} \mathrm{e}^{V(x_i)}} = \frac{\omega(\varnothing,\, \overleftarrow{\varnothing})}{\sum_{i=1}^{n} \mathrm{e}^{V(x_i)}}\,.$$

This simple formula tells us that V plays the role of potential: the higher the potential value is on the path $\{x_1,\, \cdots,\, x_n\}$, the harder it is for the walk to reach x.

We assume from now on

$$(*) \qquad \mathbf{E}\Big(\sum_{|x|=1} \mathrm{e}^{-V(x)}\Big) = 1, \qquad \mathbf{E}\Big(\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}\Big) = 0\,.$$

A general theorem by Lyons and Pemantle [85] tells us that ($*$) ensures the recurrence of the walk (for a proof using Mandelbrot's multiplicative cascades, see Menshikov and Petritis [100]). Viewing ω from the point of view of V, it is now clear that instead of the N-ary regular tree, we can define the walk on a more general supercritical Galton–Watson tree.[2] When V is associated with $\mathscr{L} = (\xi_1, \cdots, \xi_N) = (\log\lambda, \cdots, \log\lambda)$, where $\lambda > 0$ is a fixed parameter, we recover Lyons's λ-biased random walk on Galton–Watson trees ([82], [83]), who proved that the λ-biased random walk is recurrent if and only if $\lambda \geq \mathbf{E}(N)$. For the λ-biased random walk, it is known that $\frac{|X_n|}{n}$ converges a.s. to a constant, denoted by $v(\lambda)$. Lyons, Pemantle and Peres [87] conjectured that $\lambda \mapsto v(\lambda)$ is non-increasing on $[0, \mathbf{E}(N))$. This monotonicity has been established by Ben Arous, Hu, Olla and Zeitouni [18] for λ in the neighbourhood of $\mathbf{E}(N)$, by Ben Arous, Fribergh and Sidoravicius [17] for λ in the neighbourhood of 0, and by Aïdékon [4] for $\lambda \in [0, 1]$.

We often work under the "annealed measure"

$$\mathbb{P}(\,\cdot\,) := \mathbf{E}[P_\omega(\,\cdot\,)]\,,$$

by taking average over the "environment" ω.

Writing, for $n \geq 1$,

$$H_n := \inf\{i :\, |X_i| = n\}\,,$$

the first time the walk hits the n-th generation. For any $x \in \mathbb{T}$ with $|x| = n$, we have $H_n \leq T_x$, so $P_\omega\{H_n < T_\varnothing^+\} \geq P_\omega\{T_x < T_\varnothing^+\}$. Taking maximum over all vertices in the n-th generation, this leads to:

$$P_\omega\{H_n < T_\varnothing^+\} \geq \max_{|x|=n} P_\omega\{T_x < T_\varnothing^+\} = \max_{|x|=n} \frac{\omega(\varnothing, \overleftarrow{\varnothing})}{\sum_{i=1}^n \mathrm{e}^{V(x_i)}} \geq \frac{\omega(\varnothing, \overleftarrow{\varnothing})}{n}\, \mathrm{e}^{-\min_{|x|=n} \overline{V}(x)}\,,$$

where $\overline{V}(x) := \max_{1\leq i\leq |x|} V(x_i)$, as in (7.2) of Chapter I. Applying Theorem 7.4 of Chapter I, we obtain:

$$\liminf_{n\to\infty} \frac{1}{n^{1/3}} \log P_\omega\{H_n < T_\varnothing^+\} \geq -\Big(\frac{3\pi^2\sigma^2}{2}\Big)^{1/3} =: -\theta\,, \qquad \mathbf{P}\text{-a.s.},$$

where $\sigma^2 := \mathbf{E}(\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)})$ as before.

Let, for any $k \geq 1$,

$$L_k := \sum_{i=1}^k \mathbf{1}_{\{X_i = \varnothing\}}\,,$$

[2] Actually, the genealogical tree is even more general than the supercritical Galton–Watson tree, because we do not exclude the situation that a particle produces infinitely many children with positive probability.

which stands for the number of visits[3] at the root $\varnothing$ in the first k steps. Then for any $j \geq 1$ and all $\varepsilon > 0$,

$$P_\omega\{L_{H_n} > j\} = [P_\omega\{H_n > T_\varnothing^+\}]^j \leq \Big[1 - \mathrm{e}^{-(1+\varepsilon)\theta n^{1/3}}\Big]^j \leq \exp\Big(- j\,\mathrm{e}^{-(1+\varepsilon)\theta n^{1/3}}\Big)\,,$$

P-almost surely for all sufficiently large n (say $n \geq n_0(\omega)$; $n_0(\omega)$ does not depend on j). Taking $j := \lfloor \mathrm{e}^{(1+2\varepsilon)\theta n^{1/3}} \rfloor$, we see that

$$\sum_n P_\omega\{L_{H_n} \geq \lfloor \mathrm{e}^{(1+2\varepsilon)\theta n^{1/3}} \rfloor\} < \infty\,, \qquad \mathbf{P}\text{-a.s.}$$

This implies, by the Borel–Cantelli lemma, that

$$\limsup_{n\to\infty} \frac{\log L_{H_n}}{n^{1/3}} \leq \theta\,, \qquad \mathbb{P}\text{-a.s.}$$

It is known, and not hard, to check that

$$\lim_{k\to\infty} \frac{\log L_k}{\log k} = 1\,, \qquad \mathbb{P}\text{-a.s.},$$

which yields that

$$\limsup_{n\to\infty} \frac{\log H_n}{n^{1/3}} \leq \theta\,, \qquad \mathbb{P}\text{-a.s.}$$

Note that for all n and j, $\{H_n \leq k\} = \{\max_{1\leq i\leq k} |X_i| \geq n\}$. This implies that

$$\liminf_{n\to\infty} \frac{1}{(\log n)^3} \max_{1\leq i\leq n} |X_i| \geq \frac{1}{\theta^3} = \frac{2}{3\pi^2\sigma^2}\,, \qquad \mathbb{P}\text{-a.s.}$$

It turns out that $(\log n)^3$ is the correct order of magnitude for $\max_{1\leq i\leq n} |X_i|$, and the constant $\frac{2}{3\pi^2\sigma^2}$ is not exactly optimal:

Theorem 2.1. (Faraud et al. [60]) *Under Assumption* $(*)$ *and suitable integrability condition,*

$$\lim_{n\to\infty} \frac{1}{(\log n)^3} \max_{1\leq i\leq n} |X_i| = \frac{8}{3\pi^2\sigma^2}\,, \qquad \mathbb{P}\text{-a.s.}$$

Theorem 2.1 tells us that the walk (X_i) is very slow. In the next section, we study the terminal position X_n, and see that, usually, the walk is even slower.

[3] Often referred to as the (discrete) local time.

3. Weak convergence

Under Assumption $(*)$, we have seen in the previous section that the maximal displacement of the walk is of order of magnitude $(\log n)^3$. What about the terminal position X_n?

The study of the asymptotics of $|X_n|$ is more delicate. Our answer says that $|X_n|$ is usually much smaller than $\max_{1\le i\le n} |X_i|$. Let $(\mathfrak{m}(s),\, s\in [0,\,1])$ denote a standard Brownian meander, and $\overline{\mathfrak{m}}(s) := \sup_{u\in [0,\, s]} \mathfrak{m}(u)$. Recall that the standard Brownian meander can be realized as follows: $\mathfrak{m}(s) := \frac{|B(\mathfrak{g}+s(1-\mathfrak{g}))|}{(1-\mathfrak{g})^{1/2}}$, $s\in [0,\,1]$, where $(B(t),\, t\in [0,\,1])$ is a standard Brownian motion, with $\mathfrak{g} := \sup\{t\le 1 :\, B(t)=0\}$.

Theorem 3.1. *Under Assumption $(*)$ and suitable integrability condition, for all $u>0$,*

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{\sigma^2\, |X_n|}{(\log n)^2} \le u\Big) = \int_0^u \frac{1}{(2\pi r)^{1/2}}\, \mathbf{P}\Big(\eta \le \frac{1}{r^{1/2}}\Big)\, \mathrm{d}r\, ,$$

where $\sigma^2 := \mathbf{E}(\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)})$ as before, and $\eta := \sup_{s\in [0,\,1]} [\,\overline{\mathfrak{m}}(s) - \mathfrak{m}(s)]$.

We mention that $\int_0^\infty \frac{1}{(2\pi r)^{1/2}}\, \mathbf{P}(\eta \le \frac{1}{r^{1/2}})\, \mathrm{d}r = 1$ because $\mathbf{E}(\frac{1}{\eta}) = (\frac{\pi}{2})^{1/2}$, see [70]. The proof of Theorem 3.1 can be found in [69].

Chapter IV

The spinal decomposition

In Chapter I, we have introduced the fundamental tool: the many-to-one formula (Theorem 5.1). The proof is easy, but an important question remains: what does the new one-dimensional random walk (S_i) represent? We are going to answer this question with the spinal decomposition theorem. Let us start with the simple case of the Galton–Watson process.

1. Galton–Watson trees

Let $(p_i,\, i \ge 0)$ be a probability on $\{0, 1, 2, \cdots\}$, i.e., $p_i \ge 0$ for all $i \ge 0$, such that $\sum_{i=0}^{\infty} p_i = 1$. To avoid trivial discussions, we exclude the case $p_0 + p_1 = 1$.

Let $(\Omega, \mathscr{F})$ be the canonical space of rooted trees (so each $\omega \in \Omega$ is a rooted tree), and let $\mathbb{T} : \Omega \to \Omega$ be the identity mapping. There exists a probability measure $\mathbf{P}$ on $(\Omega, \mathscr{F})$ such that under $\mathbf{P}$, $\mathbb{T}$ is a Galton–Watson tree with reproduction law (p_i): each vertex has i children with probability p_i (for any $i \ge 0$), and the reproductions are mutually independent in a same generation, and also independent of everything up to that generation.

We do not describe the formalism of the canonical representation here, because it is not required for our purposes (except for the fact that $\mathscr{F} = \vee_{n=0}^{\infty} \mathscr{F}_n$,[1] where $\mathscr{F}_n$ is the sigma-field generated by the individuals in generations $0 \le i \le n$). All we need to know is its existence. For full details, see Neveu [103].

For $n \ge 0$, let Z_n be the number of individuals in the n-th generation. We also write

$$m := \sum_{i=0}^{\infty} i\, p_i\,,$$

[1] That is, $\mathscr{F}$ is the smallest field generated by all $\mathscr{F}_n$, $n \ge 0$.

and we assume that $m < \infty$. Let

$$M_n := \frac{Z_n}{m^n}, \qquad n \ge 0.$$

Then $(M_n,\, n \ge 0)$ is a non-negative martingale with respect to the filtration $(\mathscr{F}_n)$, and $\mathbf{E}(M_n) = 1$, $\forall n \ge 0$.

By Kolmogorov's extension theorem, there exists a probability measure $\mathbf{Q}$ on $(\Omega,\, \mathscr{F})$ such that for any $n \ge 0$,

$$\mathbf{Q}_{|\mathscr{F}_n} = M_n \bullet \mathbf{P}_{|\mathscr{F}_n},$$

where $\mathbf{P}_{|\mathscr{F}_n}$ and $\mathbf{Q}_{|\mathscr{F}_n}$ denote the restrictions of $\mathbf{P}$ and $\mathbf{Q}$ on $\mathscr{F}_n$, respectively. For any n,

$$\mathbf{Q}(Z_n > 0) = \mathbf{E}(\mathbf{1}_{\{Z_n > 0\}} M_n) = \mathbf{E}(M_n) = 1.$$

As such, $\mathbf{Q}(Z_n > 0,\ \forall n) = 1$, which means $\mathbf{Q}$-almost surely non-extinction of $\mathbb{T}$. The tree $\mathbb{T}$ under $\mathbf{Q}$ is called a **size-biased Galton–Watson tree**. Let us give a description of its paths.

Let N be the number of children of the root $\varnothing$. If $N \ge 1$, then there are N individuals in the first generation. We write $\mathbb{T}_1$, $\mathbb{T}_2$, $\cdots$, $\mathbb{T}_N$ for the N subtrees rooted at each of the N individuals in the first generation.

Lemma 1.1. *Let $k \ge 1$, and let A_1, A_2, $\cdots$, A_k be elements of $\mathscr{F}$. We have*

$$\begin{aligned} &\mathbf{Q}(N = k,\ \mathbb{T}_1 \in A_1, \cdots, \mathbb{T}_k \in A_k) \\ &= \frac{kp_k}{m}\, \frac{1}{k} \sum_{i=1}^k \mathbf{P}(A_1) \cdots \mathbf{P}(A_{i-1}) \mathbf{Q}(A_i) \mathbf{P}(A_{i+1}) \cdots \mathbf{P}(A_k). \end{aligned} \tag{1.1}$$

Proof. By the monotone class theorem, we may assume, without loss of generality, that A_1, A_2, $\cdots$, A_k are elements of $\mathscr{F}_n$, for some n. Then we have

$$\mathbf{Q}(N = k,\ \mathbb{T}_1 \in A_1,\ \cdots,\ \mathbb{T}_k \in A_k) = \mathbf{E}\Big(\frac{Z_{n+1}}{m^{n+1}}\, \mathbf{1}_{\{N=k,\, \mathbb{T}_1 \in A_1,\, \cdots,\, \mathbb{T}_k \in A_k\}}\Big).$$

On the event $\{N = k\}$, we can write $Z_{n+1} = \sum_{i=1}^k Z_n^{(i)}$, where $Z_n^{(i)}$ denotes the number of individuals in the n-th generation of the subtree rooted at the i-th individual in the first generation. Accordingly,

$$\mathbf{Q}(N = k,\ \mathbb{T}_1 \in A_1, \cdots, \mathbb{T}_k \in A_k) = \frac{1}{m^{n+1}} \mathbf{P}(N = k) \sum_{i=1}^k \mathbf{E}\Big\{ Z_n^{(i)}\, \mathbf{1}_{\{\mathbb{T}_1 \in A_1, \cdots, \mathbb{T}_k \in A_k\}} \,\Big|\, N = k \Big\}.$$

Since $\mathbf{P}(N=k)=p_k$, and

$$\mathbf{E}\{Z_n^{(i)}\,\mathbf{1}_{\{\mathbb{T}_1\in A_1,\cdots,\mathbb{T}_k\in A_k\}}\,|\,N=k\}=\mathbf{E}[Z_n\,\mathbf{1}_{\{\mathbb{T}\in A_i\}}]\prod_{j\neq i}\mathbf{P}(A_j)=m^n\,\mathbf{Q}(A_i)\prod_{j\neq i}\mathbf{P}(A_j),$$

proving the lemma. □

Equation (1.1) tells us the following fact about the size-biased Galton–Watson tree: The root has the biased distribution, i.e., having k children with probability $\frac{k\,p_k}{m}$; among the individuals in the first generation, one of them is chosen randomly (according to the uniform distribution) such that the subtree rooted at this vertex is a size-biased Galton–Watson tree, whereas the subtrees rooted at all other vertices in the first generation are independent copies of the usual Galton–Watson tree.

Iterating the procedure, we obtain a decomposition of the size-biased Galton–Watson tree with an (infinite) spine and with i.i.d. copies of the usual Galton–Watson tree: The root $\varnothing =: w_0$ has the biased distribution, i.e., having k children with probability $\frac{k\,p_k}{m}$. Among the children of the root, one of them is chosen randomly (according to the uniform distribution) as the element of the spine in the first generation (denoted by w_1). We attach subtrees rooted at all other children; these subtrees are independent copies of the usual Galton–Watson tree. The vertex w_1 has the biased distribution. Among the children of w_1, we choose at random one of them as the element of the spine in the second generation (denoted by w_2). Independent copies of the usual Galton–Watson tree are attached as subtrees rooted at all other children of w_1, whereas w_2 has the biased distribution. And so on. See Figure 4 below.

2. Branching random walks

Throughout this section, we assume that

$$\mathbf{E}\Big(\sum_{x:\,|x|=1}\mathrm{e}^{-V(x)}\Big)=1\,.$$

Consider the additive martingale

$$M_n:=\sum_{|x|=n}\mathrm{e}^{-V(x)}\,,\qquad n\ge 0\,.$$

Let $\mathbf{Q}$ be the probability measure on $(\Omega,\,\mathscr{F})$ such that for any $n\ge 0$,

$$\mathbf{Q}_{|\mathscr{F}_n}=M_n\bullet\mathbf{P}_{|\mathscr{F}_n}\,.$$

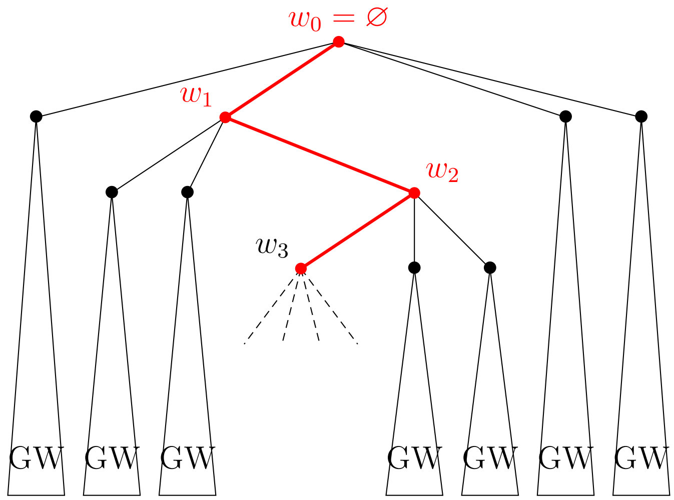

Figure 4. A size-biased Galton–Watson tree

As for the Galton–Watson tree, we see that $\mathbf{Q}(\sum_{|x|=1} 1 > 0) = 1$. The process $(V(x),\, x \in T)$ under $\mathbf{Q}$ is called a size-biased branching random walk.

The root $\varnothing =: w_0$ has the biased distribution, in the sense that $\mathbf{E}_{\mathbf{Q}}[F(V(x),\, |x| = 1)] = \mathbf{E}[F(V(x),\, |x| = 1)M_1]$. Among the children y of the root, one of them is chosen as w_1 with probability proportional to $\mathrm{e}^{-V(y)}$ (i.e., with probability $\frac{\mathrm{e}^{-V(y)}}{M_1}$). We attach subtrees rooted at all other children; these subtrees are independent copies of the usual branching random walk. The vertex w_1 has the biased distribution, shifted at position $a := V(w_1)$. Among the children y of w_1, one of them is chosen as w_2 with probability proportional to $\mathrm{e}^{-V(y)}$. We iterate the procedure. See Figure 5 below.

By the description, $V(w_n) - V(w_{n-1})$, for $n \geq 1$, are i.i.d. under $\mathbf{Q}$.[2] For any measurable function $h : \mathbb{R} \to \mathbb{R}_+$,

$$\mathbf{E}_{\mathbf{Q}}[h(V(w_1))] = \mathbf{E}_{\mathbf{Q}}\Big[\sum_{|x|=1} \mathbf{1}_{\{w_1 = x\}} h(V(x))\Big] = \mathbf{E}_{\mathbf{Q}}\Big[\mathbf{E}_{\mathbf{Q}}\Big(\sum_{|x|=1} \mathbf{1}_{\{w_1 = x\}} h(V(x)) \,|\, \mathscr{F}_1\Big)\Big].$$

We note that

$$\mathbf{E}_{\mathbf{Q}}\Big(\sum_{|x|=1} \mathbf{1}_{\{w_1 = x\}} h(V(x)) \,|\, \mathscr{F}_1\Big) = \sum_{|x|=1} h(V(x))\mathbf{Q}(w_1 = x \,|\, \mathscr{F}_1\Big) = \sum_{|x|=1} h(V(x)) \frac{\mathrm{e}^{-V(x)}}{M_1},$$

so that

$$\mathbf{E}_{\mathbf{Q}}[h(V(w_1))] = \mathbf{E}_{\mathbf{Q}}\Big[\sum_{|x|=1} h(V(x)) \frac{\mathrm{e}^{-V(x)}}{M_1}\Big] = \mathbf{E}\Big[\sum_{|x|=1} h(V(x))\mathrm{e}^{-V(x)}\Big],$$

[2]Notation: $V(\varnothing) := 0$.

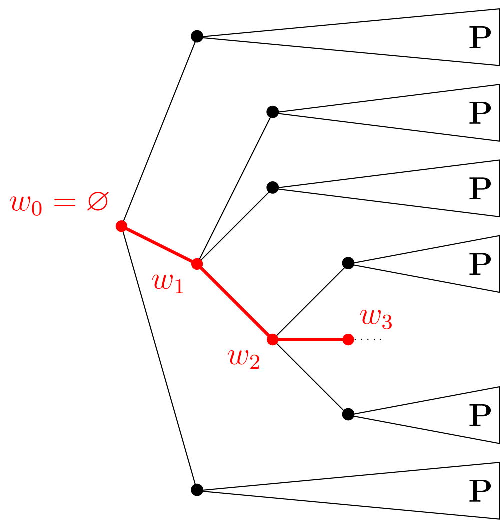

Figure 5. A size-biased branching random walk

which is $\mathbf{E}[h(S_1)]$ by definition of S_1 in (5.1) of Chapter I. Therefore, $(V(w_n),\, n \ge 1)$ under $\mathbf{Q}$ has the same distribution as $(S_n,\, n \ge 1)$ under $\mathbf{P}$: the associated one-dimensional random walk in the many-to-one formula is nothing else but the size-biased branching random walk along the spine.

Example 2.1. Under Assumption $(*)$, we have $M_n \to 0$ a.s. (see Theorem 6.1 of Chapter I).

In fact, writing $V(w_n) = \sum_{i=1}^n [V(w_i) - V(w_{i-1})]$, where $V(w_i) - V(w_{i-1})$, $i \ge 1$, are i.i.d. under $\mathbf{Q}$ with $\mathbf{E}_{\mathbf{Q}}[V(w_1)] = \mathbf{E}[S_1] = 0$, we have

$$\liminf_{n\to\infty} V(w_n) = -\infty, \qquad \mathbf{Q}\text{-a.s.}$$

Since $M_n = \sum_{|x|=n} \mathrm{e}^{-V(x)} \ge \mathrm{e}^{-V(w_n)}$, this implies that

$$\limsup_{n\to\infty} M_n = \infty, \qquad \mathbf{Q}\text{-a.s.}$$

According to Exercise 3.6 of Durrett ([54], p. 210),[3] this is equivalent to saying that $M_n \to 0$ $\mathbf{P}$-a.s. □

The idea of the spinal decomposition for branching random walks goes back at least to Kahane and Peyrière [75] and to Bingham and Doney [29]. It has appeared in various forms

[3]Let $(\mathscr{F}_n)$ be a filtration and let $\mathbf{P}$ et $\mathbf{Q}$ be probability measures on $\mathscr{F}_\infty := \vee_{n=0}^\infty \mathscr{F}_n$ such that $\mathbf{Q}_{|\mathscr{F}_n} = \eta_n \bullet \mathbf{P}_{|\mathscr{F}_n}$, $\forall n \ge 0$. Then $\limsup_{n\to\infty} \eta_n = \infty$ $\mathbf{Q}$-a.s., if and only if $\eta_n \to 0$ $\mathbf{P}$-a.s.

in the literature. The formalism we use in this chapter comes from Lyons, Pemantle and Peres [86] for Galton–Watson trees, and from Lyons [84] for branching random walks. For branching Brownian motion, see Chauvin and Rouault [45].

The spinal decomposition is the tree version of Girsanov's theorem, and is powerful in the study of the branching random walk. However, it is stated for the branching random walk under $\mathbf{Q}$, which is not equivalent to $\mathbf{P}$ on $\mathscr{F}$ (and not even on $\mathscr{F}_n$). So one needs to be careful when applying the spinal decomposition.

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