Our joint work with Miklós Csörgő

Endre Csáki
A. Rényi Institute of Mathematics
Hungarian Academy of Sciences
P.O.B. 127
H-1364 Budapest
Hungary
`csaki@renyi.hu`

Antónia Földes
Department of Mathematics
City University of New York
2800 Victory Blvd.
Staten Island, New York 10314
U.S.A.
`afoldes@gc.cuny.edu`

Zhan Shi
Laboratoire de Probabilités UMR 7599
Université Paris VI
4 Place Jussieu
F-75252 Paris Cedex 05
France
`zhan@proba.jussieu.fr`

Dedicated to Miklós Csörgő on the occasion of his 70-th birthday

Abstract. Topics in our joint work of twenty years are discussed. To name a few: asymptotic independence, strong approximation of additive functionals, iterated processes, path properties of the Cauchy principal value, Vervaat process, Ornstein-Uhlenbeck process.

1. Introduction

The work of Miklós Csörgő has had a great impact on modern probability and statistics. His books and papers are wells of informations for these fields'

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generations of young mathematicians. This survey attempts only to give a brief account of the papers, which he has co-authored with one or more of us during the 20 odd years that we have been his friends and collaborators. A number of these papers were written with Pál Révész. In the following, we summarize the contents of these papers which are focused on a few strongly related topics. The basic object of these investigations is the Wiener process or Brownian motion. The two names will be alternatively used in this paper for the same process. We call standard Wiener process (Brownian motion) a mean zero Gaussian process $\{W(t),\, t \geq 0\}$ with covariance $\mathbf{E}W(t_1)W(t_2) = \min(t_1, t_2)$. In certain cases we need a Wiener process on the whole real line. Let $\{W_1(t),\, t \geq 0\}$ and $\{W_2(t),\, t \geq 0\}$ be two independent standard Wiener processes. Then a standard Wiener process on $\mathbb{R}$ is defined as $W(t) = W_1(t)$ for $t \geq 0$ and $W(t) = W_2(-t)$ for $t < 0$.

2. Local time and additive functionals

2.1 The increments of the local time. At the beginning of the 1980-s we were fascinated with the Brownian local time. The asymptotic behaviour of the increments of the Wiener process was well understood, as Csörgő and Révész [36], [37] proved their incredible precise results in a couple of papers about how big and how small are these increments of the Wiener process. In our first joint paper our objective was to investigate the corresponding "how big" question for the increments of the local time. Before quoting their results, we introduce a pair of conditions which will be frequently used in the sequel.

Condition A:
$0 < a_t \leq t$ *is a nondecreasing function of* t *such that* t/a_t *is also nondecreasing.*

Condition B:

$$\lim_{t\to\infty} \frac{\log(t a_t^{-1})}{\log\log t} = +\infty.$$

Theorem A (Csörgő and Révész [36], [38]) *Under* Condition A *we have*

$$\begin{aligned} &\limsup_{t\to\infty} \beta_t a_t^{-1/2}(W(t+a_t) - W(t)) \\ &= \limsup_{t\to\infty} \beta_t a_t^{-1/2} \sup_{0\leq s\leq t-a_t} (W(s+a_t) - W(s)) \\ &= \limsup_{t\to\infty} \beta_t a_t^{-1/2} \sup_{0<s<a_t} (W(t+s) - W(t)) \quad = \quad 1 \qquad \text{a.s.,} \end{aligned}$$

where $\beta_t = (2(\log t a_t^{-1} + \log\log t))^{-1/2}$. *Supposing* Condition B *as well, we also have*

$$\lim_{t\to\infty} \beta_t a_t^{-1/2} \sup_{0\leq s\leq t-a_t} (W(s+a_t) - W(s)) = 1 \qquad \text{a.s.}$$

As it turned out, the increments of the local time behave very similarly, though a slightly different normalization is needed. We start with a quick definition of the local time process. For any Borel set A on the real line let

$$H(A,t) := \lambda\{s : s \leq t, W(s) \in A\}$$

be the occupation time of W, where λ is the Lebesgue measure. $H(A,t)$ is a random measure which is absolutely continuous with respect to λ with probability

1, its Radon-Nikodym derivative is called the local time of W, and will be denoted by $L(x,t)$, i.e.

$$H(A,t) = \int_A L(x,t)\,dx.$$

The joint continuity of $L(x,t)$ is a famous result of Trotter [79], who also investigated the modulus of continuity, separately for x and for t. The celebrated law of the iterated logarithm for the local time is due to Kesten [62]:

$$\limsup_{t\to\infty} \frac{L(0,t)}{(2t\log\log t)^{1/2}} = \limsup_{t\to\infty} \frac{\sup_{-\infty<x<\infty} L(x,t)}{(2t\log\log t)^{1/2}} = 1 \qquad \text{a.s.} \tag{2.1}$$

Let us denote $L(0,t)$ by $L(t)$. Our main result in [14] was the following

Theorem 2.1 *Under* Condition A *we have*

$$\limsup_{t\to\infty} \gamma_t Y(t) = \limsup_{t\to\infty} \gamma_t a_t^{-1/2}(L(t) - L(t-a_t)) = 1 \qquad \text{a.s.}, \tag{2.2}$$

where

$$Y(t) = Y(t,a_t) = a_t^{-1/2} \sup_{0<s<t-a_t} (L(s+a_t) - L(s))$$

and

$$\gamma_t = (\log t a_t^{-1} + 2\log\log t)^{-1/2}.$$

Assuming Condition B *as well, we also have*

$$\lim_{t\to\infty} \gamma_t Y(t) = 1 \qquad \text{a.s.}$$

2.2 Approximation by a Wiener sheet. Once we understood the asymptotic behaviour of the local time increments when $t \to \infty$, we turned our attention to the whole two-variate process $L(x,t) - L(0,t)$. The starting point of these investigations was a landmark paper of Dobrushin [48] formulated for random walk (instead of a Wiener process case) which we will quote later. This theorem tells us that the local time increments normalized appropriately has a distribution, which for large t is close to the distribution of the product of $N_1\sqrt{|N_2|}$ where N_1 and N_2 are independent standard normal variables. This fact is even more intriguing combined with the following insightful result of Yor [83]:

Theorem B (Yor [83]) *As* $\lambda \to \infty$,

$$\left(\frac{1}{\lambda}W(\lambda^2 t), \frac{1}{\lambda}L(a,\lambda^2 t), \frac{1}{2\sqrt{\lambda}}\left(L(a,\lambda^2 t) - L(0,\lambda^2 t)\right)\right)$$
$$\xrightarrow{w} (W(t), L(a,t), W^*(a, L(0,t)))$$

where $W^*(a,u)$ *is a Wiener sheet independent of* $W(t)$ *and* $\xrightarrow{w}$ *denotes weak convergence.*

A Wiener (Brownian) sheet $\{W^*(a,u),\ a \ge 0,\ u \ge 0\}$ is a mean-zero Gaussian process with covariance $\mathbf{E}\, W^*(a_1,\, u_1)W^*(a_2,\, u_2) = \min(a_1,a_2)\min(u_1,u_2)$.

The above two results suggested that the local time difference $L(x,t) - L(0,t)$ could be strongly approximated by $\sigma_x W^*(L^{**}(0,t))$ on such a way that

$L(0,t)$ should be close to $L^{**}(0,t)$,

$W^*(t)$ and $L^{**}(0,t)$ should be **independent,**

and σ_x is a constant depending only on x. This conjecture was confirmed in [15] by

Theorem 2.2 *There is a probability space with*

- *a standard Wiener process* $\{W(t),\, t > 0\}$ *and its two-parameter local time process* $\{L(a,t),\, a \in \mathbb{R},\, t \geq 0\}$,
- *a two-parameter Wiener process* $\{B(a,u),\, a \geq 0, u \geq 0\}$,
- *a process* $\{L^1(0,t),\, t \geq 0\}$, *with* $\{L^1(0,t),\, t \geq 0\} \overset{\mathcal{D}}{=} \{L(0,t),\, t \geq 0\}$

such that as $t \to \infty$

- $\sup_{0 \leq a \leq a^* t^{\delta/2}} |L(a,t) - L(0,t) - 2B(a, L^1(0,t))| = \mathcal{O}(t^{((1+\delta)/4 - \epsilon/2)})$ a.s.,
- $|L^1(0,t) - L(0,t)| = \mathcal{O}(t^{15/32} \log^2 t)$ a.s.,
- $\{L^1(0,t),\, t \geq 0\}$ *and* $\{B(a,u),\, a \geq 0, u \geq 0\}$ *are independent,*

and for the constants above we have; $a^* > 0,\ 0 < \delta < 7/100,\ 0 < \epsilon < 1/72 - \delta/7$, $\overset{\mathcal{D}}{=}$ *denotes equality in distribution.*

The proof of this result was based on two major ingredients. The first of these two is an approximation theorem of Berkes and Philipp [4] for weakly dependent vectors. The second ingredient is a method we developed in this paper to achieve the stated independence of $L^1(0,t)$ and $B(a,u)$ in the theorem.

As a consequence of the above results, one can conclude various limit distributions and laws of the iterated logarithm, such as

$$\frac{L(a,t) - L(0,t)}{2\sqrt{aL(0,t)}} \xrightarrow{\mathcal{D}} N_1 \qquad \text{as } t \to \infty \qquad \text{for any } a > 0, \tag{2.3}$$

$$\frac{L(a,t) - L(0,t)}{2a^{1/2}t^{1/4}} \xrightarrow{\mathcal{D}} N_1 |N_2|^{1/2} \qquad \text{as } t \to \infty \qquad \text{for any } a > 0, \tag{2.4}$$

$$\limsup_{t \to \infty} \frac{L(a,t) - L(0,t)}{2\sqrt{2aL(0,t) \log\log t}} = 1 \qquad \text{a.s.} \qquad \text{for any } a > 0, \tag{2.5}$$

$$\limsup_{t \to \infty} \frac{L(a,t) - L(0,t)}{a^{1/2} t^{1/4} (\log\log t)^{3/4}} = \frac{2^{9/4}}{3^{3/4}} \qquad \text{a.s.} \qquad \text{for any } a > 0, \tag{2.6}$$

$$\begin{aligned} &\limsup_{t \to \infty} \sup_{0 < a < a^* t^\delta} \frac{L(a,t) - L(0,t)}{2\sqrt{2a^* t^\delta L(0,t)(\log\log t)}} \\ &= \limsup_{t \to \infty} \sup_{0 < a < a^* t^\delta} \frac{3}{4} 6^{-1/4} \frac{L(a,t) - L(0,t)}{(a^* t^\delta)^{1/2} t^{1/4} (\log\log t)^{3/4}} \quad = \quad 1 \qquad \text{a.s.} \end{aligned} \tag{2.7}$$

for any $a^* > 0$ and $0 \leq \delta < 7/200$, $\xrightarrow{\mathcal{D}}$ denoting convergence in distribution.

In fact, (2.3) and (2.4) also follow from Theorem B. However the rest of the above statements do not follow from any weak invariance principle. (2.5) and (2.6) were proved directly by Csáki and Földes [28]. An important step in attaining the above strong theorems was the following result which proved to be important in its own right; If $W_1(\cdot)$ is a standard Wiener processes and $L_2(\cdot,\cdot)$ is a Wiener local time, independent of W_1, then

$$\limsup_{t \to \infty} \frac{W_1(L_2(0,t))}{t^{1/4} (\log\log t)^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \qquad \text{a.s.} \tag{2.8}$$

2.3 Additive functionals. Let us consider a sequence of i.i.d. random variables X_i, $i = 1, 2...$ taking values on the integer lattice $\mathbb{Z}$. Put $S_0 = 0$, $S_n = X_1 + X_2 + ... + X_n$. Let us denote the local time of the random walk S_n by $\xi(x, n) := \#\{k : 0 < k \leq n,\, S_k = x\}$. Define the additive functional A_n as

$$A_n := \sum_{i=1}^{n} f(S_i) = \sum_{x=-\infty}^{\infty} f(x)\xi(x, n), \tag{2.9}$$

where $f(x)$, $x \in \mathbb{Z}$ is a real valued function. Clearly in the special case $f(a) = 1$, $f(0) = -1$, and $f(x) = 0$ otherwise, $A_n = \xi(a, n) - \xi(0, n)$. Let us denote

$$\bar{f} := \sum_{k=-\infty}^{\infty} f(k).$$

The so called first-order results on A_n are establishing the following observation: If $\bar{f} \neq 0$ then the asymptotic behaviour of A_n with appropriate normalization is the same as the behaviour of $\bar{f}L(0, n)$. The interested reader should consult Kallianpur and Robbins [60], Darling and Kac [44], Skorokhod and Slobodenyuk [78] and Borodin [7] to see the history of these first order limit results. However, we were interested in the so-called second order limit theorems for A_n which are focused on the behaviour of A_n when $\bar{f} = 0$ (clearly this is the case which contains the increments of the local time). The history of this topic goes back to the above mentioned famous result of Dobrushin [48]:

Theorem C (Dobrushin [48]) *Assume that* $\mathbf{P}(X_1 = +1) = \mathbf{P}(X_1 = -1) = 1/2$ *and define the additive functional as in* (2.9). *If* $f(x)$, $x \in \mathbb{Z}$ *has finite support and* $\bar{f} = 0$, *then*

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{A_n}{dn^{1/4}} < x\right) = \mathbf{P}(N_1\sqrt{|N_2|} < x), \tag{2.10}$$

where N_1 *and* N_2 *are two independent standard normal variables, and*

$$d^2 = 4\sum_{k=-\infty}^{\infty} kf^2(k) + 8 \sum_{-\infty<i<j<\infty}\sum if(i)f(j) - \sum_{k=-\infty}^{\infty} f^2(k).$$

This result has several generalizations. The corresponding functional version was given by Kasahara [61] and Borodin [6].

Similarly to the discrete case, one can consider the additive functional of a standard Wiener process. Let $g(x)$ be an integrable function on the real line and consider

$$G_t := \int_0^t g(W(s))\,ds = \int_{-\infty}^{\infty} g(x)L(x, t)\,dx, \qquad t \geq 0.$$

Results on the additive functional G_t are parallel to the results on A_n. Let us quote the functional form of the limit theorem given by Papanicolaou et al. [69], Ikeda and Watanabe [58], Kasahara [61] and Borodin [6]. They proved (under somewhat different assumptions on g) that

$$\lambda^{-1/4}\left(\int_0^{\lambda t} g(W(s))\,ds - \bar{g}L(0, t)\right) \overset{w}{\to} \sigma W_1(L_2(t)) \qquad \text{as} \quad \lambda \to \infty, \tag{2.11}$$

where W_1 is another standard Wiener process, L_2 is a Wiener local time at zero, such that W_1 and L_2 are independent, and σ is an explicitly given constant.

Our goal was to prove the strong approximation version of (2.11) for the random walk and the Wiener case as well. In both cases the method developed in [15] proved to be the appropriate tool to achieve our results in [16]. To avoid being repetitious we only quote the Wiener case result.

Theorem 2.3 *Assume that $f(x)$ is an integrable function on $\mathbb{R}$ and*

$$\int_{-\infty}^{\infty} |x|^{1+\delta}|f(x)|\,dx < \infty \qquad for \quad some \quad \delta > 0. \tag{2.12}$$

Then on a suitable probability space one can define a standard Wiener process $W(t)$ with two other standard Wiener processes $W_1(t)$ and $W_2(t)$ such that

- *$W_1(t)$ and $W_2(t)$ are independent,*
- $|\int_0^t f(W(s))\,ds - \bar{f}L(0,t) - \sigma W_1(L_2(0,t))| = O(t^{\tau/2}) \qquad \text{a.s.} \quad (t \to \infty)$,
- $|L(0,t) - L_2(0,t)| = O(t^{\kappa/4}) \qquad \text{a.s.} \quad (t \to \infty)$,

where $\bar{f} = \int_{-\infty}^{\infty} f(x)\,dx$ and

$$\sigma^2 = 4\int_{-\infty}^{0}\left(\int_{-\infty}^{x} f(y)\,dy\right)^2 dx + 4\int_{0}^{\infty}\left(\int_{x}^{\infty} f(y)\,dy\right)^2 dx, \tag{2.13}$$

$L(x,t)$ and $L_2(x,t)$ resp., are the local times of $W(\cdot)$ and $W_2(\cdot)$ resp., and κ, τ are any numbers satisfying $\kappa < 4\tau$,

$$\frac{1}{4} + \frac{1}{2(2+\delta)} < \tau < \frac{1}{2}, \qquad \frac{7}{4} + \frac{1}{2(2+\delta)} < \kappa < 2. \tag{2.14}$$

Remark It was shown in [22] that the condition (2.12) can be relaxed to

$$\int_{-\infty}^{\infty} |x|^{1/2+\delta}|f(x)|\,dx < \infty \qquad \text{for} \quad \text{some} \quad \delta > 0. \tag{2.15}$$

As a consequence of the above theorem we get the following LIL type result for the additive functionals.

Under the conditions of the above theorem we have

$$\limsup_{t\to\infty} \frac{|\int_0^t f(W(s))\,ds - \bar{f}L(0,t)|}{t^{1/4}(\log\log t)^{3/4}} = \sigma\,\frac{2^{5/4}}{3^{3/4}} \qquad \text{a.s.}$$

Both of the above two results and their random walk counterparts became the starting point of many further investigations in this direction. The method of proof was successfully used to generalize these results for the additive functionals of various processes. Extensions were given for Markov chains by Csáki and Csörgő [13], for diffusions by Csáki and Salminen [32], for Markov processes by Eisenbaum and Földes [50], for simple symmetric random walk on the plane by Csáki et al. [30]. In [29] additive functionals of more general random walks in one and two dimensions were strongly approximated under various conditions. As a consequence of these results one always gets both LIL-type and weak convergence results.

2.4 Principal value of Brownian local time. An important special type of additive functionals is the following

$$Y_\alpha(t) := \int_0^t \frac{ds}{W^\alpha(s)} = \int_0^\infty \frac{L(x,t) - L(-x,t)}{x^\alpha}\, dx, \tag{2.16}$$

where the integral $\int_0^t ds/W^\alpha(s)$ (notation: $z^\alpha = |z|^\alpha \operatorname{sgn}(z)$) is in the sense of Cauchy's principal value). Strictly speaking, the first integral is defined as Cauchy's principal value for $1 \le \alpha < 3/2$ and as Riemann integral for $\alpha < 1$. The investigation of the process $Y_1(t)$ which is called the Cauchy principal value of the Brownian local time goes back at least to Itô and McKean [59] and has become very active since the late 70s, due to applications in various branches of stochastic analysis. For example, it is a natural example in Fukushima [55] theory for Dirichlet processes and zero-energy additive functionals. Also, the principal values of Brownian local times are the key ingredient in establishing Bertoin [5]'s excursion theory for Bessel processes of small dimensions. For a detailed account on these facts and general properties of principal values of local times, we refer to the collection of research papers in Yor [84] and to the survey paper by Yamada [82].

Hu and Shi [56] proved the following LIL-s for the local and global behaviour of the principal value:

Theorem D (Hu and Shi [56])

$$\limsup_{t\to\infty} \frac{Y_1(t)}{\sqrt{t \log\log t}} = 2\sqrt{2} \qquad \text{a.s.} \tag{2.17}$$

and

$$\limsup_{h\to 0} \frac{Y_1(h)}{\sqrt{h \log\log(1/h)}} = 2\sqrt{2} \qquad \text{a.s.}$$

This result supports the common belief that the principal value process $Y_1(t)$ is very similar in behaviour to the Brownian motion. To explore further this phenomenon we investigated some path properties of $Y_\alpha(\cdot)$ and especially $Y_1(\cdot)$. We studied the modulus of continuity and large increment properties (including the LIL) of $Y_\alpha(\cdot)$, as well as appropriate properties of a simple symmetric random walk along these lines. Due however to lack of precise distributional properties of $Y_\alpha(\cdot)$, when $\alpha \neq 1$, we could not obtain the desirable exact constants, though the rates we established are optimal. In our first theorem [22] we proved the upper bounds for the LIL, large increments and modulus of continuity.

Theorem 2.4 *Under* Condition A *for* $0 < \alpha < 3/2$ *we have*

$$\limsup_{t\to\infty} \frac{\sup_{0\le u\le t-a_t} \sup_{0\le s\le a_t} |Y_\alpha(u+s) - Y_\alpha(u)|}{a_t^{1-\alpha/2} \left(\log(t/a_t) + \log\log t\right)^{\alpha/2}} \le c_1(\alpha) \qquad \text{a.s.} \tag{2.18}$$

$$\limsup_{h\to 0} \frac{|Y_\alpha(h)|}{h^{1-\alpha/2} (\log\log(1/h))^{\alpha/2}} \le c_1(\alpha) \qquad \text{a.s.} \tag{2.19}$$

$$\limsup_{h\to 0} \frac{\sup_{0\le t\le 1-h} \sup_{0\le s\le h} |Y_\alpha(t+s) - Y_\alpha(t)|}{h^{1-\alpha/2} \left(\log(1/h)\right)^{\alpha/2}} \le c_1(\alpha) \qquad \text{a.s.} \tag{2.20}$$

Here, the constant $c_1(\alpha)$ *is given by*

$$c_1(\alpha) = \frac{3 \cdot 2^{7\alpha/6}}{\alpha^{2\alpha/3}(3-2\alpha)^{1-\alpha/3}(2-\alpha)^{\alpha/3}}. \tag{2.21}$$

Remark In the particular case $a_t = t$ we get

$$\limsup_{t\to\infty} \frac{|Y_\alpha(t)|}{t^{1-\alpha/2}(\log\log t)^{\alpha/2}} \le c_1(\alpha), \qquad \text{a.s.} \tag{2.22}$$

Concerning the constant in LIL, we have the following result.

Theorem 2.5 *For* $0 < \alpha < 3/2$, *there exists a finite positive constant* $c_2(\alpha)$ *such that*

$$\limsup_{t\to\infty} \frac{|Y_\alpha(t)|}{t^{1-\alpha/2}(\log\log t)^{\alpha/2}} = c_2(\alpha) \in \left[2^{3\alpha/2}\,\Gamma(3-\alpha),\, c_1(\alpha)\right] \qquad \text{a.s.} \tag{2.23}$$

The LIL holds true also for random walks via the following invariance principle [22]. Let S_i, $i = 1, 2, \ldots$ be a simple symmetric random walk on the line, starting from 0, and let $\xi(x, n)$ be its local time. Define

$$G_\alpha(n) := \sum_{k=1}^{n} \frac{\mathbf{1}_{[S_k \neq 0]}}{S_k^\alpha} = \sum_{i=1}^{\infty} \frac{\xi(i,n) - \xi(-i,n)}{i^\alpha}. \tag{2.24}$$

Theorem 2.6 *On a suitable probability space one can define a Wiener process* $\{W(t),\, t \ge 0\}$ *and a simple symmetric random walk* $\{S_n,\, n = 1, 2, \ldots\}$ *such that for any* $0 < \alpha < 3/2$ *and sufficiently small* $\varepsilon > 0$ *we have*

$$|Y_\alpha([t]) - G_\alpha([t])| = o(t^{1-\alpha/2-\varepsilon}) \qquad \text{a.s.,} \tag{2.25}$$

as $t \to \infty$.

As a consequence of our Theorem 2.6, the LIL-s in (2.17), (2.22) and (2.23) remain true if Y_α is replaced by G_α.

As it is easily seen, Y_α is not defined for $\alpha \ge 3/2$. In this case, we considered instead the process

$$Z_\alpha(t) := \int_0^t \frac{\mathbf{1}_{[|W(s)| \ge 1]}}{W^\alpha(s)}\, ds = \int_1^\infty \frac{L(x,t) - L(-x,t)}{x^\alpha}\, dx. \tag{2.26}$$

This is a "nice" additive functional, for which Theorem 2.3 can be applied. The limit process associated with such functionals is $V(t) = W_1(L_2(t))$, where $W_1(\cdot)$ is a standard Wiener process and $L_2(\cdot)$ is a Wiener local time at zero, independent of W_1.

Considering the special case of Y_1, in [21] we characterized the modulus of continuity as follows;

Theorem 2.7 *With probability one,*

$$\lim_{h\to 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \frac{|Y_1(t+s) - Y_1(t)|}{\sqrt{h\log(1/h)}} = 2.$$

Remark $Y(t)/\sqrt{2}$ and $W(t)$ have the same moduli of continuity (and the same remark applies to our next theorem below). We have already seen that $Y(t)/2$ and $W(t)$ satisfy the same LIL. Heuristically speaking, that a factor $\sqrt{2}$ is missing in the modulus of continuity, this comes from the fact that the Hausdorff dimension of the zero set of W is 1/2.

As to the large increments of $Y(\cdot)$, in [21] we proved

Theorem 2.8 *Under* Conditions A *and* B *we have*

$$\lim_{t\to\infty}\sup_{0\le u\le t-a_t}\sup_{0\le s\le a_t}\frac{|Y_1(u+s)-Y_1(u)|}{\sqrt{a_t\log(t/a_t)}}=2 \qquad \text{a.s.}$$

Remark Recently Csáki and Hu [31] was able to fill the gap in the above increment results by showing that Condition A is enough to get a limsup.

To look at the the corresponding two-dimensional question, let

$$\{\mathbf{W}(t):=(W_1(t),W_2(t)),\ t\ge 0\}$$

be a two-dimensional Wiener process, where $W_1(t)$ and $W_2(t)$ are two independent one-dimensional Wiener processes, with $W_1(0)=W_2(0)=0$. Put

$$R(t):=\|\mathbf{W}(t)\|=\sqrt{W_1^2(t)+W_2^2(t)}\,.$$

It is well-known that $\{R(t),\ t\ge 0\}$ is a two-dimensional Bessel process. In [23] we were interested in the additive functional

$$\widetilde{Z}_\alpha(t):=\int_0^t\frac{ds}{R^\alpha(s)}, \tag{2.27}$$

the critical case being $\alpha=2$ (instead of 3/2). It can be seen that the integral in (2.27) converges for $\alpha<2$, but diverges for $\alpha\ge 2$ almost surely. In the latter case we defined the modified process

$$Z_\alpha^*(t):=\int_0^t\frac{1}{R^\alpha(s)}\mathbf{1}_{\{R(s)\ge 1\}}\,ds. \tag{2.28}$$

Considering the random walk counterpart, let $\{\mathbf{S}_n\}_{n=1}^\infty$ be a simple symmetric random walk on the integer lattice $\mathbb{Z}^2$, i.e. $\mathbf{S}_n=\sum_{k=1}^n\mathbf{X}_k$, where the random variables $\mathbf{X}_i$, $i=1,2,\ldots$ are i.i.d., with

$$\mathbf{P}(\mathbf{X}_1=(0,1))=\mathbf{P}(\mathbf{X}_1=(0,-1))=\mathbf{P}(\mathbf{X}_1=(1,0))=\mathbf{P}(\mathbf{X}_1=(-1,0))=\frac{1}{4}.$$

We also proposed to study the discrete process

$$U_\alpha(n):=\sum_{k=1}^n\frac{1}{\|\mathbf{S}_k\|^\alpha}\mathbf{1}_{\{\mathbf{S}_k\neq\mathbf{0}\}}. \tag{2.29}$$

Define

$$\xi(\mathbf{x},n):=\#\{k;\ 1\le k\le n,\ \ \mathbf{S}_k=\mathbf{x}\},$$

for any lattice point $\mathbf{x}\in\mathbb{Z}$. This is the local time process of $\{\mathbf{S}_n\}_{n=1}^\infty$.

First we considered the case $0<\alpha<2$, for which we managed to show that the processes $\widetilde{Z}_\alpha(\cdot)$ and $2^{-\alpha/2}U_\alpha(\cdot)$ are close enough to each other to share many of their properties. Based on some results of Revuz and Yor [72], Azencott [1] and Borovkov and Mogulskii [8], in [23] we proved the following lim sup and lim inf results for both of these processes.

Theorem 2.9 *For* $2/3\le\alpha<2$ *we have*

$$\limsup_{t\to\infty}\frac{\widetilde{Z}_\alpha(t)}{t^{1-\alpha/2}(\log\log t)^{\alpha/2}}=\limsup_{n\to\infty}\frac{2^{-\alpha/2}U_\alpha(n)}{n^{1-\alpha/2}(\log\log n)^{\alpha/2}}=K_1(\alpha) \qquad \text{a.s.}$$

and

$$\liminf_{t\to\infty}t^{-(1-\alpha/2)}(\log\log t)^{\alpha/2}\widetilde{Z}_\alpha(t)$$
$$=\liminf_{n\to\infty}n^{-(1-\alpha/2)}(\log\log n)^{\alpha/2}2^{-\alpha/2}U_\alpha(n)=K_2(\alpha) \qquad \text{a.s.,}$$

whith certain positive constants $K_1(\alpha)$ *and* $K_2(\alpha)$.

On the other hand, it turned out that when $\alpha > 2$ then the two processes have to be investigated separately. However both processes, suitably centered, are close to certain iterated processes. We only quote the results from [23] in random walk case, and some of its consequences, parallel results are true for $Z^*_\alpha(t)$.

Theorem 2.10 *Let* $\alpha > 2$. *There exists a probability space where one can define*

- *a two-dimensional simple symmetric random walk* $\{\mathbf{S}_n\}_0^\infty$ *with its local time* $\xi(\mathbf{x}, n)$, *and with the corresponding additive functional* $\{U_\alpha(n),\ n = 1, 2...\}$ *as in* (2.29);
- *a process* $\{\xi^{(1)}(\mathbf{0}, n),\ n = 1, 2, ...\} \stackrel{\mathcal{D}}{=} \{\xi(\mathbf{0}, n),\ n = 1, 2, ...\}$;
- *a standard Wiener process* $\{W(t),\ t \geq 0\}$, *independent of* $\{\xi^{(1)}(\mathbf{0}, n),\ n = 1, 2, ...\}$;

such that, for some $\varepsilon > 0$, *as* $n \to \infty$,

- $U_\alpha(n) - \bar{f}_\alpha \xi(\mathbf{0}, n) = \sigma_\alpha W(\xi^{(1)}(\mathbf{0}, n)) + \mathcal{O}(\log^{1/2-\epsilon} n)$ a.s.,
- $\xi(\mathbf{0}, n) = \xi^{(1)}(\mathbf{0}, n) + \mathcal{O}(\log^{1-\epsilon} n)$ a.s.

where $\bar{f}_\alpha := \sum_{\mathbf{x} \in \mathbb{Z}^2 - \{\mathbf{0}\}} \frac{1}{\|x\|^\alpha}$, $\quad \sigma_\alpha := \sqrt{\mathrm{Var}(U_\alpha(\rho_1))}$.

The above theorem has both weak and strong implications.

Theorem 2.11 *For* $\alpha > 2$ *we have*

$$\frac{\pi U_\alpha(n)}{\bar{f}_\alpha \log n} \xrightarrow{\mathcal{D}} |E|, \quad n \to \infty, \tag{2.30}$$

$$\frac{U_\alpha(n) - \bar{f}_\alpha \xi(\mathbf{0}, n)}{\sigma_\alpha \sqrt{\log n}} \sqrt{2\pi} \xrightarrow{\mathcal{D}} E, \quad n \to \infty, \tag{2.31}$$

$$\limsup_{n \to \infty} \frac{U_\alpha(n)}{\log n \log_3 n} = \frac{\bar{f}_\alpha}{\pi} \quad \text{a.s.}, \tag{2.32}$$

$$\limsup_{n \to \infty} \frac{U_\alpha(n) - \bar{f}_\alpha \xi(\mathbf{0}, n)}{\sqrt{\log n} \log_3 n} = \frac{\sigma_\alpha}{\sqrt{2\pi}} \quad \text{a.s.}, \tag{2.33}$$

where E *is a bilateral exponential random variable with density* $e^{-|x|}/2$, $x \in \mathbb{R}$, *and* $|E|$ *is exponential with parameter 1.*

2.5 Integral functionals. In [20] we studied the following two types of integral functionals of geometric stochastic processes which are of interest in financial modelling:

$$A(t) := \int_0^t \exp(X(u))\, du, \qquad B(t) := \int_0^\infty \exp\left(Y(u) - \frac{u^\alpha}{t}\right) du, \quad 0 < t < \infty. \tag{2.34}$$

We managed to show, that under fairly general conditions on $X(t)$ and $Y(t)$ respectively, $\log A(t)$ and $\log B(t)$ behave like $\sup_{0 \leq u \leq t} X(u)$ and $\sup_{0 \leq u < \infty}(Y(u) - u^\alpha/t)$. We only quote our first strong invariance theorem which deals with $X(t)$.

Theorem 2.12 *Let the stochastic process* $\{X(t); 0 \le t < \infty\}$ *have almost surely continuous sample paths,* $\mathbf{P}(X(0)=0)=1$ *and put*

$$Z(t) := \log A(t) \qquad \text{and} \qquad U(t) := \sup_{0\le u\le t} X(u).$$

Assume that for the increment of $X(t)$ *we have*

$$\sup_{0\le s\le t-a_t}\ \sup_{0\le v\le a_t} |X(s+v)-X(s)| = O(r(t,a_t)) \qquad \text{a.s.}$$

as $t\to\infty$*, with some non-decreasing* a_t $(1\le a_t\le t)$ *and rate* $r(t,a_t)$*. Then as* $t\to\infty$,

$$|Z(t)-U(t)| = O(r(t,a_t)+\log t) \qquad \text{a.s.}$$

We applied these strong approximation theorems for a number of processes, such as Wiener process, fractional Brownian motion, Gaussian processes, and diffusion processes.

3. Iterated processes, and their local times

3.1 Iterated processes. C. Burdzy [9] proposed to investigate the process

$$Z(t) := \{W_1(W_2(t)),\ 0\le t<\infty\}, \tag{3.1}$$

where $\{W_1(t),\ t\in\mathbb{R}\}$ and $\{W_2(t),\ t\ge 0\}$ are two independent standard Brownian motions. He called this process an iterated Brownian motion (IBM), and proved the following LIL:

Theorem E (Burdzy [9])

$$\limsup_{t\to 0} \frac{Z(t)}{t^{1/4}(\log\log(1/t))^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \qquad \text{a.s.} \tag{3.2}$$

Closely related processes to $Z(t)$ are

$$H(t) := \{W_1(|W_2(t)|),\ 0\le t<\infty\} \tag{3.3}$$

and

$$V(t) := \{W_1(L_2(t)),\ 0\le t<\infty\},$$

where L_2 is a Wiener local time at 0, independent of W_1.

In 1993-94 many people got interested in these processes, one should consult [17] for proper references. In the above Theorem E we have an LIL for $t\to 0$ and in (2.8) we have an LIL for the process $V(t)$ as $t\to\infty$ with the very same constant. The latter result combined with a famous result of Paul Lévy, mentioned earlier, implies that the same is true for the process $Y(t) := W_1(\max_{0\le s\le t} W_2(s))$, and $H(t)$, as well. It is easy to see that

$$\frac{V(t)}{t^{1/4}} \overset{\mathcal{D}}{=} \frac{H(t)}{t^{1/4}} \overset{\mathcal{D}}{=} \frac{Y(t)}{t^{1/4}} \overset{\mathcal{D}}{=} N_1\sqrt{|N_2|}, \tag{3.4}$$

where N_1 and N_2 are two independent standard normal variables. We have seen this distribution to appear in Dobrushin's theorem (2.10) and in (2.4) as well. In all of these results we have in the above sense an iterated process created from a pair of independent processes. This gave us the idea that there must be a common way to investigate these three processes and started to study these iterated processes more closely. To introduce our first result in this direction, we recall the following definition: Let $\mathcal{S}$ be the Strassen class of functions, i.e., $\mathcal{S} \subset C[0,1]$ is the class of absolutely continuous functions (with respect to the Lebesgue measure) on $[0,1]$ for which

$$f(0) = 0 \qquad \text{and} \qquad \int_0^1 \dot{f}^2(x)dx \le 1. \tag{3.5}$$

The set of $\mathbb{R}^2$-valued, absolutely continuous functions

$$\{(g(y), h(x)),\ 0 \le y \le 1,\ 0 \le x \le 1\} \tag{3.6}$$

for which $g(0) = h(0) = 0$ and

$$\int_0^1 \dot{g}^2(y)dy + \int_0^1 \dot{h}^2(x)dx \le 1 \tag{3.7}$$

will be called *Strassen class* $\mathcal{S}^2$.

Now let $C_0[0,1] \subset C[0,1]$ be the set of continuous functions $f(\cdot)$ on $[0,1]$ for which $f(0) = 0$. Let A be an operator on $C_0[0,1]$, satisfying

(C.1) $Acf = c^\rho Af \qquad (\rho \ge 1,\ c > 0)$,
(C.2) $Af \ge 0$,
(C.3) $Af \in C_0[0,1]$,
(C.4) A is uniformly continuous on bounded subsets of $C_0[0,1]$, i.e., $\forall \varepsilon > 0,\ K > 0,\ \exists \delta = \delta(\varepsilon, K) > 0$ such that if $f, g \in C_0[0,1]$, $\sup_{0\le x\le 1} |f(x)| \le K$, $\sup_{0\le x\le 1} |g(x)| \le K$ and $\sup_{0\le x\le 1} |f(x) - g(x)| < \delta$, then $\sup_{0\le x\le 1} |Af(x) - Ag(x)| \le \varepsilon$,
(C.5) $\sup_{f\in\mathcal{S}} Af(x) = \lambda(A, x) = \lambda_x \qquad 0 < \lambda_x \le 1.$

Some of our examples for $Af(x)$ are the following: $|f(x)|$, $\max_{0<y\le x} f(y)$, and $\max_{0<y\le x} |f(y)|$.

Theorem 3.1 *Let $W_1(\cdot)$ and $W_2(\cdot)$ be two independent standard Wiener processes starting from zero, and let A be an operator satisfying conditions* (C.1)–(C.5). *Then for $0 \le x \le 1$, $0 \le y \le 1$, the limit set of the vector*

$$\left(\frac{W_1(yAW_2(xT))}{T^{\rho/4}(2\log\log T)^{(\rho+2)/4}}, \frac{W_2(xT)}{(2T\log\log T)^{1/2}}\right) \tag{3.8}$$

is $(g(yAh(x)), h(x))$, where $(g,h) \in \mathcal{S}^2$.

This theorem gives an easy way to show the above LIL-s, and it has many more consequences. Here we mention only one of them as an example.

Theorem 3.2 *For $0 \le x \le 1$, $0 \le y \le 1$, $\rho \ge 1$ we have*

$$\limsup_{T\to\infty} \frac{W_1(yAW_2(xT))}{T^{\rho/4}(2\log\log T)^{(\rho+2)/4}} = 2^{1/2}\lambda_x^{1/2} y^{1/2} \rho^{\rho/4} (\rho+2)^{-(\rho+2)/4} \qquad \text{a.s.} \tag{3.9}$$

Using the invariance principle of Komlós et. al. [65], [66] it was also shown that if we construct an iterated random walk $U(n) := S_1(|S_2(n)|)$ from two independent simple symmetric random walks S_1 and S_2, then the iterated random walk is close to the iterated Brownian motion.

Theorem 3.3 *On a rich enough probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *one can construct an iterated Wiener process* $\{H(t),\, t \geq 0\}$ *and an iterated random walk* $\{U(n),\, n = 1, 2, ..\}$ *such that*

$$\max_{1 \leq k \leq n} |U(k) - H(k)| = \mathcal{O}(\log n) \qquad \text{a.s.} \tag{3.10}$$

This theorem enables us to carry over many limit theorems from the IBM processes to the iterated random walk.

3.2 Local time and occupation time. In [18] we defined the local time $L^*(x, t)$ of $H(t) = W_1(|W_2(t)|)$ as follows:

$$L^*(x,t) := \int_0^\infty \overline{L}_2(s,t)\, d_s L_1(x,s), \qquad x \in \mathbb{R}, \quad t \geq 0. \tag{3.11}$$

where $\overline{L}_2(\cdot,\cdot)$, $L_1(\cdot,\cdot)$ are the local time processes of $|W_2(\cdot)|$ and $W_1(\cdot)$, respectively. In particular, $\overline{L}_2(x,t) := L_2(x,t) + L_2(-x,t),\ x \geq 0$, where $L_2(\cdot,\cdot)$ is the local time process of $W_2(\cdot)$. At about the same time Burdzy and Khoshnevisan [10] studied the local time of the process $Z(t) = W_1(W_2(t))$ and proved its Hölder continuity. Concerning L^*, we established its joint continuity and studied its path behaviour aiming at the four classical Lévy classes of functions. However, these results are far from being optimal yet, and leave open many problems for further considerations, including even that of proving an LIL for $L^*(x,t)$ at $x = 0$. Indeed, a systematic study of the fine analytic properties of the process $\{L^*(x,t),\ x \in \mathbb{R},\ t \geq 0\}$ along the lines of those of the classical Brownian local time of P. Lévy seems to be a challenging problem. For further liminf type results we refer to [75].

We also considered the corresponding iterated random walk $U(n) = S_1(|S_2(n)|)$ and defined its local time by

$$\xi^*(x,n) := \#\{k : 0 < k \leq n,\, U(k) = x\}.$$

Then we established that on an appropriate probability space, as $t \to \infty$,

$$\sup_{x \in \mathbb{Z}} |\xi^*(x,t) - L^*(x,t)| = \mathcal{O}(t^{11/16+\varepsilon}) \qquad \text{a.s.} \tag{3.12}$$

which implies that all the above mentioned Lévy class type results are inherited by $\xi^*(x,t)$.

It is quite interesting to note that even though $Z(t) = W_1(|W_2(t)|)$ and $V(t) = W_1(L_2(t))$ share many properties, the investigation of their respective local times reveals how different they really are. We started our investigation with studying the occupation time of $V(t)$, and it turned out that we must confine our attention to it as $V(t)$ has no local time. Actually because of the non-Markovian nature of $V(t)$ it is more appropriate to talk about the non-existence of its occupation density. Another surprise was to realize that we were unable to establish a strong approximation result similar to (3.10), hence each result had to be established separately for $V(t)$ and the corresponding iterated random walk. For simplicity, here we only explain how to define the occupation time of the iterated random walk. Let $S_1(\cdot)$ and $S_2(\cdot)$ be two independent simple symmetric random walks as above and denote the local time at zero of $S_2(\cdot)$ by $\xi_2(n)$. In the spirit of $V(t) = W_1(L_2(t))$ we define $R(n) := S_1(\xi_2(n-1))$ and the corresponding occupation time of $R(n)$ is defined as

$$\widetilde{\xi}(r,n) := \#\{k : 1 \leq k \leq n, R(k) = r\}. \tag{3.13}$$

Then clearly

$$\begin{aligned}\widetilde{\xi}(r,n) &= \sum_{k=1}^{n} I\{S_1(\xi_2(k-1))=r\}\\ &= \sum_{0\le s\le \xi_2(n-1)} (\rho_2(s+1)\wedge n-\rho_2(s))I\{S_1(s)=r\},\end{aligned} \qquad (3.14)$$

where $I(\cdot)$ is an indicator function and $0=\rho_2(0)<\rho_2(1)<...$ are the consecutive return epochs to zero of our second walk $S_2(\cdot)$. Thus we have

$$\widetilde{\xi}(r,\rho_2(n))=\sum_{s=1}^{n-1}(\rho_2(s+1)-\rho_2(s))I\{S_1(s)=r\}. \qquad (3.15)$$

Further studying (3.15) led us to the right way of interpreting $\widetilde{\xi}(r,n)$ and the occupation time of $V(t)$ as well. It turned out that these occupation times has interesting limiting distributions. Here we only mention the following one.

Theorem 3.4 *For any fixed integer $r\ge 0$, as $n\to\infty$, we have*

$$\frac{\widetilde{\xi}(r,n)}{n^{1/2}} \xrightarrow{\mathcal{D}} N_1^2|N_2|T_1 \overset{\mathcal{D}}{=} C^2|N_2|, \qquad (3.16)$$

where N_1 and N_2 are independent standard normal random variables that are also independent of the stable $(1/2)$ random variable T_1, and C is a standard Cauchy random variable independent of N_2.

As it was indicated above, the Lévy class type results for the occupation time of $V(s)$ and $R(s)$ were separately established. For further LIL-type results for $\widetilde{\xi}$ we refer to Révész [71].

4. Empirical processes

4.1 Level crossings. Let $U_1, U_2, \ldots$ be a sequence of i.i.d. random variables uniformly distributed in $(0,1)$. The empirical process α_n based on the first n observations is defined as:

$$\alpha_n(t):=n^{1/2}(F_n(t)-t), \quad 0\le t\le 1,$$

where $F_n(t):=\#\{i:\ 1\le i\le n,\ U_i\le t\}$, $0\le t\le 1$, is the empirical distribution function.

Consider the (normalized) level crossings of the empirical process:

$$\mathbb{L}_n(x):=n^{-1/2}\#\left\{t\in[0,1]:\ \alpha_n(t)=x\right\}, \qquad x\in\mathbb{R}.$$

Let us also define the maximal level crossings:

$$\mathbb{L}_n^*:=\sup_{x\in\mathbb{R}}\mathbb{L}_n(x).$$

We recall the following results.

Theorem F (Révész [70]) *Almost surely,*

$$\limsup_{n\to\infty}\frac{\mathbb{L}_n(0)}{(2\log\log n)^{1/2}}=1.$$

Theorem G (Bass and Khoshnevisan [3]) *We have,*

$$\begin{aligned}\limsup_{n\to\infty}\frac{\mathbb{L}_n^*}{(2\log\log n)^{1/2}} &= 1 \qquad \text{a.s.}\\ \liminf_{n\to\infty}(\log\log n)^{1/2}\,\mathbb{L}_n^* &= 2^{1/2}\pi \qquad \text{a.s.}\end{aligned}$$

In [41], we proved the following results.

Theorem 4.1 *Almost surely,*

$$\left\{\left(\frac{\mathbb{L}_n(0)}{(2\log\log n)^{1/2}},\frac{\mathbb{L}_n^*}{(2\log\log n)^{1/2}}\right)\right\}$$

is relatively compact, with limit set equal to $\{(x,y):\ 0\le x\le y\le 1\}$.

Theorem 4.2 *We have,*

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{(\log\log n)^{1/2}}\int_{-\infty}^{\infty}\mathbb{L}_n(x)^2dx &= \left(\frac{2}{3}\right)^{1/2} \qquad \text{a.s.,}\\ \liminf_{n\to\infty}(\log\log n)^{1/2}\int_{-\infty}^{\infty}\mathbb{L}_n(x)^2dx &= \left(\frac{2|a_1|}{3}\right)^{3/2} \qquad \text{a.s.,}\end{aligned}$$

where $a_1<0$ *is the largest real zero of the Airy function* $\mathrm{Ai}(\cdot)$.

Theorem 4.3 *For any* $p\ge 3$, *with probability one,*

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{(\log\log n)^{(p-1)/2}}\int_{-\infty}^{\infty}\mathbb{L}_n(x)^p dx\\ &\quad = 2^{(p+1)/2}(p-1)^{(p-1)/2}(p+1)^{(p-3)/2}\,B\left(\frac{1}{2},\frac{1}{p-1}\right)^{1-p},\end{aligned}$$

where $B(\cdot,\cdot)$ *stands for the usual beta function.*

4.2 Vervaat error process. Let $F_n(t)$ be as before the empirical distribution function from a uniform [0,1] sample. Let F_n^{-1} be the left-continuous inverse of F_n. We denote the empirical and quantile processes over the interval $[0,1]$ by

$$\begin{aligned}\alpha_n(t) &:= n^{1/2}(F_n(t)-t), \quad 0\le t\le 1,\\ \beta_n(t) &:= n^{1/2}(F_n^{-1}(t)-t), \quad 0\le t\le 1,\end{aligned}$$

respectively. The sum

$$R_n(t) := \alpha_n(t)+\beta_n(t), \quad 0\le t\le 1,$$

of the empirical and quantile processes is known in the literature as the Bahadur–Kiefer process (cf. Bahadur [2], Kiefer [63], [64]). This process enjoys some remarkable asymptotic properties, which are of interest in statistical quantile data analysis (cf., e.g., Csörgő [33], Shorack and Wellner [77]). We summarize the most relevant results of Kiefer [63], [64], Shorack [76], Deheuvels and Mason [46] in the following theorem. For further developments one can consult Deheuvels and Mason [47], Einmahl [49], Csörgő and Szyszkowicz [42].

Theorem H *For every fixed* $t\in(0,1)$, *we have*

$$n^{1/4}R_n(t) \xrightarrow{\mathcal{D}} (t(1-t))^{1/4}N_1(|N_2|)^{1/2}, \quad n\to\infty, \tag{4.1}$$

$$\limsup_{n\to\infty}\frac{n^{1/4}|R_n(t)|}{(\log_2 n)^{3/4}} = (t(1-t))^{1/4}\frac{2^{5/4}}{3^{3/4}} \qquad \text{a.s.,} \tag{4.2}$$

where N_1 and N_2 are independent standard normal variables. Also,

$$\lim_{n\to\infty} n^{1/4}(\log n)^{-1/2}\frac{\|R_n\|}{(\|\alpha_n\|)^{1/2}} = 1 \qquad \text{a.s.,} \tag{4.3}$$

where $\|f\| := \sup_{0\le t\le 1}|f(t)|$ denotes the sup-norm of f.

Via using the usual and the other laws of the iterated logarithm for α_n, (4.3) immediately implies

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{-1/4}\|R_n\| \;=\; 2^{-1/4} \qquad \text{a.s.,} \tag{4.4}$$

$$\liminf_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{1/4}\|R_n\| \;=\; \frac{\pi^{1/2}}{8^{1/4}} \qquad \text{a.s.,} \tag{4.5}$$

while a direct application of (4.3) together with the weak convergence of α_n to a Brownian bridge B gives

$$n^{1/4}(\log n)^{-1/2}\|R_n\| \overset{\mathcal{D}}{\to} (\|B\|)^{1/2}, \quad n\to\infty. \tag{4.6}$$

Nevertheless, the following result, which one can immediately conclude also by combining (4.1) with (4.6), is true, and it was first formulated and proved directly by Vervaat [81].

Theorem I (Vervaat [81]) *The statement*

$$a_n R_n \overset{\mathcal{D}}{\to} Y, \quad n\to\infty$$

cannot hold true in the space $D[0,1]$ (endowed with the Skorokhod topology) for any sequence $\{a_n\}$ of positive real numbers and any non-degenerate random element Y of $D[0,1]$.

In view of Theorems H and I, it is of interest to see the asymptotic behaviour of the Bahadur–Kiefer process possibly in other norms as well. In this regard the following theorem was proved in [39], [40].

Theorem 4.4 *For any $p\in[2,\infty)$, we have*

$$\lim_{n\to\infty} n^{1/4}\frac{\|R_n\|_p}{(\|\alpha_n\|_{p/2})^{1/2}} = c_0(p) \qquad \text{a.s.,} \tag{4.7}$$

where

$$c_0(p) := (\mathbf{E}|N_1|^p)^{1/p} = \sqrt{2}\left(\frac{\Gamma((p+1)/2)}{\sqrt{\pi}}\right)^{1/p}, \tag{4.8}$$

and N_1 stands for a standard normal variable, and $\|f\|_p := \left(\int_0^1 |f(t)|^p\,dt\right)^{1/p}$, the L_p norm of f.

Vervaat's [81] proof of Theorem I was based, in an elegant way, on the following integrated Bahadur–Kiefer process

$$I_n(t) := \int_0^t R_n(s)ds, \quad 0\le t\le 1.$$

Concerning the latter process, he established the weak convergence of

$$V_n(t) := 2n^{1/2}I_n(t) \tag{4.9}$$

to B^2, the square of a Brownian bridge, as well as a functional LIL for V_n, via proving the following theorem.

Theorem J (Vervaat [80], [81]) *We have*

$$\begin{aligned}\lim_{n\to\infty}(\log\log n)^{-1}\|V_n-\alpha_n^2\| &= 0 \qquad \text{a.s.} &(4.10)\\ \lim_{n\to\infty}\|V_n-\alpha_n^2\| &= 0 \qquad \textit{in probability.} &(4.11)\end{aligned}$$

In particular, in the space $C[0,1]$,

$$V_n \xrightarrow{\mathcal{D}} B^2, \quad n\to\infty. \tag{4.12}$$

We call the process V_n of (4.9) the uniform Vervaat process.

Bahadur [2] introduced R_n as the remainder term in the representation

$$\beta_n = -\alpha_n + R_n$$

of the quantile process β_n in terms of the empirical process α_n. As we have seen above, the remainder term R_n, i.e., the Bahadur–Kiefer process, is asymptotically smaller than the main term α_n, i.e., the empirical process, in both the L_p and sup-norm topologies.

Similarly, one can consider the process

$$Q_n(t) := V_n(t) - \alpha_n^2(t), \quad 0\le t\le 1, \tag{4.13}$$

that appears in both statements (4.10) and (4.11) of Theorem J as the remainder term Q_n in the following representation

$$V_n = \alpha_n^2 + Q_n \tag{4.14}$$

of the uniform Vervaat process V_n in terms of the square of the empirical process. It is well-known (cf. Zitikis, [85], for details and references) that the remainder term Q_n in (4.14) is asymptotically smaller than the main term α_n^2. Thus, just like in the case of R_n, one may like to know how small the remainder term Q_n is.

In view of Theorems H and 4.4, one suspects that there should be substantial differences between the asymptotic pointwise, sup- and L_p-norm behaviour of the process Q_n. Indeed, Csörgő and Zitikis [43] established the following strong convergence result for $\|Q_n\|_p$.

Theorem K (Csörgő and Zitikis [43]) *For any* $p\in[1,\infty)$, *we have*

$$\lim_{n\to\infty} n^{1/4}\frac{\|Q_n\|_p}{(\|\alpha_n\|_{3p/2})^{3/2}} = \frac{1}{\sqrt{3}}c_0(p) \qquad \text{a.s.,} \tag{4.15}$$

where $c_0(p)$ *is defined in* (4.8).

For a comparison of this result to that of Theorem 4.4, as well as for that of their consequences, we refer to Csörgő and Zitikis [43], who have also conjectured that in sup-norm the analogue statement of (4.15) should be of the following form:

$$\lim_{n\to\infty} b_n n^{1/4}\frac{\|Q_n\|}{\|\alpha_n\|^{3/2}} = c \qquad \text{a.s.,} \tag{4.16}$$

where b_n is a slowly varying function converging to 0 and c is a positive constant.

One of our aims in [24] was to prove that this conjecture is true with $b_n = (\log n)^{-1/2}$. In addition, we also studied the pointwise behaviour of the Vervaat error process Q_n. We summarize our results in the following theorem, which parallels Theorem H concerning the process R_n.

Theorem 4.5 *For every fixed* $t \in (0,1)$*, we have*

$$\begin{aligned} n^{1/4}Q_n(t) &\xrightarrow{\mathcal{D}} (4/3)^{1/2}(t(1-t))^{3/4}N_1(|N_2|)^{3/2}, \quad n \to \infty, && (4.17) \\ \limsup_{n\to\infty} \frac{n^{1/4}|Q_n(t)|}{(\log\log n)^{5/4}} &= (t(1-t))^{3/4}\frac{2^{11/4}3^{1/4}}{5^{5/4}} \qquad \text{a.s.}, && (4.18) \end{aligned}$$

where N_1 *and* N_2 *are independent standard normal variables. Also,*

$$\lim_{n\to\infty} n^{1/4}(\log n)^{-1/2}\frac{\|Q_n\|}{(\|\alpha_n\|)^{3/2}} = (4/3)^{1/2} \qquad \text{a.s.} \tag{4.19}$$

As a consequence of this theorem, as well as that of Theorem K combined with (4.19), we have the following corollary, which confirms the above conjecture.

Corollary 4.1 *The statement*

$$a_nQ_n \to_d Y, \quad n \to \infty,$$

cannot hold true in the space $D[0,1]$ *for any sequence* $\{a_n\}$ *of positive real numbers and for any non-degenerate random element* Y *of the space* $D[0,1]$*.*

Another consequence of (4.19) is the following corollary.

Corollary 4.2 *We have*

$$\begin{aligned} \limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{-3/4}\|Q_n\| &= \frac{2^{1/4}}{3^{1/2}} \qquad \text{a.s.}, \\ \liminf_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{3/4}\|Q_n\| &= \frac{\pi^{3/2}}{3^{1/2}2^{5/4}} \qquad \text{a.s.}, \\ n^{1/4}(\log n)^{-1/2}\|Q_n\| &\xrightarrow{\mathcal{D}} (4/3)^{1/2}\|B\|^{3/2}, \quad n \to \infty, \end{aligned}$$

where B *is a standard Brownian bridge.*

5. Banach space valued stochastic processes

Let $\{Y(t),\, t \in \mathbb{R}\} := \{X_k(t),\, t \in \mathbb{R}\}_{k=1}^{\infty}$ be a sequence of independent Ornstein-Uhlenbeck processes with coefficients γ_k and λ_k, i.e. X_k is a stationary, mean zero Gaussian process with $\mathbf{E}X_k(s)\,X_k(t) = (\gamma_k/\lambda_k)\exp(-\lambda_k|t-s|)$. This process was introduced by Dawson [45] and its path properties were studied by Csörgő and Lin [34], [35], Fernique [52], [53], [54], Iscoe et al. [57], Schmuland [73], [74]. For further development we refer to the books [67], [68] and the references therein. The basic ingredient in these investigations was the celebrated inequality of Fernique [51] and its various extensions.

In [25] we studied the infinite series

$$X(t) := \sum_{k=1}^{\infty} X_k(t), \qquad -\infty < t < \infty$$

and established certain moduli of continuity and large increment results.

In subsequent papers [11], [12], [26], [27] investigations on moduli of continuity and large increments were extended to Banach space valued processes, ℓ^2- and ℓ^p-valued processes in particular. We quote two general theorems from [26].

Theorem 5.1 *Let* $\{\Gamma(t),\, -\infty < t < \infty\}$ *be a stochastic process with values in a separable Banach space* $\mathcal{B}$ *with norm* $\|\cdot\|$*. Let* $\mathbf{P}$ *be the probability measure generated by* $\Gamma(\cdot)$*. Assume that* $\Gamma(\cdot)$ *is* $\mathbf{P}$*-almost surely continuous with respect to*

$\|\cdot\|$ *and that for* $|t| \le t_0$, $0 < x^* \le x$, *and* $0 < h \le h_0$ *there exist non-negative non-decreasing functions* $\sigma_1(h)$ *and* $\sigma_2(h)$ *such that*

$$\mathbf{P}(\|\Gamma(t+h) - \Gamma(t)\| \ge x\sigma_1(h) + \sigma_2(h)) \le K\ \exp(-\gamma x^\beta)$$

with some $K, \gamma, \beta > 0$. *Then we have*

$$\mathbf{P}\Big(\sup_{0\le t\le T}\ \sup_{0\le s\le a} \|\Gamma(t+s) - \Gamma(t)\| \ge x(\sigma_1(a) + \sigma_1(a,k))$$

$$+\sigma_1^*(a,k) + \sigma_2(a) + \sigma_2(a,k)\Big)$$

$$\le 4\left(\frac{T}{a} + 1\right) K\, 2^{2^{k+1}} \exp(-\gamma x^\beta)$$

for any $0 \le T \le t_0$, $0 < a \le h_0$, $x \ge x^*$ *and any* $k \ge 3$, *where*

$$\begin{aligned}
\sigma_1(a,k) &= 2^{3+(1/\beta)} \int_{2^{k-3}}^{\infty} \frac{\sigma_1(ae^{-z})}{z}\, dz,\\
\sigma_2(a,k) &= 6\int_{2^{k-3}}^{\infty} \frac{\sigma_2(ae^{-z})}{z}\, dz,\\
\sigma_1^*(a,k) &= 4\left(\frac{14}{\gamma}\right)^{1/\beta} \beta \int_{2^{(k-2)/\beta}}^{\infty} \sigma_1(ae^{-z^\beta})\, dz.
\end{aligned}$$

Before stating the next theorem, we give a definition: A function $f(x)$ is called *quasi-increasing on* (a,b) if there exists a positive number c such that

$$f(x) \le cf(y) \quad \text{for all} \quad a < x < y < b.$$

Theorem 5.2 *Assume the conditions of* Theorem 5.1 *with* $t_0 = \infty$ *and that* $\sigma_1(h)$ *and* $\sigma_2(h)$ *are continuous functions such that* $\sigma_1(h)/h^\alpha$ *and* $\sigma_2(h)/h^\alpha$ *are quasi-increasing for some* $\alpha > 0$. *Let* a_T *and* b_T *be continuous functions of* T *such that*

$$\frac{b_T}{a_T} + \sigma_1(a_T) + \frac{1}{\sigma_1(a_T)} \to \infty \quad \text{as} \quad T \to \infty$$

and

$$\limsup_{T\to\infty} a_T \le h_0.$$

Then we have

$$\limsup_{T\to\infty}\ \sup_{0\le t\le b_T}\ \sup_{0\le s\le a_T} \beta_T \|\Gamma(t+s) - \Gamma(t)\| \le 1 \qquad \text{a.s.},$$

where

$$\beta_T = \sigma_1(a_T)A_T + \sigma_2(a_T),$$

$$A_T = \left(\frac{1}{\gamma}\left(\log\left(1 + \frac{b_T}{a_T}\right) + \log\log\left(\sigma_1(a_T) + \frac{1}{\sigma_1(a_T)}\right)\right)\right)^{1/\beta}.$$

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