Level Crossings of a Two-Parameter Random Walk

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Abstract

We prove that the number $Z(N)$ of level crossings of a two-parameter simple random walk in its first $N \times N$ steps is almost surely $N^{\frac{3}{2}+o(1)}$ as $N \to \infty$. The main ingredient is a strong approximation of $Z(N)$ by the crossing local time of a Brownian sheet. Our result provides a useful algorithm for simulating the level sets of the Brownian sheet.

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1 Introduction

Recall that a two-parameter real-valued Brownian sheet $W = \{W(s,t);\ s,t \geq 0\}$ is a centered Gaussian process with covariance

$$\mathrm{E}\{W(s,t)W(s',t')\} = \min(s,s') \times \min(t,t'), \qquad \forall s,t,s',t' \geq 0. \tag{1.1}$$

It is known that the level sets of W have a rich and complicated structure. For instance, if $W^{-1}\{a\} := \{s,t) \in \mathbb{R}_+^2 : W(s,t) = a\}$, then it follows that with probability one,

$$\dim\left(W^{-1}\{a\}\right) = \frac{3}{2}, \qquad \forall a \in \mathbb{R}; \tag{1.2}$$

cf. [1, 12]. Here, dim refers to Hausdorff dimension. Other, more delicate, features of the level sets can be found in [5–11, 13].

One expects that the level sets of 2-parameter random walks are uniform in local time. Informally speaking, this and (1.2) together imply that for any reasonable discrete approximation $\mathcal{A}_N$ of $W^{-1}\{0\} \cap [0,1]^2$, one might expect that $\#\mathcal{A}_N \approx N^{\frac{3}{2}}$, as $N \to \infty$; here, # denotes cardinality, and "≈" stands for any reasonable notion of asymptotic equivalence.

This paper is motivated, in part, by our desire to find a good algorithm for simulating the zero-set of W inside a given box that we take to be $[0,1]^2$ to be concrete. A natural way to try

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and do this is by first performing a random-walk approximation to W, and then approximating the zero-set of W by that of the walk.

With this in mind, let $\{X_{i,j};\ i,j \geq 1\}$ denote an array of i.i.d. random variables with $\mathrm{P}\{X_{i,j} = 1\} = \mathrm{P}\{X_{i,j} = -1\} = \frac{1}{2}$, and consider the two-parameter random walk $\{S_{m,n};\ m,n \geq 0\}$ defined as

$$S_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{i,j}, \qquad \forall m,n \geq 1, \tag{1.3}$$

with the added stipulation that $S_{m,n} = 0$ whenever $mn = 0$. It is then possible to show that as $N \to \infty$,

$$\{N^{-1} S_{\lfloor Ns \rfloor, \lfloor Nt \rfloor};\ 0 \leq s,t \leq 1\} \Longrightarrow \{W(s,t);\ 0 \leq s,t \leq 1\}, \tag{1.4}$$

where $\Rightarrow$ denotes weak convergence in a suitable space. Here, convergence in $\mathcal{D}_{\mathcal{D}_{\mathbb{R}}[0,1]}([0,1])$ will do, but we will not need this fact in the sequel; see [14, Theorem 4.1.1, Chapter 6] for a variant of this statement. Suffice it to say that the factor of N^{-1} is the central limit scaling that comes from adding $O(N^2)$ i.i.d. variates.

A natural approximation of the zero-set $W^{-1}\{0\} \cap [0,1]^2$ would then be the random set

$$\Upsilon_N = \{(i,j) \in \{0, \dots, N\}^2 :\ S_{i,j} = 0\}, \tag{1.5}$$

where N is a large integer. While this algorithm is intuitively attractive, it does *not* perform well. Indeed, by the local limit theorem ([19, Theorem 2.8]),

$$\mathrm{E}\{\#\Upsilon_N\} \sim (2\pi)^{-\frac{1}{2}} \sum_{i=1}^{N} \sum_{j=1}^{N} (ij)^{-\frac{1}{2}} \sim 4(2\pi)^{-\frac{1}{2}} N, \qquad \text{as } N \to \infty. \tag{1.6}$$

One can use this, in conjunction with a monotonicity argument, to show that with probability one,

$$\lim_{N \to \infty} N^{-\frac{3}{2}} \#\Upsilon_N = 0. \tag{1.7}$$

In light of (1.2) and its proceeding discussion, (1.7) suggests that Υ_N might be too thin to properly simulate the zero-set $W^{-1}\{0\} \cap [0,1]^2$ of the Brownian sheet.

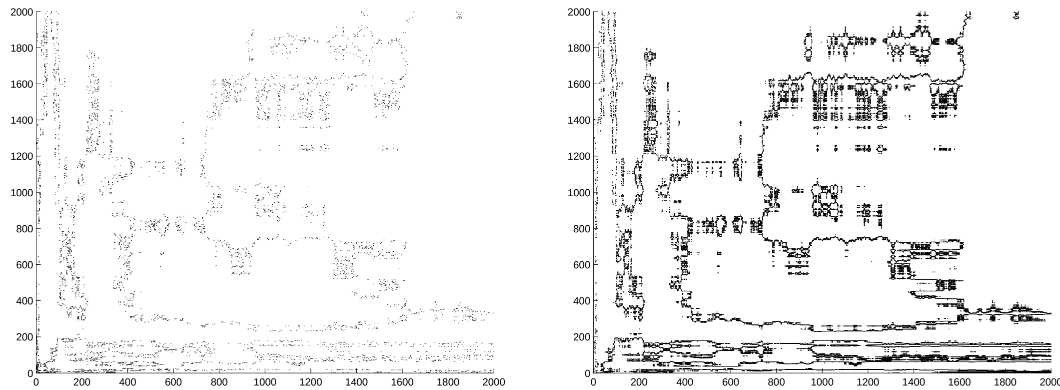
In this paper, we present an alternative algorithm for simulating the zero-set of W, and show that our approximation has the correct size of $N^{\frac{3}{2}+o(1)}$ as $N \to \infty$. Our suggested approximation is a natural one that is based on the "crossing numbers" of the approximating two-parameter random walk S.

A lattice point (i,j) is called a (vertical) *crossing* for the random walk S if

$$S_{i,j} S_{i,j+1} < 0. \tag{1.8}$$

Define

$$\begin{aligned}
\Xi_N &:= \big\{(i,j) \in [0,N]^2 \cap \mathbb{Z}^2 :\ (i,j) \text{ is a crossing}\big\} \\
\zeta_i(N) &:= \#\big\{j \in [0,N] \cap \mathbb{Z}^2 :\ (i,j) \text{ is a crossing}\big\} \\
Z(N) &:= \zeta_1(N) + \cdots + \zeta_N(N).
\end{aligned} \tag{1.9}$$



(a) The actual zeros

(b) The vertical crossings

Figure 1: The zeros vs. the vertical crossings

In words, $Z(N) = \#\Xi_N$ is the total number of crossings of the random walk in the first $N \times N$ steps. We propose to show that Ξ_N is a good approximation to the zero-set of the Brownian sheet in $[0,1]^2$, at least in the sense that $Z(N) = \#\Xi_N$ is sufficiently thick in the following asymptotically sense.

Theorem 1.1 *With probability one,* $Z(N) = N^{\frac{3}{2}+o(1)}$ *as* $N \to \infty$.

Figure 1 shows the simulation of the level set of a two-parameter simple walk, together with the vertical crossings of the same random walk. The figure speaks for itself, and the Matlab code is added as a brief appendix at the end of the paper.

Throughout this paper, we write $\log x := \ln(x \vee e)$.

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2 Brownian sheet and invariance

To prove Theorem 1.1, we need to analyse the crossings of the walk simultaneously at all levels. With this in mind, for each $x \in \mathbb{R}$ we say that (i,j) is a (vertical) *x-crossing* for the random walk if

$$(S_{i,j} - x)(S_{i,j+1} - x) < 0. \tag{2.1}$$

Next, we can define

$$\begin{aligned} \zeta_i(x;n) &:= \#\{j : 0 \le j < n : (i,j) \text{ is an } x\text{-crossing}\} \\ Z(x;m,n) &:= \sum_{i=0}^{m} \zeta_i(x;n). \end{aligned} \tag{2.2}$$

Thus, $Z(x; m, n)$ denotes the number of x-crossings in the first $m \times n$ steps. We remark that $Z(N) = Z(0; N, N)$.

When N is large, the entire process $(x, s, t) \mapsto Z(x; \lfloor Ns \rfloor, \lfloor Nt \rfloor)$ is close to the crossing local times of a Brownian sheet that we describe next.

For any fixed $s > 0$, $\{L_s(x; t);\, t \geq 0\}$ denotes the local time at 0 of the process $t \mapsto W(s, t)$. This is the density of the occupation measure, as described by the formula,

$$\int_{-\infty}^{+\infty} f(x) L_s(x; t)\, dx = \int_0^t f(W(s, u))\, du. \tag{2.3}$$

Reference [21] introduces the process L_s as the *line local time* of W.

We define the *crossing local time* of W at level x as

$$C(x; s, t) = \int_0^s \sqrt{u}\, L_u(x; t)\, du. \tag{2.4}$$

The following strong approximation constitutes a central portion of this paper.

Theorem 2.1 *Possibly in an enlarged probability space, there exists a coupling for the two-parameter random walk S and the Brownian sheet W such that for any $\varepsilon > 0$, the following holds almost surely: As $N \to \infty$,*

$$\begin{aligned} \max_{1 \leq m \leq N} \max_{1 \leq n \leq N} |S_{m,n} - W(m, n)| &= O\left(N^{\frac{1}{2}} (\log N)^{\frac{3}{2}}\right) \\ \max_{\substack{1 \leq m \leq N \\ 1 \leq n \leq N}} \sup_{x \in \mathbb{R}} |Z(x; m, n) - C(x; m, n)| &= o\left(N^{\frac{4}{3}+\varepsilon}\right), \end{aligned} \tag{2.5}$$

where $Z(x; m, n)$ is the x-crossing number of the random walk, and $C(x; s, t)$ is the crossing local time of W.

Let us say a few words about the proof of (2.5). Recall that $Z(x; m, n) = \sum_{i=0}^m \zeta_i(x; n)$, where for each i, $n \mapsto \zeta_i(x; n)$ is the number of x-level crossings of the one-parameter, simple random walk $\{S_{i,j}\}_{j=0,1,2,\dots}$. There is a rich literature for level crossings of such random walks. For example, we can appeal to [4, Theorem 1.2] to see that for each fixed i,

$$\zeta_i(x; n) \approx \sqrt{i}\, L_i(x; n). \tag{2.6}$$

In words, this means that $\zeta_i(x; n)$ can be well approximated by $\sqrt{i}\, L_i(x; n)$, where $L_i(x; n)$ is the local time of a Brownian motion with infinitesimal variance i. One might then conjecture that such local times can be embedded in the line local times L_i of a single Brownian sheet W; cf. (2.3). If this were so,

$$Z(x; m, n) = \sum_{i=0}^m \zeta_i(x; n) \approx \sum_{i=0}^m \sqrt{i}\, L_i(x; n) \approx \int_0^m \sqrt{s}\, L_s(x; n)\, ds = C(x; m, n). \tag{2.7}$$

Such a route is fraught with technical difficulties. For one, it is not clear why (2.6) should hold jointly for all i if L_i is the line local time of a single Brownian sheet. Moreover, the rate of approximation in (2.6) depends in a subtle way on i, and it is not at all clear what information can be gleaned from this about the rate of approximation in (2.7).

Our method provides an approach that is based on our attempt at solving the following loosely-stated problem: "*How close are the local times* $L(X_1)$ *and* $L(X_2)$ *of two processes* X_1 *and* X_2 *if* $X_1 \approx X_2$, *and if* $L(X_1)$ *and* $L(X_2)$ *are sufficiently smooth?*". Interestingly enough, our solution to the mentioned problem uses (2.6), but only for a fixed value of i.

The remainder of the paper is organized as follows: The two parts of Theorem 2.1, namely (2.5) and (2.5), are proved in distinct sections. The proof of (2.5), which is straightforward, is given in Section 3. We prove (2.5) in Section 4 by means of two technical lemmas. The proofs of these lemmas are provided in Sections 5 and 6, respectively. Finally, we prove Theorem 1.1 in Section 7, by applying (2.5), and a recent estimate of the authors on the explosion rate of the local time along lines of W; cf. [15].

3 Proof of Theorem 2.1 (2.5): First Part

Our proof of the first part of (2.5) relies on *Bernstein's inequality* that we now recall; cf. [20, p. 855].

Let $\{\eta_k;\, k \geq 1\}$ be a sequence of independent mean-zero variables such that for some $c > 0$, $\mathrm{E}\{|\eta_k|^n\} \leq \frac{1}{2} v_k n!\, c^{n-2}$ for all $n \geq 2$. Then, for any $x > 0$ and $n \geq 1$,

$$\mathrm{P}\left\{\left|\sum_{k=1}^{n} \eta_k\right| \geq x\right\} \leq 2\exp\left(-\frac{1}{2}\frac{x^2}{\sum_{k=1}^{n} v_k + cx}\right). \tag{3.1}$$

Fix $j \geq 1$, and consider the sequence $\{X_{i,j};\, i \geq 1\}$ of i.i.d. random variables. According to [16], possibly in an enlarged probability space, there exists a standard Brownian motions W_j such that for all $x > 0$ and $m \geq 1$,

$$\mathrm{P}\left\{\left|\sum_{i=1}^{m} X_{i,j} - W_j(m)\right| > c_1 \log m + x\right\} \leq c_2 e^{-c_3 x}, \tag{3.2}$$

where c_1, c_2 and c_3 are constants that do not depend on x, m. Since $\{X_{i,j}\,;\, i, j \geq 1\}$ are i.i.d., we can arrange things so that $\{W_j\,;\, j \geq 1\}$ are independent processes.

Now, let us fix $m \geq 1$, and consider the process $\eta = \{\eta_j\}_{j \geq 1}$ where $\eta_j = \sum_{i=1}^{m} X_{i,j} - W_j(m)$. Clearly, η is a sequence of independent (but not necessarily i.d.) mean zero variables. According to [16] for any $n \geq 2$ and $j \geq 1$,

$$\begin{aligned} \mathrm{E}\{|\eta_j|^n\} &= \int_0^\infty n y^{n-1} \mathrm{P}\{|\eta_j| > y\}\, dy \\ &\leq \int_0^{c_1 \log m} n y^{n-1}\, dy + \int_0^\infty n(c_1 \log m + x)^{n-1} c_2\, e^{-c_3 x}\, dx. \end{aligned} \tag{3.3}$$

The first integral equals $(c_1 \log m)^n$, whereas the second is

$$\begin{aligned}
&\left(\int_0^{c_1 \log m} + \int_{c_1 \log m}^{\infty}\right) n(c_1 \log m + x)^{n-1} c_2\, e^{-c_3 x}\, dx \\
&\qquad \le n(2c_1 \log m)^{n-1} c_2 \int_0^{c_1 \log m} e^{-c_3 x}\, dx + nc_2 \int_{c_1 \log m}^{\infty} (2x)^{n-1} e^{-c_3 x}\, dx \\
&\qquad \le n(2c_1 \log m)^{n-1} c_2 \int_0^{\infty} e^{-c_3 x}\, dx + nc_2 \int_0^{\infty} (2x)^{n-1} e^{-c_3 x}\, dx \\
&\qquad \le \frac{n(2c_1 \log m)^{n-1} c_2}{c_3} + \frac{n!\, c_2 2^{n-1}}{(c_3)^n}.
\end{aligned} \tag{3.4}$$

Therefore, there exists an absolute constant c_4 such that

$$\mathrm{E}\left(|\eta_j|^n\right) \le \frac{c_4\,(\log m)^2}{2}\,(c_4 \log m)^{n-2} n!\,, \qquad \forall n \ge 2. \tag{3.5}$$

By Bernstein's inequality (cf. 3.1), for any $x > 0$ and $m, n \ge 1$,

$$\mathrm{P}\left\{\left|\sum_{j=1}^{n}\sum_{i=1}^{m} X_{i,j} - \sum_{j=1}^{n} W_j(m)\right| \ge x\right\} \le 2\exp\left(-\frac{1}{2c_4}\frac{x^2}{n(\log m)^2 + x\log m}\right). \tag{3.6}$$

We can embed $\{\sum_{j=1}^{\ell} W_j(k);\ j \ge 1,\ k \ge 1\}$ in a two-parameter Brownian sheet W, possibly in an enlarged probability space. (In the paper, we often use several embedding schemes in enlarged spaces. This can be justified by a coupling argument as in [3, p. 53].) Therefore, for any $x > 0$ and $N \ge 1$,

$$\mathrm{P}\left\{\max_{1\le m\le N}\max_{1\le n\le N} |S_{m,n} - W(m,n)| > x\right\} \le 2N^2 \exp\left(-\frac{1}{2c_4}\frac{x^2}{N(\log N)^2 + x\log N}\right). \tag{3.7}$$

Take $x = (7c_4)^{\frac{1}{2}} N^{\frac{1}{2}} (\log N)^{\frac{3}{2}}$, so that the expression on the right hand side is summable in N. Applying the Borel–Cantelli lemma readily yields the first assertion of (2.5). □

4 Proof of Theorem 2.1 (2.5): The Second Part

In this section, we prove the second assertion of (2.5) of Theorem 2.1. This is done by virtue of two technical estimates—Propositions 4.1 and 4.2—as well as two supporting lemmas.

Our first proposition controls the oscillations of the process C, viz.,

Proposition 4.1 *For any $\alpha, \varepsilon > 0$, the following holds almost surely: As $N \to \infty$,*

$$\max_{1\le m\le N}\max_{1\le n\le N}\sup_{|x-y|\le N^\alpha} |C(x;m,n) - C(y;m,n)| = o\left(N^{1+\max(\frac{\alpha}{2},\frac{1}{4})+\varepsilon}\right). \tag{4.1}$$

Theorem 2.1 asserts that the processes Z and C are asymptotically close to one another. The following analogue of Propostion 4.1 states that their asymptotic moduli of continuity are also close, viewed on an appropriate scale.

Proposition 4.2 *Given $\alpha, \varepsilon > 0$, the following holds almost surely: As $N \to \infty$,*

$$\max_{0 \le m \le N} \max_{1 \le n \le N} \sup_{|x-y| \le N^\alpha} |Z(x;m,n) - Z(y;m,n)| = o\left(N^{1+\max(\frac{\alpha}{2},\frac{1}{4})+\varepsilon}\right). \tag{4.2}$$

We postpone proving these Propositions until Sections 5 and 6, respectively. In the remainder of this section, we use the latter propositions to prove the second assertion of (2.5) of Theorem 2.1.

Lemma 4.3 *For any fixed $\theta > \frac{1}{2}$, the following is almost surely valid:*

$$\max_{0 \le i,j \le N} \sup_{i \le u \le i+1} \sup_{j \le v \le j+1} |W(u,v) - W(i,j)| \le N^\theta, \qquad \forall N \text{ large.} \tag{4.3}$$

Proof We can appeal to the proof of [18, Lemma 1.2] to deduce that for any $a, b, \lambda > 0$,

$$\mathrm{P}\left\{ \sup_{(s,t) \in [0,a] \times [0,b]} |W(s,t)| > \lambda \right\} \le 4\mathrm{P}\{|W(a,b)| > \lambda\}. \tag{4.4}$$

Therefore, for any $i, j \le N$,

$$\begin{aligned} &\mathrm{P}\left\{ \sup_{i \le u \le i+1} \sup_{j \le v \le j+1} |W(u,v) - W(i,j)| > N^\theta \right\} \\ &\quad \le 4\mathrm{P}\left\{|W(1,1)| > N^\theta\right\} + 4\mathrm{P}\left\{|W(i,1)| > N^\theta\right\} + 4\mathrm{P}\left\{|W(1,j)| > N^\theta\right\} \\ &\quad \le 12 \exp\left(-\tfrac{1}{2} N^{2\theta - 1}\right). \end{aligned} \tag{4.5}$$

Consequently,

$$\sum_{N \ge 1} \mathrm{P}\left\{ \max_{0 \le i,j \le N} \sup_{i \le u \le i+1} \sup_{j \le v \le j+1} |W(u,v) - W(i,j)| > N^\theta \right\} \le 12 \sum_{N \ge 1} N^2 \exp\left(-\tfrac{1}{2} N^{2\theta-1}\right), \tag{4.6}$$

which is finite. The Borel–Cantelli lemma completes our verification. □

Our final lemma is an almost-sure uniform bound for Z.

Lemma 4.4 *For any fixed $\varepsilon > 0$, the following holds almost surely: As $N \to \infty$,*

$$\sup_{x \in \mathbb{R}} Z(x; N, N) = o\big(N^{\frac{3}{2}+\varepsilon}\big). \tag{4.7}$$

Proof Applying the local limit theorem, it is not hard to see that

$$\mathrm{E}\{Z(0;N,N)\} = \sum_{i=0}^{N} \sum_{j=0}^{N-1} \mathrm{P}\{S_{i,j} S_{i,j+1} < 0\} \le \sum_{i=0}^{N} \sum_{j=0}^{N-1} \frac{c_5}{\sqrt{j+1}} \le c_6 N^{\frac{3}{2}}, \tag{4.8}$$

where c_5 and c_6 are two unimportant constants; cf. [19, Theorem 2.8] for the local limit theorem. To this, we apply Markov's inequality to obtain

$$\mathrm{P}\big\{Z(0;N,N) > N^{\frac{3}{2}+\varepsilon}\big\} \le c_6 N^{-\varepsilon}. \tag{4.9}$$

Let $N_k = \lfloor k^{2/\varepsilon} \rfloor$ and use the Borel–Cantelli lemma to see that, with probability one, for all large k, $Z(0; N_k, N_k) \le N_k^{\frac{3}{2}+\varepsilon}$. By the monotonicity of $N \mapsto Z(0; N, N)$,

$$Z(0; N, N) = O(N^{\frac{3}{2}+\varepsilon}), \qquad \text{a.s.} \tag{4.10}$$

Proposition 4.2 then shows that

$$\sup_{|x| \le N^{1+\varepsilon}} |Z(x; N, N)| = o(N^{\frac{3}{2}+2\varepsilon}), \qquad \text{a.s.} \tag{4.11}$$

It remains to replace $\sup_{|x| \le N^{1+\varepsilon}}$ by $\sup_{x \in \mathbb{R}}$ in the above.

By the law of the iterated logarithm law for Brownian sheet,

$$\limsup_{s,t \to \infty} \frac{|W(s,t)|}{\sqrt{4st \log\log(st)}} = 1, \qquad \text{a.s.;} \tag{4.12}$$

cf. [22]. This, together with (2.5), implies that

$$\max_{1 \le m \le N} \max_{1 \le n \le N} |S_{m,n}| = o(N^{1+\varepsilon}), \qquad \text{a.s.} \tag{4.13}$$

Therefore, with probability one, when N is sufficiently large, $Z(x; N, N) = 0$ for all $|x| \ge N^{1+\varepsilon}$. Accordingly, (4.11) implies our lemma since $\varepsilon > 0$ is arbitrary. □

We are in position for presenting our

Proof of Theorem 2.1 (2.5): Second Part. By our definition (cf. 2.4) of the crossing local time $C(x; s, t)$, for any Borel set $A \subseteq \mathbb{R}$,

$$\int_A C(a; s, t)\, da = \int_0^s du \int_0^t dv\, \sqrt{u}\, \mathbf{1}_{\{W(u,v) \in A\}}. \tag{4.14}$$

In particular, for any $x \in \mathbb{R}$ and $\beta > \frac{1}{2}$,

$$\int_x^{x+N^\beta} C(a; m, n)\, da = \int_0^m du \int_0^n dv\, \sqrt{u}\, \mathbf{1}_{\{x \le W(u,v) \le x+N^\beta\}}. \tag{4.15}$$

We apply Proposition 4.1 to this and deduce that with probability one the following exists for all N large: For all $m, n \le N$,

$$N^\beta C(x; m, n) + o\left(N^{\beta+1+\max(\frac{\beta}{2},\frac{1}{4})+\varepsilon}\right) = \int_0^m du \int_0^n dv \sqrt{u}\, \mathbf{1}_{\{x \le W(u,v) \le x+N^\beta\}}. \tag{4.16}$$

Lemma 4.3 then shows us that almost surely, as $N \to \infty$,

$$\begin{aligned} &\sum_{i=0}^m \sum_{j=0}^n \sqrt{i}\, \mathbf{1}_{\{x+N^\theta \le W(i,j) \le x+N^\beta - N^\theta\}} + O\big(N^{\frac{3}{2}}\big) \\ &\qquad \le \int_0^m du \int_0^n dv\, \sqrt{u}\, \mathbf{1}_{\{x \le W(u,v) \le x+N^\beta\}} \\ &\qquad\quad \le \sum_{i=0}^m \sum_{j=0}^n \sqrt{i}\, \mathbf{1}_{\{x-N^\theta \le W(i,j) \le x+N^\beta+N^\theta\}} + O\big(N^{\frac{3}{2}}\big). \end{aligned} \tag{4.17}$$

On the other hand, according to (2.5), for any $a < b$, almost surely as $N \to \infty$,

$$\mathbf{1}_{\{a+N^\theta \le S_{i,j} \le b-N^\theta\}} \le \mathbf{1}_{\{a \le W(i,j) \le b\}} \le \mathbf{1}_{\{a-N^\theta \le S_{i,j} \le b+N^\theta\}}, \tag{4.18}$$

uniformly in $i, j \le N$. Recall $Z(x; m, n)$ from (2.2) and note that

$$\int_A Z(a; m, n)\, da = \sum_{i=0}^{m} \sum_{j=0}^{n} \sqrt{i}\, \mathbf{1}_{\{S_{i,j} \in A\}}. \tag{4.19}$$

¿From this it follows that with probability one, as $N \to \infty$,

$$\begin{aligned} \int_{x+2N^\theta}^{x-2N^\theta+N^\beta} Z(a; m, n)\, da &\le N^\beta C(x; m, n) + o\left(N^{\beta+1+\max(\frac{\beta}{2}, \frac{1}{4})+\varepsilon}\right) \\ &\le \int_{x-2N^\theta}^{x+2N^\theta+N^\beta} Z(a; m, n)\, da, \end{aligned} \tag{4.20}$$

uniformly in $m, n \le N$. Combining Proposition 4.2 with the above, we arrive at

$$\begin{aligned} (N^\beta - 4N^\theta) Z(x; m, n) &\le N^\beta C(x; m, n) + o\left(N^{\beta+1+\max(\frac{\beta}{2}, \frac{1}{4})+\varepsilon}\right) \\ &\le (N^\beta + 4N^\theta) Z(x; m, n), \end{aligned} \tag{4.21}$$

uniformly in $m, n \le N$ and in $x \in \mathbb{R}$. Consequently, by Lemma 4.4, for any $\varepsilon > 0$,

$$\begin{aligned} &\max_{0 \le m \le N} \max_{1 \le n \le N} \sup_{x \in \mathbb{R}} |Z(x; m, n) - C(x; m, n)| \\ &\quad \le 4N^{-(\beta-\theta)} \sup_{x \in \mathbb{R}} Z(x; N, N) + o\left(N^{1+\max(\frac{\beta}{2}, \frac{1}{4})+\varepsilon}\right) \\ &\quad = o\left(N^{-(\beta-\theta-\frac{3}{2}-\varepsilon)}\right) + o\left(N^{1+\max(\frac{\beta}{2}, \frac{1}{4})+\varepsilon}\right). \end{aligned} \tag{4.22}$$

We could take $\beta = \frac{2}{3} + \varepsilon$ and $\theta = \frac{1}{2} + \varepsilon$. Since $\varepsilon > 0$ can be chosen arbitrarily small, this yields (2.5). □

5 Proof of Proposition 4.1

Let us start by fixing the basic notation in this section. For any real-valued random variable η, we write

$$\|\eta\|_p = \{\mathrm{E}[|\eta|^p]\}^{\frac{1}{p}}, \quad \text{and} \quad \langle \eta \rangle_{\mathsf{Orl}} = \inf\left\{a > 0 :\ \mathrm{E}\left[\exp\left(a^{-1}|\eta|\right)\right] \le 2\right\}. \tag{5.1}$$

They stand for the $L^p(\mathrm{P})$-norm and the Orlicz (pseudo-)norm (associated with the convex function $f(x) = e^x - 1$) of η, respectively. (To be precise, $\langle \bullet \rangle_{\mathsf{Orl}}$ is *not* a norm, but it is equivalent to one.) The two norms are related to each other as follows: There exists an absolute constant c_7 such that

$$\langle \eta \rangle_{\mathsf{Orl}} \le c_7 \sup_{m \ge 1} \frac{\|\eta\|_m}{m}. \tag{5.2}$$

Our proof of Proposition 4.1 is based on the following convenient formulation of Dudley's metric entropy theorem.

Lemma 5.1 ([17, Theorem 3.1]) *Let $\{X(t);\, t \in T\}$ be a real-valued stochastic process, and let $d_X(s,t) = \langle X_s - X_t \rangle_{\mathsf{Orl}}$ be the natural pseudo-metric on T induced by X. Let $\mathcal{N}(r) = N(r, T, d_X)$ be the minimum number of d_X-balls of radius r needed to cover T, and let*

$$D := \sup_{(s,t)\in T^2} d_X(s,t) \quad \textit{and} \quad m_D := \int_0^D \log \mathcal{N}(r)\, dr. \tag{5.3}$$

If $m_D < \infty$, then there exists a universal constant c_8, such that for all $\lambda > 0$,

$$\mathrm{P}\left\{ \sup_{(s,t)\in T^2} |X_s - X_t| > c_8\left(\lambda + D + m_D\right) \right\} \le c_8 e^{-\lambda/D}. \tag{5.4}$$

Fix $1 \le i, n \le N$, and choose $k \in \{0, \pm 1, \pm 2, \cdots \pm k_{\max}\}$ where $k_{\max} := \lfloor N^2/n^{\frac{1}{2}} \rfloor$. We intend to apply Fact 5.1 to the process $\{L_s(x;n);\, (s,x) \in T = [i, i+1] \times [k\sqrt{n}\,, (k+1)\sqrt{n}\,]\}$, where $L_s(x;n)$ is the line local time of the Brownian sheet W. This was defined in (2.3).

We begin by estimating the entropy number $\mathcal{N}(r)$ induced by d_X. This will be done in successive steps.

Choose some $(s,x) \in T$ and $(t,y) \in T$. In order to bound $d_X((s,x),(t,y))$, we start by estimating $\|L_s(x;n) - L_t(y;n)\|_p$ for $p \ge 1$. This can be reduced to estimating

$$\begin{aligned} I_1(p) &= \|L_s(x;n) - L_s(y;n)\|_p, \quad \text{and} \\ I_2(p) &= \|L_s(y;n) - L_t(y;n)\|_p. \end{aligned} \tag{5.5}$$

Lemma 5.2 *For every $\nu \in (0, \frac{1}{2})$, there exists a constant $c_9 = c_9(\nu) \in (0,\infty)$ such that for all $x, y \in \mathbb{R}$, for all integers $i, n \ge 1$, $s \in [i, i+1]$, and $p \ge 1$,*

$$I_1(p) \le c_9\, p\, \frac{n^{\frac{1}{2}(1-\nu)}}{i^{\frac{1}{2}(1+\nu)}}\, |x-y|^\nu. \tag{5.6}$$

Proof Writing $\overset{(d)}{=}$ for equality of distributions, and use scaling to deduce that

$$L_s(x;n) - L_s(y;n) \overset{(d)}{=} \sqrt{\frac{n}{s}}\left[L_1\left(\frac{x}{\sqrt{sn}}, 1\right) - L_1\left(\frac{y}{\sqrt{sn}}, 1\right)\right]. \tag{5.7}$$

Since $L_1(x,t)$ is a standard Brownian local time, we can use the following inequality of [2]: For any $\nu \in [0, \frac{1}{2})$ and $p \ge 1$,

$$c_9(\nu) = \sup_{p\ge 1} \frac{1}{p} \left\| \sup_{0\le t\le 1} \sup_{x\ne y} \frac{|L_1(x,t) - L_1(y,t)|}{|x-y|^\nu} \right\|_p < \infty. \tag{5.8}$$

Consequently, and writing $c_9 = c_9(\nu)$ for brevity, we have

$$I_1(p) \le c_9\, p\, \frac{n^{\frac{1}{2}(1-\nu)}}{s^{\frac{1}{2}(1+\nu)}}\, |x-y|^\nu. \tag{5.9}$$

Since $s \in [i, i+1]$, this has the desired result. $\square$

Our estimate for $I_2(p)$ of (5.5) is derived by similar methods.

Lemma 5.3 *For every $\lambda \in (0, \frac{1}{4})$, there exists a constant $c_{10} = c_{10}(\lambda) \in (0, \infty)$ such that for all $p \geq 1$, integers $n, i \geq 1$, and $s, t \in [i, i+1]$,*

$$I_2(p) \leq c_{10}\, p\, n^{\frac{1}{2}} i^{-\lambda-\frac{1}{2}} (t-s)^{\lambda}. \tag{5.10}$$

Proof We recall the following result of [17, Proposition 4.2]: For any $p \geq 1$, $0 < h < 1$ and $0 < \lambda < \frac{1}{4}$,

$$\sup_{x \in \mathbb{R}} \| L_{1+h}(x;1) - L_1(x;1) \|_p \leq c_{10}\, h^{\lambda} p. \tag{5.11}$$

Assuming $s \leq t$ without loss of generality, and using the scaling property, we have: For all $i \leq s \leq t \leq i+1$ and $y \in \mathbb{R}$,

$$I_2(p) = \sqrt{\frac{n}{s}} \left\| L_1\left(\frac{y}{\sqrt{sn}}, 1\right) - L_{t/s}\left(\frac{y}{\sqrt{sn}}, 1\right) \right\|_p \leq c_{10}\, p\, n^{\frac{1}{2}} s^{-\lambda-\frac{1}{2}} (t-s)^{\lambda}. \tag{5.12}$$

Apply this to $s, t \in [i, i+1]$ to finish. □

Lemma 5.4 *For every $\nu \in (0, \frac{1}{2})$, there exists a constant $c_{12} = c_{12}(\nu) \in (0, \infty)$ such that for all integers $i, n \geq 1$, $x, y \in \mathbb{R}$, and $(s,t) \in [i, i+1]^2$,*

$$\big\langle L_s(x;n) - L_t(y;n) \big\rangle_{\mathsf{Orl}} \leq c_{12} \frac{n^{\frac{1}{2}(1-\nu)} |x-y|^{\nu} + n^{\frac{1}{2}} |t-s|^{\frac{1}{2}\nu}}{i^{\frac{1}{2}(1+\nu)}}. \tag{5.13}$$

Proof We choose $\lambda = \frac{\nu}{2}$, and appeal to Lemmas 5.2 and 5.3, as well as Minkowski's inequality to deduce that

$$\begin{aligned} \| L_s(x;n) - L_t(y;n) \|_p &\leq I_1(p) + I_2(p) \\ &\leq c_{11} p\, \frac{n^{\frac{1}{2}(1-\nu)} |x-y|^{\nu} + n^{\frac{1}{2}} (t-s)^{\frac{1}{2}\nu}}{i^{\frac{1}{2}(1+\nu)}}, \end{aligned} \tag{5.14}$$

where $c_{11} := c_9 + c_{10}$. This immediately yields our lemma. □

We now use this to estimate the asymptotic modulus of continuity of the line local times.

Lemma 5.5 *Given a fixed $\nu \in (0, \frac{1}{2})$, there exists a constant $c_{16} = c_{16}(\nu) \in (0, \infty)$ such that almost surely for all large N, and $1 \leq i, n \leq N$,*

$$\sup_{(s,t) \in [i,i+1]^2} \sup_{x \in \mathbb{R}} | L_s(x;n) - L_t(x;n) | \leq c_{16} \frac{N^{\frac{1}{2}} \log N}{i^{\frac{1}{2}(1+\nu)}}. \tag{5.15}$$

Proof We prove this by applying Lemma 5.1 to the process $(x, s) \mapsto X_{x,s} = L_s(x;n)$, where $n \geq 1$ is an arbitrary integer.

Recall D and m_D from (5.3), and note that thanks to Lemma 5.4,

$$D \leq c_{12} \frac{n^{\frac{1}{2}(1-\nu)} (\sqrt{n})^{\nu} + n^{\frac{1}{2}}}{i^{\frac{1}{2}(1+\nu)}} \leq 2 c_{12} \frac{N^{\frac{1}{2}}}{i^{\frac{1}{2}(1+\nu)}}. \tag{5.16}$$

Next, we estimate m_D: Owing to Lemma 5.4, the minimum number $\mathcal{N}(r)$ of d_X-balls of radius r needed to cover T satisfies

$$\mathcal{N}(r) \le c_{13}\, n^{\frac{1}{2}} \left(\frac{\frac{1}{2} n^{(3-\nu)}}{r^3 i^{\frac{3}{2}(1+\nu)}}\right)^{\frac{1}{\nu}} \le c_{13}\, N^{\frac{1}{2}} \left(\frac{N^{\frac{1}{2}(3-\nu)}}{r^3}\right)^{\frac{1}{\nu}}. \tag{5.17}$$

Thus, $m_D \le \int_0^D \log(\mathcal{N}(r))\, dr \le c_{14}\, D \log N$, where c_{14} depends only on ν. Applying Lemma 5.1 with $\lambda := 6D \log N$ yields the following:

$$\begin{aligned} &\mathrm{P}\{A_{ikn}\} \le c_8 N^{-6}, \quad \text{where} \\ &A_{ikn} := \left\{ \sup_{(s,t)\in[i,i+1]^2} \sup_{(x,y)\in[k\sqrt{n},(k+1)\sqrt{n}]^2} |L_s(x;n) - L_t(y;n)| > c_{15}\, D \log N \right\}, \end{aligned} \tag{5.18}$$

and $c_{15} := c_8\,(7 + c_{14})$. Therefore,

$$\mathrm{P}\left\{ \bigcup_{i=1}^{N} \bigcup_{n=1}^{N} \bigcup_{|k|\le k_{\max}} A_{ikn} \right\} \le N^2(2N^2+1)\frac{c_8}{N^6} \le \frac{3c_8}{N^2}. \tag{5.19}$$

By the Borel–Cantelli lemma, almost surely for all large N, $1 \le i, n \le N$, $(s,t) \in [i, i+1]^2$, and $|x| \le N^2$, we have

$$|L_s(x;n) - L_t(x;n)| \le c_{15}\, D \log N \le c_{16} \frac{N^{\frac{1}{2}} \log N}{i^{\frac{1}{2}(1+\nu)}}, \tag{5.20}$$

with $c_{16} = 2c_{12}c_{15}$. On the other hand, it follows from (4.12) that with probability one, when N is sufficiently large, $\sup_{0\le s\le N} \sup_{0\le t\le N} |W(s,t)| < N^2$, so that $L_s(x;n) = 0$ for $s, n \le N$ and $|x| > N^2$. Our lemma follows as a consequence. $\square$

We present one final supporting lemma.

Lemma 5.6 *If $\alpha, \varepsilon > 0$ and $\nu \in (0, \frac{1}{2})$ are held fixed, then with probability one for all large N and all $i, n \le N$,*

$$\sup_{|x-y|\le N^\alpha} |L_i(x;n) - L_i(y;n)| \le i^{-\frac{1}{2}(1+\nu)} N^{\alpha\nu + \frac{1}{2}(1-\nu)+\varepsilon}. \tag{5.21}$$

Proof Let $\alpha > 0$ and use scaling to see that for each i and n, $\sup_{|x-y|\le N^\alpha} |L_i(x;n) - L_i(y;n)|$ is distributed as $(n/i)^{\frac{1}{2}} \sup_{|x-y|\le N^\alpha/\sqrt{in}} |L_1(x;1) - L_1(y;1)|$. Hence, for any $b > 0$ and $p \ge 1$,

$$\begin{aligned} \mathrm{P}\left\{ \sup_{|x-y|\le N^\alpha} |L_i(x;n) - L_i(y;n)| > b \right\} &\le b^{-p} \left(\frac{n}{i}\right)^{\frac{p}{2}} \left\| \sup_{|x-y|\le N^\alpha/\sqrt{in}} |L_1(x;1) - L_1(y;1)| \right\|_p^p \\ &\le b^{-p} \left(\frac{n}{i}\right)^{\frac{p}{2}} \left(\frac{N^\alpha}{\sqrt{in}}\right)^{\nu p} \left\| \sup_{x\ne y} \frac{|L_1(x;1) - L_1(y;1)|}{|x-y|^\nu} \right\|_p^p \\ &\le (pc_9(\nu))^p\, b^{-p} \left(\frac{n}{i}\right)^{\frac{p}{2}} \left(\frac{N^\alpha}{\sqrt{in}}\right)^{\nu p}. \end{aligned} \tag{5.22}$$

The last inequality follows from (5.8).

Let $\varepsilon \in (0,1)$, $p := 4\varepsilon^{-1}$, and $b := (n/i)^{\frac{1}{2}}(N^\alpha/\sqrt{in})^\nu N^\varepsilon$ to obtain,

$$\mathrm{P}\left(\bigcup_{i=1}^{N}\bigcup_{n=1}^{N}\left\{\sup_{|x-y|\leq N^\alpha}|L_i(x;n)-L_i(y;n)| > \left(\frac{n}{i}\right)^{\frac{1}{2}}\left(\frac{N^\alpha}{\sqrt{in}}\right)^\nu N^\varepsilon\right\}\right) \leq N^2\left(\frac{4c_9(\nu)}{\varepsilon}\right)^{\frac{4}{\varepsilon}} N^{-4}. \quad (5.23)$$

Since this is summable in N, we can use the Borel–Cantelli lemma to see that with probability one, for all large N and all $i, n \leq N$,

$$\sup_{|x-y|\leq N^\alpha}|L_i(x;n)-L_i(y;n)| \leq n^{\frac{1}{2}(1-\nu)} i^{-\frac{1}{2}(1+\nu)} N^{\alpha\nu+\varepsilon}, \quad (5.24)$$

thus leading to our lemma. □

We are ready to present our

Proof of Proposition 4.1 According to [17, Theorem 2.2], the following holds with probability one:

$$\begin{aligned}\int_0^1 \sqrt{u}\,\sup_{x\in\mathbb{R}} L_u(x;N)\,du &= O\left(\sqrt{N\log\log N}\right)\\ \sup_{1\leq u\leq N}\sup_{x\in\mathbb{R}}\sqrt{u}L_u(x;N) &= O\left(\sqrt{N\log\log N}\right).\end{aligned} \quad (5.25)$$

Let $C(x;s,t)$ be the crossing local time of W, as in (2.4). For $1 \leq m, n \leq N$,

$$\begin{aligned}C(x;m,n) &= \int_0^m \sqrt{u}\,L_u(x;n)\,du\\ &= O\left(\sqrt{N\log\log N}\right) + \int_1^m \sqrt{u}\,L_u(x;n)\,du\\ &= \sum_{i=1}^m \sqrt{i}\,L_i(x;n) + O\left(N^{\frac{1}{2}(3-\nu)}\log N\right).\end{aligned} \quad (5.26)$$

The last inequality following from Lemma 5.5 and (5.25), and $O(\cdots)$ is uniform in $1 \leq m, n \leq N$ and $x \in \mathbb{R}$. From this and Lemma 5.6, we can see that for any $\nu \in [0, \frac{1}{2})$, almost surely,

$$\sup_{|x-y|\leq N^\alpha}|C(x;m,n)-C(y;m,n)| = O\left(N^{\frac{1}{2}(3-\nu)}\log N + N^{\alpha\nu+\frac{3}{2}-\nu+\varepsilon}\right), \quad (5.27)$$

uniformly in $1 \leq m, n \leq N$. Since $\max(\frac{1}{2}(3-\nu), \alpha\nu+\frac{3}{2}-\nu+\varepsilon)$ can be as close to $1+\max(\frac{\alpha}{2},\frac{1}{4})$ as possible, Proposition 4.1 follows. □

6 Proof of Proposition 4.2

This section is devoted to estimating the increments of $x \mapsto Z(x;m,n) = \sum_{i=0}^{m} \zeta_i(x;n)$. By the definition of $\zeta_i(x;n)$ in (2.2),

$$\zeta_i(x;n) = \sum_{j=0}^{n-1} \mathbf{1}_{\{S_{i,j}>x,\, S_{i,j+1}<x\}} + \sum_{j=0}^{n-1} \mathbf{1}_{\{S_{i,j}<x,\, S_{i,j+1}>x\}}. \tag{6.1}$$

The two sums on the right hand side represent the numbers of down- and up-crossings, respectively. Thus,

$$\left| \zeta_i(x;n) - 2\sum_{j=0}^{n-1} \mathbf{1}_{\{S_{i,j}>x,\, S_{i,j+1}<x\}} \right| \le 1. \tag{6.2}$$

It remains to study the increments of $x \mapsto \sum_{j=0}^{n-1} \mathbf{1}_{\{S_{i,j}>x,\, S_{i,j+1}<x\}}$. Our first step is to estimate $S_{i,j}$ by a Wiener process. An obvious candidate would be the Brownian sheet in (3.6) that was used in §3 to prove a strong approximation of $S_{i,j}$. Unfortunately, the error term in this approximation scheme is too large for our needs. So, we proceed very carefully in replacing $S_{i,j}$ by another Brownian motion.

Fix $i \le N$. Since $S_{i,j}$ is the sum of (ij) i.i.d. symmetric Bernoulli random variables, by the KMT theorem (3.2) there exists a standard Wiener process $\{W(t);\, t \ge 0\}$ such that for any $z > 0$ and some absolute constants c_1, c_2 and c_3,

$$\mathrm{P}\left\{ |S_{i,j} - W(ij)| > c_1 \log(ij) + z \right\} \le c_2\, e^{-c_3 z}. \tag{6.3}$$

Strictly speaking, we really should write W_i instead of W (our Wiener process W depends on i). The same remark applies to the forthcoming event A and Wiener process B.

Fix $\varepsilon \in (0, \frac{1}{2})$ and $\delta \in (0, \frac{\varepsilon}{2})$, and consider the event

$$E := \bigcap_{j=1}^{N} \left\{ |S_{i,j} - W(ij)| \le N^{\delta} \right\}, \qquad \forall i \le N. \tag{6.4}$$

By (6.3), for large N, say $N \ge N_0$,

$$\mathrm{P}\left\{E^{\complement}\right\} \le \sum_{j=1}^{N} c_{17} N^{-5} = c_{17} N^{-4}. \tag{6.5}$$

On the event E, we have for $i \le N$, $n \le N$ and $x \in \mathbb{R}$,

$$\begin{aligned} \sum_{j=0}^{n-1} \mathbf{1}_{\{S_{i,j}>x,\, S_{i,j+1}<x\}} &\le \sum_{j=0}^{n-1} \mathbf{1}_{\{W(ij)>x-N^{\delta},\, W(i(j+1))<x+N^{\delta}\}} \\ &\le \sum_{j=0}^{n-1} \mathbf{1}_{\{W(ij)>x,\, W(i(j+1))<x\}} + 2\sup_{a\in\mathbb{R}} \sum_{j=0}^{n} \mathbf{1}_{\{a \le W(ij) \le a+N^{\delta}\}}. \end{aligned} \tag{6.6}$$

Similarly, on E,

$$\sum_{j=0}^{n-1} \mathbf{1}_{\{S_{i,j}>x,\, S_{i,j+1}<x\}} \geq \sum_{j=0}^{n-1} \mathbf{1}_{\{W(ij)>x,\, W(i(j+1))<x\}} - 2\sup_{a\in\mathbb{R}} \sum_{j=0}^{n} \mathbf{1}_{\{a\leq W(ij)\leq a+N^\delta\}}. \tag{6.7}$$

If we write

$$\Delta_{i,n} = \sup_{|x-y|\leq N^\alpha} \left| \sum_{j=0}^{n-1} \mathbf{1}_{\{S_{i,j}>x,\, S_{i,j+1}<x\}} - \sum_{j=0}^{n-1} \mathbf{1}_{\{S_{i,j}>y,\, S_{i,j+1}<y\}} \right|, \tag{6.8}$$

then

$$\begin{aligned}\Delta_{i,n}\mathbf{1}_E \leq & \sup_{|x-y|\leq N^\alpha} \left| \sum_{j=0}^{n-1} \mathbf{1}_{\{W(ij)>x,\, W(i(j+1))<x\}} - \sum_{j=0}^{n-1} \mathbf{1}_{\{W(ij)>y,\, W(i(j+1))<y\}} \right| \\ & + 4\sup_{a\in\mathbb{R}} \sum_{j=0}^{n} \mathbf{1}_{\{a\leq W(ij)\leq a+N^\delta\}}.\end{aligned} \tag{6.9}$$

Let us consider the process $x \mapsto \sum_{j=0}^{n-1} \mathbf{1}_{\{W(ij)>x,\, W(i(j+1))<x\}}$: For each i, it is distributed as $x \mapsto \sum_{j=0}^{n-1} \mathbf{1}_{\{W(j)>x/\sqrt{i},\, W(j+1)<x/\sqrt{i}\}}$. Therefore, by Borodin (1986, equation 1.13), there exists a coupling of W and a standard Wiener process B such that for any $\lambda > 1$, there exist constants c_{18} and c_{19} such that for all $n \geq 1$,

$$\mathrm{P}\left\{ \sup_{x\in\mathbb{R}} \left| \sum_{j=0}^{n-1} \mathbf{1}_{\{W(j)>x,\, W(j+1)<x\}} - c_{20}\sqrt{n}\, L\left(\frac{x}{\sqrt{n}}, 1\right) \right| \geq c_{18}\, n^{\frac{1}{4}} \log n \right\} \leq c_{19} n^{-\lambda}, \tag{6.10}$$

where L is the local time of B, and $c_{20} := \mathrm{E}(\lfloor B^+(1) \rfloor)$. We also mention that in Borodin (1986, equation 1.13), the preceding inequality was stated only for some $\lambda > 1$, which sufficed for the intended applications there. However, it is clear from its proof (Borodin 1986, pp. 272–273) that by altering $c_{19} = c_{19}(\lambda)$ suitably, λ can be made to be arbitrarily large.

A straightforward consequence of (6.9) and (6.10), with $\lambda = \frac{4}{\delta}$ in (6.10), is as follows: For any $b > 0$ and $k > 0$,

$$\begin{aligned} & \mathrm{P}\left\{ \Delta_{i,n}\mathbf{1}_E > b + k + 2c_{18}\, n^{\frac{1}{4}} \log n \right\} \\ & \leq \mathrm{P}\left\{ \sup_{|x-y|\leq N^\alpha} \left| c_{20}\sqrt{n}\, L\left(\frac{x}{\sqrt{in}}, 1\right) - c_{20}\sqrt{n}\, L\left(\frac{y}{\sqrt{in}}, 1\right) \right| > b \right\} \\ & \quad + \mathrm{P}\left\{ 4\sup_{a\in\mathbb{R}} \sum_{j=0}^{n} \mathbf{1}_{\{a\leq W(ij)\leq a+N^\delta\}} > k \right\} + c_{19} n^{-\frac{4}{\delta}} \\ & = P_1 + P_2 + c_{19} n^{-\frac{4}{\delta}}. \end{aligned} \tag{6.11}$$

Plugging this into (6.5) yields

$$\mathrm{P}\left\{ \Delta_{i,n} > b + k + 2c_{18}\, n^{\frac{1}{4}} \log n \right\} \leq P_1 + P_2 + c_{19} n^{-\frac{4}{\delta}} + c_{17} N^{-4}. \tag{6.12}$$

To bound P_1, we use (5.8) to see that for any $\nu \in [0, \frac{1}{2})$ and $p \geq 1$,

$$\begin{aligned} P_1 &= \mathrm{P}\left\{ \sup_{|x-y| \leq N^{\alpha}/\sqrt{in}} |L(x,1) - L(y,1)| > \frac{b}{2c_{20}\sqrt{n}} \right\} \\ &\leq \mathrm{P}\left\{ \sup_{x \neq y} \frac{|L(x,1) - L(y,1)|}{|x-y|^{\nu}} > \frac{b}{2c_{20}\sqrt{n}\,(N^{\alpha}/\sqrt{in}\,)^{\nu}} \right\} \\ &\leq (pc_9(\nu))^p \left(\frac{2c_{20}\sqrt{n}\,(N^{\alpha}/\sqrt{in}\,)^{\nu}}{b} \right)^p . \end{aligned} \tag{6.13}$$

Choosing $b = \sqrt{n}\,(N^{\alpha}/\sqrt{in}\,)^{\nu} N^{\delta}$ and $p = \frac{4}{\delta}$ gives

$$P_1 \leq c_{21} N^{-4}, \tag{6.14}$$

where $c_{21} = (8c_{20}\, c_9(\nu)/\delta)^{\frac{4}{\delta}}$ depends only on (ν, δ).

We now estimate P_2: The standard Gaussian density is bounded above by $(2\pi)^{-\frac{1}{2}} \leq \frac{1}{2}$. Therefore, for any $a \in \mathbb{R}$, we have $\mathrm{P}\{a \leq W(ij) \leq a + 2N^{\delta}\} \leq N^{\delta}(ij)^{-\frac{1}{2}}$. We can apply this together with the Markov property to deduce that for any integer $p \geq 1$,

$$\left\| \sum_{j=1}^{n} \mathbf{1}_{\{a \leq W(ij) \leq a + 2N^{\delta}\}} \right\|_p^p \leq p! \left(\sum_{j=1}^{n} \frac{N^{\delta}}{\sqrt{ij}} \right)^p \leq c_{22} \left(\frac{N^{\delta}\sqrt{n}}{\sqrt{i}} \right)^p, \tag{6.15}$$

for some constant $c_{22} = c_{22}(p)$ that depends only on p. By Markov's inequality, for any $k > 0$,

$$\sup_{a \in \mathbb{R}} \mathrm{P}\left\{ \sum_{j=1}^{n} \mathbf{1}_{\{a \leq W(ij) \leq a + 2N^{\delta}\}} > k \right\} \leq \frac{c_{22}}{k^p} \left(\frac{N^{\delta}\sqrt{n}}{\sqrt{i}} \right)^p . \tag{6.16}$$

Now let $a_{\ell} = 2N^{\delta}\ell$, for $\ell = 0, \pm 1, \pm 2, \cdots, \pm \ell_{\max}$, where $\ell_{\max} = \lfloor N^{\delta}\sqrt{in}\,\rfloor$. By the usual estimate for Gaussian tails,

$$\mathrm{P}\left\{ \max_{1 \leq j \leq n} |W(ij)| > a_{\ell_{\max}} \right\} \leq 2\exp\left(-\frac{(2N^{\delta}\lfloor N^{\delta}\sqrt{in}\,\rfloor)^2}{2in} \right) \leq 2\exp(-N^{4\delta}). \tag{6.17}$$

On the other hand, if $\max_{1 \leq j \leq n} |W(ij)| > a_{\ell_{\max}}$, and if $\sum_{j=1}^{n} \mathbf{1}_{\{a \leq W(ij) \leq a + N^{\delta}\}} > k$ for some $a \in \mathbb{R}$, then there exists ℓ, with $|\ell| \leq \ell_{\max}$ such that $\sum_{j=1}^{n} \mathbf{1}_{\{a_{\ell} \leq W(ij) \leq a_{\ell} + 2N^{\delta}\}} > k$. By (6.16), this yields,

$$\mathrm{P}\left\{ \sup_{a \in \mathbb{R}} \sum_{j=1}^{n} \mathbf{1}_{\{a \leq W(ij) \leq a + N^{\delta}\}} > k \right\} \leq (2\ell_{\max} + 1) \frac{c_{22}}{k^p} \left(\frac{N^{\delta}\sqrt{n}}{\sqrt{i}} \right)^p + 2\exp(-N^{4\delta}). \tag{6.18}$$

We replace k by $\frac{1}{4}k$ to deduce that for all $i, n \leq N$,

$$\begin{aligned} P_2 &\leq c_{23} \frac{N^{\delta}\sqrt{in}}{k^p} \left(\frac{N^{\delta}\sqrt{n}}{\sqrt{i}} \right)^p + 2\exp(-N^{4\delta}) \\ &\leq c_{23} \frac{N^{\delta}\sqrt{in}}{k^p} \left(\frac{N^{\delta}\sqrt{n}}{\sqrt{i}} \right)^p + 2\exp(-N^{4\delta}). \end{aligned} \tag{6.19}$$

Taking $k = N^{2\delta}(n/i)^{\frac{1}{2}}$ and $p = \frac{1}{\delta}(5+\delta)$ yields

$$P_2 \le c_{23}N^{-4} + 2\exp(-N^{4\delta}). \tag{6.20}$$

Plugging this and (6.14) into (6.12) implies that for $i, n \le N$,

$$\begin{aligned} &\mathrm{P}\left\{\Delta_{i,n} > \sqrt{n}\Big(\frac{N^\alpha}{\sqrt{in}}\Big)^\nu N^\delta + N^{2\delta}\sqrt{\frac{n}{i}} + 2c_{18}\, n^{\frac{1}{4}} \log n\right\} \\ &\qquad \le c_{24}N^{-4} + 2\exp(-N^{4\delta}) + c_{19}n^{-\frac{4}{\delta}}, \end{aligned} \tag{6.21}$$

where $c_{24} := c_{21} + c_{23} + c_{17}$. As a consequence,

$$\begin{aligned} &\mathrm{P}\left(\bigcup_{i=1}^{N}\bigcup_{n=\lfloor N^\delta\rfloor}^{N}\left\{\Delta_{i,n} > \sqrt{n}\left(\frac{N^\alpha}{\sqrt{in}}\right)^\nu N^\delta + N^{2\delta}\sqrt{\frac{n}{i}} + 2c_{18}\, n^{\frac{1}{4}}\log n\right\}\right) \\ &\qquad \le c_{24}N^{-2} + 2N^2\exp(-N^{4\delta}) + c_{19}N^{-2}, \end{aligned} \tag{6.22}$$

which is summable for N. By the Borel–Cantelli lemma, almost surely for all large N, $i \le N$, and every $N^\delta \le n \le N$,

$$\Delta_{i,n} \le \sqrt{n}\left(\frac{N^\alpha}{\sqrt{in}}\right)^\nu N^\delta + N^{2\delta}\sqrt{\frac{n}{i}} + 2c_{18}\, n^{\frac{1}{4}}\log n. \tag{6.23}$$

Thus, whenever $m \le N$ and $N^\delta \le n \le N$,

$$\begin{aligned} \sum_{i=1}^{m}\Delta_{i,n} &\le \sqrt{n}\, N^{1-\frac{\nu}{2}}\left(\frac{N^\alpha}{\sqrt{n}}\right)^\nu N^\delta + N^{2\delta+\frac{1}{2}}\sqrt{n} + N^{\frac{5}{4}+\delta} \\ &\le N^{\frac{3}{2}-\nu+\alpha\nu+\delta} + 2N^{\frac{5}{4}+\delta}, \qquad \text{eventually (in } N). \end{aligned} \tag{6.24}$$

This last inequality uses the facts that $n \le N$ and that $\delta < \frac{1}{4}$.

If $n < N^\delta$, we can use the trivial inequality $\Delta_{i,n} \le 2n$ to see that $\sum_{i=1}^m \Delta_{i,n} \le 2N^{1+\delta}$. Therefore, whenever $\nu \in [0, \frac{1}{2})$, $\varepsilon \in (0, \frac{1}{4})$ and $\delta \in (0, \frac{\varepsilon}{2})$, with probability one, for all large N and $m, n \le N$,

$$\sum_{i=1}^{m}\Delta_{i,n} \le N^{\frac{3}{2}-\nu+\alpha\nu+\delta} + 3N^{\frac{5}{4}+\delta}. \tag{6.25}$$

Since $\nu \in [0, \frac{1}{2})$ is arbitrary, it can be chosen such that $\frac{3}{2} - \nu + \alpha\nu + \delta < 1 + \frac{\alpha}{2} + 2\delta$. In view of the definition of $\Delta_{i,n}$ in (6.8), and the relation (6.2), we have proved that, almost surely, for any $\varepsilon \in (0, \frac{1}{2})$, when $N \to \infty$,

$$\max_{0\le m\le N}\max_{1\le n\le N}\sup_{|x-y|\le N^\alpha}\left|\sum_{i=0}^{m}\zeta_i(x;n) - \sum_{i=0}^{m}\zeta_i(y;n)\right| = o\big(N^{1+\frac{\alpha}{2}+\varepsilon} + N^{\frac{5}{4}+\varepsilon}\big). \tag{6.26}$$

This completes our proof of Proposition 4.2. □

7 Proof of Theorem 1.1

In view of (4.10), we need to check only the lower bound; namely that for any $\varepsilon > 0$, $Z(N) \geq N^{\frac{3}{2}-\varepsilon}$ almost surely for all large N. By means of (2.5), we have to prove that for large N,

$$C(0; N, N) \geq N^{\frac{3}{2}-\varepsilon}, \qquad \text{a.s.} \tag{7.1}$$

We start by choosing $a \in (\frac{1}{2}, 1)$ and $b > 1$ such that $2a > b$. We also recall that $C(0; N, N) = \int_0^N \sqrt{u}\, L_u(0; N)\, du$. Thus,

$$C(0; N, N) \geq \int_{aN}^{N} \sqrt{aN}\, L_u(0; N)\, du \geq \sqrt{a}\,(1-a) N^{\frac{3}{2}} \inf_{aN \leq u \leq N} L_u(0; N). \tag{7.2}$$

We now use the following recent estimate of the authors (cf. Khoshnevisan et al. 2001, Th. 3.3): For any $\nu \in (0, \frac{1}{2})$, there exists a small constant $h_0 = h_0(\nu, a)$ such that for all $N > 0$ and $h \in (0, h_0)$,

$$\mathrm{P}\left\{ \inf_{\frac{1}{2}N \leq u \leq N} L_u\left(0; \frac{N}{2a}\right) \leq h \right\} \leq \exp\left(-\frac{\nu |\log h|}{\log|\log h|} \right). \tag{7.3}$$

In [15] we obtained the above for $N = 1$. This formulation is a consequence of the case $N = 1$ and scaling. Taking $N_k = \lfloor b^k \rfloor$ and $h = N_k^{-\frac{\varepsilon}{2}}$ yields

$$\sum_{k=1}^{\infty} \mathrm{P}\left\{ \inf_{\frac{1}{2}N_k \leq u \leq N_k} L_u\left(0; \frac{N_k}{2a}\right) \leq N_k^{\frac{\varepsilon}{2}} \right\} < +\infty. \tag{7.4}$$

By the Borel–Cantelli lemma, almost surely for all large k,

$$\inf_{\frac{1}{2}N_k \leq u \leq N_k} L_u\big(0; \frac{N_k}{2a}\big) > N_k^{-\frac{\varepsilon}{2}}. \tag{7.5}$$

Consequently, whenever $N_{k-1} \leq N \leq N_k$,

$$\begin{aligned} \inf_{aN \leq u \leq N} L_u(0; N) &\geq \inf_{aN_{k-1} \leq u \leq N_k} L_u(0; N_{k-1}) \geq \inf_{\frac{1}{2}N_k \leq u \leq N_k} L_u\big(0; \frac{N_k}{2a}\big) \\ &> N_k^{-\frac{\varepsilon}{2}} \geq (2aN)^{-\frac{\varepsilon}{2}}. \end{aligned} \tag{7.6}$$

We have used the fact that $N_{k-1} \geq (2a)^{-1} N_k$ for all sufficiently large k; this, in turn, follows from the inequality $2a > b$. In summary, for all large N,

$$C(0; N, N) \geq \sqrt{a}\,(1-a) N^{\frac{3}{2}} (2aN)^{-\frac{\varepsilon}{2}} \geq N^{\frac{3}{2}-\varepsilon}, \tag{7.7}$$

yielding (7.1), whence Theorem 1.1. □

Appendix: The Matlab Code

The Matlab code for Figure 1(a) follows. It presupposes the existence of a seed for the random number generator, and that the seed is stored in a Matlab file called "`seed.dat`."

```
 1. % run a 1-dimensional simple walk for n time-steps
 2. load seed              % The same random walk is always used
 3. rand('state',seed)
 4. figure;
 5. x = rand(n);
 6. X = 2*round(x)-1;              % Generate Rademacher variables
 7. S=cumsum(X);               % Sum the columns separately
 8. W=S;
 9. W(:,1) = S(:,1);
10. for i=2:n
        W(:,i) = W(:,i-1) + S(:,i);             % W is the walk
    end
11. for k=1:n
        for j=1:n
            if W(k,j) == 0
              plot(k,j)
            end
        end
    end
12. print -deps file.ps
```

In order to simulate the vertical crossings (for the same walk), the third line of 11 above (i.e., "`if W(k,j)==0`") needs to be replaced by "`if W(k,j)*W(k,j+1) < 0`."

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