The most visited sites of symmetric stable processes

by

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Summary. Let X be a symmetric stable process of index $\alpha \in (1,2]$ and let L_t^x denote the local time at time t and position x. Let $V(t)$ be such that $L_t^{V(t)} = \sup_{x\in\mathbb{R}} L_t^x$. We call $V(t)$ the most visited site of X up to time t. We prove the transience of V, that is, $\lim_{t\to\infty} |V(t)| = \infty$ almost surely. An estimate is given concerning the rate of escape of V. The result extends a well-known theorem of Bass and Griffin for Brownian motion. Our approach is based upon an extension of the Ray–Knight theorem for symmetric Markov processes, and relates stable local times to fractional Brownian motion and further to the winding problem for planar Brownian motion.

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1. Introduction

Let $X = \{X(t);\, t \geq 0\}$ be a symmetric stable process of index α, with $X(0) = 0$. That is, X has independent and stationary increments, with characteristic function

$$\mathbb{E}(\mathrm{e}^{izX(t)}) = \exp(-c_0\, |z|^\alpha\, t), \tag{1.1}$$

where $c_0 > 0$ is a constant. We assume $\alpha \in (1, 2]$, so that X admits a jointly continuous local time process $\{L_t^x;\, t \geq 0,\, x \in \mathbb{R}\}$ (see for example Boylan [3]), which we may normalize so that for any $t \geq 0$ and Borel function $f \geq 0$,

$$\int_0^t f(X(s))\, \mathrm{d}s = \int_{-\infty}^{\infty} f(x)\, L_t^x\, \mathrm{d}x.$$

Clearly, when $\alpha = 2$, X is Brownian motion.

We are interested in the set

$$\mathbb{V}(t) \stackrel{\text{def}}{=} \Big\{ x \in \mathbb{R} : \, L_t^x = \sup_{y \in \mathbb{R}} L_t^y \Big\},$$

which usually is referred to as the set of the "most visited sites" of X or the "favorite points" of X up to time t, see Erdős and Révész [7]. It is known (Eisenbaum [5]) that $\mathbb{V}(t)$ is either a singleton, or, for countably many t, composed of two points, but we will not use this property.

Let us choose

$$V(t) = \max_{x \in \mathbb{V}(t)} x, \tag{1.2}$$

which will be called the (maximal) most visited site. We mention that the choice of $V(t)$ is irrelevant, in the sense that all the results in this paper remain unchanged if we replace $V(t)$ by any element of $\mathbb{V}(t)$.

Erdős and Révész [7] were the first to study the most visited site, for simple random walk. In the case of Brownian motion, we recall the following somewhat surprising result of Bass and Griffin [2].

Theorem A (Bass and Griffin [2]). *If X is a Brownian motion, i.e. if $\alpha = 2$, then for any $\gamma > 11$,*

$$\lim_{t \to \infty} \frac{(\log t)^\gamma}{t^{1/2}}\, |V(t)| = \infty, \qquad \text{a.s.}$$

In particular, Theorem A confirms the **transience** of the process V, in the sense that $\lim_{t\to\infty} |V(t)| = \infty$ almost surely. The problem of determining the exact rate of escape of V remains open to the best of our knowledge, though it is also proved in [2] that almost surely, $\liminf_{t\to\infty} t^{-1/2}(\log t)^\gamma |V(t)| = 0$, for all $\gamma < 1$.

It is natural to ask if the most visited site is still transient for stable processes. The answer is in the affirmative. Here is the main result of this paper.

Theorem 1.1. *Let* $1 < \alpha \le 2$. *For* $\gamma > 9/(\alpha - 1)$,

$$\lim_{t\to\infty} \frac{(\log t)^\gamma}{t^{1/\alpha}} |V(t)| = \infty, \qquad \text{a.s.}$$

We say a few words about our method. The proof of Theorem A by Bass and Griffin relies on the Ray–Knight theorem and a path decomposition for the Bessel process, together with some martingale properties related to the Brownian local time. In the case of a stable non-Brownian process, no such path decomposition or martingale property is available. Therefore, we have to adopt a different approach. Our starting point is an extension, which has been obtained in [6], of the classical Ray–Knight theorem to symmetric Markov processes. In particular, this relates stable local times to fractional Brownian motion. It is therefore natural that we shall be using some Gaussian techniques. Our method shows a relationship between fractional Brownian motion and the winding angle of planar Brownian motion; this may be of independent interest.

Our key estimate (Theorem 3.1) is the one where the Gaussian techniques are most heavily used and says that

$$\mathbb{P}(\sup_{|x|\le 1} L^x_{\tau(1)} < 1 + \lambda) \le c\,\lambda^{5/4} |\log \lambda|^2$$

for λ sufficiently small, where c is a constant and $\tau(r)$ is the first time L^0_t is equal to r. This is then used to obtain Lemma 4.1, which says that for $b > 4$

$$\lim_{r\to\infty} \frac{(\log r)^b}{r} (\sup_x L^x_{\tau(r)} - r) = \infty, \qquad \text{a.s.}$$

Somewhat simpler is Lemma 4.5, which says that if $b > 4$ and $\mu > 2b/(\alpha - 1)$, then

$$\lim_{r\to\infty} \frac{(\log r)^b}{r} \sup_{|x|\le r^{1/(\alpha-1)}/(\log r)^\mu} (L^x_{\tau(r)} - r) = 0, \qquad \text{a.s.}$$

We deduce Theorem 1.1. from Lemmas 4.1 and 4.5.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminaries on fractional Brownian motion, the Ray–Knight theorem and Brownian winding angles. They lead in Section 3 to our main probability estimate, Theorem 3.1. The proof of Theorem 1.1 is completed in Section 4.

Throughout the paper, the letter c with subscripts denotes unimportant (but finite and positive) constants.

2. Preliminaries

2.1. Fractional Brownian motion

By fractional Brownian motion of index β (abbreviated FBM(β) or simply FBM), we mean a centered Gaussian process $\eta = \{\eta(x);\, x \in \mathbb{R}\}$, whose covariance function is given by

$$\mathbb{E}(\eta(x)\eta(y)) = \frac{1}{2}\Big(|x|^\beta + |y|^\beta - |x-y|^\beta \Big).$$

(In particular, FBM(1) is Brownian motion). Unless stated otherwise, we always assume the FBM to start from 0, i.e., $\eta(0) = 0$. The self-similarity of FBM will be frequently used without further mention: if η is an FBM(β), then for any $a > 0$,

$$\eta(\cdot) \stackrel{\text{law}}{=} a^{-\beta/2}\eta(a\,\cdot\,),$$

where "$\stackrel{\text{law}}{=}$" denotes identity in law.

We shall also need the following law of the iterated logarithm (LIL): if η is an FBM(β) with $\beta \in (0,1]$,

$$\limsup_{t\to 0^+} \frac{\eta(t)}{\sqrt{2t^\beta \log|\log t|}} = 1, \qquad \text{a.s.} \tag{2.1}$$

This can be found in Marcus [11]. We mention that, in this paper, we shall only need (2.1) in the case $\beta \in (0,1]$, though the latter condition is not necessary.

2.2. Ray–Knight theorem for stable processes

Our basic tool is an extension of the Ray–Knight theorem for symmetric strong Markov processes, which bears a relatively simple form in the case of stable processes. This extension has been established in [6], and can be seen as a consequence of Dynkin's isomorphism theorem. The latter, which relates the local time of Markov processes to Gaussian processes, turns out to be a powerful tool in the study of local times and additive functionals. See Marcus and Rosen [12] and [13] for a deep study of this subject.

As before, let X be a symmetric process of index $\alpha \in (1, 2]$, with local time denoted by L. Let

$$\tau(r) \stackrel{\text{def}}{=} \inf\Big\{t > 0 : L_t^0 > r\Big\}, \qquad r \geq 0, \tag{2.2}$$

which is the (right-continuous version of the) inverse local time at 0.

Theorem B ([6]). *Let $1 < \alpha \leq 2$, and let η be an* FBM$(\alpha - 1)$ *independent of X. It is possible to choose a value for the normalizing constant c_0 in* (1.1), *such that*

$$L_{\tau(1)}^{\cdot} + \frac{1}{2}\eta^2 \stackrel{\text{law}}{=} \frac{1}{2}(\eta + \sqrt{2})^2. \tag{2.3}$$

Remark. In the case $\alpha = 2$, Theorem B takes the form of the additivity property of squared Bessel processes, which is an equivalent form of the usual Ray–Knight theorem for Brownian local time at $\tau(1)$; see Ray [15], Knight [10]. We also mention that, unlike Dynkin's isomorphism theorem, (2.3) does not involve signed measures. This will enable us to obtain some sharp inequalities for various suprema of $L_{\tau(1)}$.

2.3. Scaling for stable local times

There are some easy scaling properties for the local time process of X. Though they are very simple, we present a list of these properties here in order to facilitate applications. As before, X is symmetric stable of index α, with local time L, and inverse local time τ at 0 as in (2.2). Then we have the following identities in law: for any $c > 0$,

$$\Big\{L_{ct}^x;\, t \geq 0,\, x \in \mathbb{R}\Big\} \stackrel{\text{law}}{=} \Big\{c^{(\alpha-1)/\alpha} L_t^{x/c^{1/\alpha}};\, t \geq 0,\, x \in \mathbb{R}\Big\},$$
$$\Big\{\tau(cr);\, r \geq 0\Big\} \stackrel{\text{law}}{=} \Big\{c^{\alpha/(\alpha-1)}\tau(r);\, r \geq 0\Big\},$$
$$\Big\{L_{\tau(cr)}^x;\, r \geq 0,\, x \in \mathbb{R}\Big\} \stackrel{\text{law}}{=} \Big\{c\, L_{\tau(r)}^{x/c^{1/(\alpha-1)}};\, r \geq 0,\, x \in \mathbb{R}\Big\}.$$

2.4. Brownian windings

Let $\{Z(t);\, t \geq 0\}$ be a planar Brownian motion, starting from $(1, 0)$. It is known that every point is polar for Z. In particular, with probability one, Z never hits the origin. So there exists a continuous determination of $\theta(t)$, the total angle wound by Z around the origin up to time t (with, say, $\theta(0) = 0$). Thus θ records the angle and keeps track of the number of times the Brownian path has wound around the origin, counting clockwise loops (-2π) and counterclockwise loops (2π).

We need the following result: for any $a > 0$, there exists a constant $c_1 = c_1(a) \in (0, \infty)$ such that for all $u > 0$,

$$\mathbb{P}\Big(\sup_{0 \le s \le u} |\theta(s)| < a \Big) \le \frac{c_1}{u^{\pi/(4a)}}. \tag{2.4}$$

A somewhat weaker version of this estimate can be found in Theorem 2 of Spitzer [18], though his argument actually yields (2.4). For the statement of the estimate in this form, together with further extensions for the general exit problem of Brownian motion from a cone, see for example Bañuelos and Smits [1].

3. Key estimate

Throughout the section, X is a symmetric stable process of index $\alpha \in (1, 2]$, starting from 0, with local time L. The inverse local time at 0 is denoted by τ, as in (2.2).

Since the normalizing constant c_0 (defined in (1.1)) has no influence on Theorem 1.1, we shall from now on choose c_0 to be the one satisfying Theorem B (see Section 2.2), without further mention.

Here is the main probability estimate of the paper.

Theorem 3.1. *There exists a constant $c_2 \in (0, \infty)$ such that for all $0 < \lambda \le \mathrm{e}^{-2}$,*

$$\mathbb{P}\Big(\sup_{|x| \le 1} L^x_{\tau(1)} < 1 + \lambda \Big) \le c_2 \, \lambda^{5/4} \, |\log \lambda|^2. \tag{3.1}$$

Proof. We only have to treat the situation where λ is sufficiently close to 0.

Let $\{\eta(x);\, x \in \mathbb{R}\}$ be an FBM$(\alpha - 1)$, independent of X. Consider the following measurable events:

$$\begin{aligned}
E_1 &\stackrel{\text{def}}{=} \Big\{ \sup_{|x| \le 1} L^x_{\tau(1)} < 1 + \lambda \Big\}, \\
E_2 &\stackrel{\text{def}}{=} \Big\{ \eta^2(x) < c_3 \, \lambda, \ \text{ for all } |x| \le \lambda^{1/(\alpha-1)} \Big\}, \\
E_3 &\stackrel{\text{def}}{=} \Big\{ \eta^2(x) < c_3, \ \text{ for all } |x| \le 1 \Big\}, \\
E_4 &\stackrel{\text{def}}{=} \Big\{ \eta^2(x) < c_3 \, |x|^{\alpha-1} \, |\log \lambda|, \ \text{ for all } \lambda^{1/(\alpha-1)} \le |x| \le 1 \Big\}.
\end{aligned}$$

(So the probability term on the left hand side of (3.1) is $\mathbb{P}(E_1)$). By self-similarity,

$$\mathbb{P}(E_2) = \mathbb{P}\Big(\eta^2(x) < c_3, \text{ for all } |x| \le 1 \Big) = \mathbb{P}(E_3),$$

so we can choose c_3 sufficiently large such that

$$\mathbb{P}(E_2 \cap E_3) \ge \frac{2}{3}. \tag{3.2}$$

On the other hand, by the LIL for FBM (see (2.1)) and symmetry, the random variable

$$\sup_{0<u<1/3} \sup_{u^{1/(\alpha-1)}<|x|\le 1} \frac{\eta^2(x)}{|x|^{\alpha-1} \, |\log u|}$$

is almost surely finite. (Actually the $|\log u|$ term here can be replaced by $\log|\log u|$). Therefore, it is possible to pick c_3 so large (how large depending only on α) that $\mathbb{P}(E_4) \ge 2/3$. Jointly considering this and (3.2), we are able to fix a choice for c_3 such that

$$\mathbb{P}(E_2 \cap E_3 \cap E_4) \ge \frac{1}{3}. \tag{3.3}$$

Now, observe that E_1 and $E_2 \cap E_3 \cap E_4$ are independent, and that

$$\bigcap_{i=1}^{4} E_i \ \subset E_5,$$

where

$$E_5 \stackrel{\text{def}}{=} \Big\{ L^x_{\tau(1)} + \frac{1}{2}\eta^2(x) - 1 < \lambda + c_3\, f_0(x), \text{ for all } |x| \le 1 \Big\},$$

$$f_0(x) \stackrel{\text{def}}{=} \begin{cases} \lambda & \text{if } 0 \le |x| \le \lambda^{1/(\alpha-1)}, \\ \min(|x|^{\alpha-1}\, |\log\lambda|, \ 1) & \text{if } \lambda^{1/(\alpha-1)} < |x| \le 1. \end{cases}$$

Therefore,

$$\begin{aligned} \mathbb{P}(E_1)\, \mathbb{P}\Big(\bigcap_{i=2}^{4} E_i \Big) &= \mathbb{P}\Big(\bigcap_{i=1}^{4} E_i \Big) \\ &\le \mathbb{P}(E_5) \\ &= \mathbb{P}\Big(\frac{1}{2}\, (\eta(x) + \sqrt{2}\,)^2 - 1 < \lambda + c_3\, f_0(x), \ \text{ for } |x| \le 1 \Big) \\ &\le \mathbb{P}\Big(\frac{1}{2}\, (\eta(x) + \sqrt{2}\,)^2 - 1 < (1 + c_3)\, f_0(x), \ \text{ for } |x| \le 1 \Big), \end{aligned}$$

where we have used (2.3) in the last equality. It is easily checked that for $a \in (0, 1 + c_3)$, if $(y + \sqrt{2}\,)^2/2 - 1 < a$, then $y < c_4\, a$, where we write c_4 for $1/\sqrt{2}$. Taking into account (3.3), we arrive at:

$$\mathbb{P}(E_1) \le 3\mathbb{P}\Big(\eta(x) < c_4\, (1 + c_3)\, f_0(x), \quad |x| \le 1 \Big). \tag{3.4}$$

The next step is to use Slepian's lemma to estimate the probability term on the right hand side of (3.4).

To this end, let $\{W_1(t);\, t \ge 0\}$ and $\{W_2(t);\, t \ge 0\}$ be two independent real-valued Brownian motions, with $W_1(0) = W_2(0) = 0$. Define the process $U = \{U(x);\, x \in [-1,1]\}$ by

$$U(x) \stackrel{\text{def}}{=} \begin{cases} W_1(x^{\alpha-1}), & \text{if } 0 \le x \le 1, \\ \frac{1}{2} W_1(|x|^{\alpha-1}) + \frac{\sqrt{3}}{2} W_2(|x|^{\alpha-1}), & \text{if } -1 \le x \le 0. \end{cases}$$

Clearly, for any $x \in [-1,1]$,

$$\mathbb{E}(U^2(x)) = |x|^{\alpha-1} = \mathbb{E}(\eta^2(x)). \tag{3.5}$$

In order to apply Slepian's lemma, we have to compare the covariance functions of U and η. Let $(x,y) \in [-1,1]^2$. There are two possible situations. the first case is when $xy \ge 0$. We first assume $x \ge 0$ and $y \ge 0$. Then (writing $a \wedge b$ for $\min(a,b)$)

$$\mathbb{E}(U(x)U(y)) = (x \wedge y)^{\alpha-1} \ge \frac{1}{2}\left(x^{\alpha-1} + y^{\alpha-1} - |x-y|^{\alpha-1}\right) = \mathbb{E}(\eta(x)\eta(y)).$$

The situation where $x \le 0$ and $y \le 0$ is similar.
Another possibility is that $xy < 0$. Without loss of generality, we assume $x > 0$ and $y < 0$. In this case,

$$\begin{aligned}\mathbb{E}(U(x)U(y)) &= \frac{1}{2}\,(x \wedge |y|)^{\alpha-1} \\ &\ge \frac{1}{2}\left(x^{\alpha-1} + |y|^{\alpha-1} - (x+|y|)^{\alpha-1}\right) \\ &= \mathbb{E}(\eta(x)\eta(y)).\end{aligned}$$

Therefore, we have proved that

$$\mathbb{E}(U(x)U(y)) \ge \mathbb{E}(\eta(x)\eta(y)), \tag{3.6}$$

for any $-1 \le x, y \le 1$.

In view of (3.5) and (3.6), we can apply Slepian's lemma (see [17]), to see that for any non-negative Borel function f,

$$\begin{aligned}&\mathbb{P}\Big(\, \eta(x) < f(x), \quad |x| \le 1 \Big) \\ &\quad \le \mathbb{P}\Big(\, U(x) < f(x), \quad |x| \le 1 \Big) \\ &\quad = \mathbb{P}\Big(\, W_1(t) < f(t^{1/(\alpha-1)}),\, W_{1,2}(t) < 2f(-t^{1/(\alpha-1)}), \quad 0 \le t \le 1 \Big),\end{aligned}$$

where we have written $W_{1,2}(t) \stackrel{\text{def}}{=} W_1(t) + \sqrt{3}\, W_2(t)$ for brevity.

Taking $f(x) = c_4\,(1+c_3)\, f_0(x) \stackrel{\text{def}}{=} c_5\, f_0(x)$, and going back to (3.4),

$$\begin{aligned}
\mathbb{P}(E_1) &\le 3\mathbb{P}\Big(\, W_1(t) < c_5\, f_0(t^{1/(\alpha-1)}),\, W_{1,2}(t) < 2c_5\, f_0(-t^{1/(\alpha-1)}), \quad 0 \le t \le 1 \Big) \\
&\le 3\mathbb{P}\Big(\, W_1(t) < c_5\, \lambda,\, W_{1,2}(t) < 2c_5\, \lambda, \quad 0 \le t \le \lambda; \\
&\qquad\qquad W_1(s) < c_5\, s\, |\log \lambda|, \quad \lambda < s \le 1 \Big) \\
&\le 3(\mathbb{P}(E_6) + \mathbb{P}(E_7)\, \mathbb{P}(E_8)), \qquad (3.7)
\end{aligned}$$

where

$$\begin{aligned}
E_6 &\stackrel{\text{def}}{=} \Big\{\, W_1(\lambda) < -\sqrt{\lambda}\, |\log \lambda| \,\Big\}, \\
E_7 &\stackrel{\text{def}}{=} \Big\{\, W_1(t) < c_5\, \lambda,\, W_{1,2}(t) < 2c_5\, \lambda, \quad 0 \le t \le \lambda \,\Big\}, \\
E_8 &\stackrel{\text{def}}{=} \Big\{\, W_1(s) - W_1(\lambda) < c_5\, s\, |\log \lambda| + \sqrt{\lambda}\, |\log \lambda|, \quad \lambda < s \le 1 \,\Big\}.
\end{aligned}$$

It remains to estimate $\mathbb{P}(E_i)$ for $i = 6,\, 7,\, 8$. By the well-known Mill's ratio for Gaussian tails (see for example Shorack and Wellner [16, p. 850]),

$$\mathbb{P}(E_6) \le \frac{1}{\sqrt{2\pi}\, |\log \lambda|}\, \exp\Big(-\frac{1}{2}\, |\log \lambda|^2 \Big). \qquad (3.8)$$

To estimate $\mathbb{P}(E_7)$, observe that by scaling,

$$\begin{aligned}
\mathbb{P}(E_7) &= \mathbb{P}\Big(\, W_1(t) < c_5\, \sqrt{\lambda},\, W_1(t) + \sqrt{3}\, W_2(t) < 2c_5\, \sqrt{\lambda}, \quad 0 \le t \le 1 \Big) \\
&= \mathbb{P}\Big(\, (W_1(t), W_2(t)) \in \mathcal{D}, \quad 0 \le t \le 1 \Big),
\end{aligned}$$

where $\mathcal{D} \stackrel{\text{def}}{=} \{(x,y) :\ x < c_5\, \sqrt{\lambda},\, x + \sqrt{3}\, y < 2c_5\, \sqrt{\lambda}\}$. In words, the event on the right-hand side says that, starting from $(0,0)$, the planar Brownian motion (W_1, W_2) stays in $\mathcal{D}$ during $[0,1]$. A geometric observation (using translation and rotation) reveals that

$$\begin{aligned}
\mathbb{P}(E_7) = P\Big(\ &\text{starting from } (\sqrt{c_6\, \lambda}\,, 0), \text{ the angular part of} \\
&\qquad \text{planar Brownian motion lies in } (-\pi/3, \pi/3) \text{ during } [0,1] \Big),
\end{aligned}$$

with $c_6 \stackrel{\text{def}}{=} 4(c_5)^2/3$. Let Z denote a planar Brownian motion starting from $(1,0)$, with angular part θ (see Section 2.4). By scaling,

$$\begin{aligned}
\mathbb{P}(E_7) &= \mathbb{P}\Big(\sup_{0 \le s \le 1/(c_6\, \lambda)} |\theta(s)| < \frac{\pi}{3} \Big) \\
&\le c_7\, \lambda^{3/4}, \qquad (3.9)
\end{aligned}$$

the last inequality following from (2.4).

It remains to estimate $\mathbb{P}(E_8)$. By the stationarity of Brownian increments,

$$\begin{aligned}\mathbb{P}(E_8) &= \mathbb{P}\Big(W_1(t) < c_5\,(t+\lambda)\,|\log\lambda| + \sqrt{\lambda}\,|\log\lambda|, \quad 0 < t \le 1-\lambda \Big)\\ &\le \mathbb{P}\Big(W_1(t) < c_5\,t\,|\log\lambda| + (c_5+1)\sqrt{\lambda}\,|\log\lambda|, \quad 0 < t \le 1/2 \Big).\end{aligned}$$

For any $a > 0$ and $b > 0$,

$$\mathbb{P}\Big(W_1(s) < a + bs, \quad s > 0 \Big) = 1 - \mathrm{e}^{-2ab}, \tag{3.10}$$

(see for example Karatzas and Shreve [9, p. 197]). It follows that

$$\mathbb{P}\Big(W_1(t) - W_1(1/2) < (t-1/2)\,|\log\lambda| + 1, \quad t \ge 1/2 \Big) \ge c_8.$$

Therefore, by the independence of Brownian increments,

$$\begin{aligned}\mathbb{P}(E_8) &\le \frac{1}{c_8}\,\mathbb{P}\Big(W_1(t) < c_5\,t\,|\log\lambda| + (c_5+1)\sqrt{\lambda}\,|\log\lambda|, \quad 0 < t \le 1/2,\\ &\qquad W_1(t) < (t-1/2)\,|\log\lambda| + 1 + \frac{c_5}{2}\,|\log\lambda| + (c_5+1)\sqrt{\lambda}\,|\log\lambda|, \quad t \ge 1/2 \Big)\\ &\le \frac{1}{c_8}\,\mathbb{P}\Big(W_1(t) < (c_5+1)t\,|\log\lambda| + (c_5+1)\sqrt{\lambda}\,|\log\lambda|, \quad t > 0 \Big)\\ &\le c_9\,\sqrt{\lambda}\,\,|\log\lambda|^2, \end{aligned}\tag{3.11}$$

the last inequality following from (3.10). Combining (3.7)–(3.9) and (3.11) completes the proof of Theorem 3.1. □

4. Proof of Theorem 1.1

Before starting the proof of Theorem 1.1, we recall some notation which was already used in the previous sections: X is a symmetric stable process of index $\alpha \in (1,2]$, whose local time is L_t^x. The inverse local time at 0 is denoted by $\tau(r)$. Also, η will denote the FBM$(\alpha-1)$ introduced in (2.3). As before, we assume without loss of generality that c_0 satisfies Theorem B (see Section 2.2).

For brevity, we write

$$L_t^* \stackrel{\text{def}}{=} \sup_{x\in\mathbb{R}} L_t^x, \qquad t \ge 0.$$

The proof of Theorem 1.1 is divided into several small steps.

Lemma 4.1. *For any $b > 4$,*

$$\lim_{r\to\infty} \frac{(\log r)^b}{r} \Big(L^*_{\tau(r)} - r \Big) = \infty, \qquad \text{a.s.} \tag{4.1}$$

Proof. Since $b > 4$, it is possible to find a constant $0 < a < 1/5$ such that $b > 4/(5a)$. Define the sequence $r_n = \exp(n^a)$. By scaling,

$$\begin{aligned}\mathbb{P}\Big(L^*_{\tau(r_n)} < r_{n+1} + \frac{r_{n+1}}{(\log r_n)^b}\Big) &= \mathbb{P}\Big(L^*_{\tau(1)} < \frac{r_{n+1}}{r_n} + \frac{r_{n+1}}{r_n (\log r_n)^b}\Big) \\ &\le P\Big(L^*_{\tau(1)} < 1 + \frac{c_{10}}{n^{1-a}} + \frac{c_{11}}{n^{ab}}\Big).\end{aligned}$$

Applying Theorem 3.1 to $\lambda = c_{10}\, n^{-(1-a)} + c_{11}\, n^{-ab}$ yields that

$$\mathbb{P}\Big(L^*_{\tau(r_n)} < r_{n+1} + \frac{r_{n+1}}{(\log r_n)^b}\Big) \le c_{12} \Big(\frac{1}{n^{5(1-a)/4}} + \frac{1}{n^{5ab/4}}\Big) (\log n)^2,$$

which is summable for n (recalling that $a < 1/5$ and that $b > 4/(5a)$). An application of the Borel–Cantelli lemma, together with the monotonicity, gives that

$$\liminf_{r\to\infty} \frac{(\log r)^b}{r} \Big(L^*_{\tau(r)} - r \Big) \ge 1, \qquad \text{a.s.}$$

Since $b > 4$ is arbitrary, this completes the proof of Lemma 4.1. □

Lemma 4.2. *For $0 < \lambda < 1$ and $h > 0$,*

$$\mathbb{P}\Big(\sup_{|x|\le h} L^x_{\tau(1)} > 1 + \lambda \Big) \le c_{13} \exp\Big(-\frac{\lambda^2}{9h^{\alpha-1}}\Big). \tag{4.2}$$

Proof. Let η be an FBM$(\alpha - 1)$, independent of the underlying stable process X. Clearly,

$$\mathbb{P}\Big(\sup_{|x|\le h} L^x_{\tau(1)} > 1 + \lambda \Big) \le \mathbb{P}\Big(\sup_{|x|\le h} \big(L^x_{\tau(1)} + \frac{1}{2}\, \eta^2(x)\big) > 1 + \lambda \Big).$$

By Theorem B (see Section 2.2), we have

$$\begin{aligned}\mathbb{P}\Big(\sup_{|x|\le h} L^x_{\tau(1)} > 1 + \lambda \Big) &\le \mathbb{P}\Big(\frac{1}{2} \sup_{|x|\le h} \big(\eta(x) + \sqrt{2}\big)^2 > 1 + \lambda \Big) \\ &\le \mathbb{P}\Big(\sup_{|x|\le h} |\eta(x)| > \frac{\lambda}{2}\Big) \\ &= \mathbb{P}\Big(\sup_{|x|\le 1} |\eta(x)| > \frac{\lambda}{2h^{(\alpha-1)/2}}\Big).\end{aligned}$$

This yields (4.2) by means of a tail estimate for general Gaussian processes due to Marcus and Shepp [14]: if $\{Y(t);\, t \in \mathbb{T}\}$ is a bounded real-valued centered Gaussian process, then

$$\log \mathbb{P}\Big(\sup_{t\in\mathbb{T}} |Y(t)| > x \Big) \sim -\frac{x^2}{2\sigma_*^2}, \qquad x \to \infty,$$

with $\sigma_*^2 \overset{\text{def}}{=} \sup_{t\in\mathbb{T}} \mathbb{E}(Y^2(t))$. □

Lemma 4.3. *For any $M > 0$, there exists $c_{14} > 0$ and $h_* \in (0,1)$ such that, whenever $0 < h < h_*$ and $\lambda \geq M\, h^{(\alpha-1)/2}$,*

$$P\Big(\sup_{|x|\le h} |L^x_{\tau(1)} - 1| < \lambda \Big) \geq c_{14}. \tag{4.3}$$

Proof. Without loss of generality, we can assume that $M \le 1/2$. Let η be an FBM$(\alpha-1)$, independent of X. Clearly,

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{|x|\le h} |L^x_{\tau(1)} - 1| < \lambda \Big) \\
&\quad \geq \mathbb{P}\Big(\sup_{|x|\le h} |L^x_{\tau(1)} - 1| < \lambda,\ \sup_{|x|\le h} \eta^2(x) < \lambda \Big) \\
&\quad \geq \mathbb{P}\Big(\sup_{|x|\le h} |L^x_{\tau(1)} - 1 + \frac{1}{2}\eta^2(x)| < \frac{\lambda}{2},\ \sup_{|x|\le h} \eta^2(x) < \lambda \Big) \\
&\quad \geq \mathbb{P}\Big(\sup_{|x|\le h} |L^x_{\tau(1)} - 1 + \frac{1}{2}\eta^2(x)| < \frac{\lambda}{2} \Big) - \mathbb{P}\Big(\sup_{|x|\le h} \eta^2(x) \geq \lambda \Big) \\
&\quad \overset{\text{def}}{=} \mathbb{P}(E_9) - \mathbb{P}(E_{10}),
\end{aligned} \tag{4.4}$$

with obvious notation. By (2.3),

$$\begin{aligned}
\mathbb{P}(E_9) &= \mathbb{P}\Big(\sup_{|x|\le h} |\frac{1}{2}\eta^2(x) + \sqrt{2}\,\eta(x)| < \frac{\lambda}{2} \Big) \\
&\geq \mathbb{P}\Big(\sup_{|x|\le h} |\eta(x)| < \frac{\lambda}{4} \Big) \\
&= \mathbb{P}\Big(\sup_{|x|\le 1} |\eta(x)| < \frac{\lambda}{4h^{(\alpha-1)/2}} \Big) \\
&\geq \mathbb{P}\Big(\sup_{|x|\le 1} |\eta(x)| < \frac{M}{4} \Big) \\
&\overset{\text{def}}{=} c_{15}.
\end{aligned} \tag{4.5}$$

On the other hand,

$$\begin{aligned}\mathbb{P}(E_{10}) &= \mathbb{P}\Big(\sup_{|x|\le 1} \eta^2(x) \ge \frac{\lambda}{h^{\alpha-1}} \Big) \\ &\le \mathbb{P}\Big(\sup_{|x|\le 1} \eta^2(x) \ge \frac{M}{h^{(\alpha-1)/2}} \Big) \\ &\le \mathbb{P}\Big(\sup_{|x|\le 1} \eta^2(x) \ge \frac{M}{h_*^{(\alpha-1)/2}} \Big) \\ &\stackrel{\text{def}}{=} c_{16}. \end{aligned} \tag{4.6}$$

We can choose h_* sufficiently small such that $c_{16} < c_{15}$. The lemma follows by jointly considering (4.4), (4.5) and (4.6). □

Lemma 4.4. *It is possible to choose a sufficiently small $h_* \in (0,1)$ such that, for all $0 < h < h_*$ and $\lambda > 0$,*

$$\mathbb{P}\Big(\sup_{1\le r\le 2} \sup_{|x|\le h} (L^x_{\tau(r)} - r) > \lambda \Big) \le c_{17} \exp\Big(-\frac{\lambda^2}{243\, h^{\alpha-1}} \Big). \tag{4.7}$$

Proof. We assume without loss of generality that $\lambda \ge h^{(\alpha-1)/2}$, otherwise (4.7) holds trivially. Define,

$$\begin{aligned} Y_h(r) &\stackrel{\text{def}}{=} \sup_{|x|\le h} \Big(L^x_{\tau(r)} - L^0_{\tau(r)} \Big), \\ T &\stackrel{\text{def}}{=} \inf\Big\{ r \ge 1 :\ Y_h(r) > \lambda \Big\}, \\ E_{11} &\stackrel{\text{def}}{=} \Big\{ T \le 2;\ \sup_{|x|\le (3-T)^{1/(\alpha-1)} h} \Big| L^x_{\tau(3)} - L^x_{\tau(T)} - (3-T) \Big| < \frac{\lambda(3-T)}{3} \Big\}. \end{aligned}$$

Since $r \mapsto Y_h(r)$ is right-continuous, we have $Y_h(T) \ge \lambda$ when $T < \infty$. By the triangle inequality, on the event E_{11} (thus $1 \le T \le 2$),

$$\begin{aligned} \sup_{|x|\le h} \Big(L^x_{\tau(3)} - L^0_{\tau(3)} \Big) &\ge Y_h(T) - \sup_{|x|\le h} \Big| L^x_{\tau(3)} - L^x_{\tau(T)} - (3-T) \Big| \\ &> \lambda - \frac{\lambda(3-T)}{3} \\ &\ge \frac{\lambda}{3}. \end{aligned}$$

In other words, $E_{11} \subset \{Y_h(3) > \lambda/3\}$. Let $(\mathcal{F}_t)_{t\ge 0}$ be the natural filtration of X. Observe that on $\{T < \infty\}$, $T = L^0_{\tau(T)}$ is measurable with respect to $\mathcal{F}_{\tau(T)}$; and so is the event

$\{T \le 2\}$. Therefore, given $T \le 2$, E_{11} is independent of $\mathcal{F}_{\tau(T)}$. By the strong Markov and scaling properties,

$$\begin{aligned}\mathbb{P}(T \le 2)\,\mathbb{P}\Big(\sup_{|x|\le h} |L^x_{\tau(1)} - 1| < \frac{\lambda}{3}\Big) &= \mathbb{P}(E_{11}) \\ &\le \mathbb{P}(Y_h(3) > \lambda/3) \\ &= \mathbb{P}\Big(\sup_{|x|\le h/3^{1/(\alpha-1)}} L^x_{\tau(1)} > 1 + \frac{\lambda}{9}\Big),\end{aligned}$$

which, in view of Lemmas 4.2 and 4.3, yields: for small h and $\lambda \ge h^{(\alpha-1)/2}$,

$$\mathbb{P}(T \le 2) \le c_{18} \exp\Big(-\frac{\lambda^2}{243\, h^{\alpha-1}}\Big).$$

Since $\{\sup_{1\le r\le 2} \sup_{|x|\le h} (L^x_{\tau(r)} - r) > \lambda\} \subset \{T \le 2\}$, this completes the proof of the lemma. □

Lemma 4.5. *For any $\mu > 0$ and $b > 0$ such that $(\alpha - 1)\mu > 2b$, we have*

$$\lim_{r\to\infty} \frac{(\log r)^b}{r} \sup_{|x|\le r^{1/(\alpha-1)}/(\log r)^\mu} \big(L^x_{\tau(r)} - r\big) = 0, \qquad \text{a.s.} \tag{4.8}$$

Proof. Define $r_n = 2^n$. By scaling and Lemma 4.4, for all sufficiently large n,

$$\begin{aligned}&\mathbb{P}\Big(\sup_{r_n\le r\le r_{n+1}} \sup_{|x|\le r_{n+1}^{1/(\alpha-1)}/(\log r_n)^\mu} \big(L^x_{\tau(r)} - r\big) > \frac{r_n}{(\log r_n)^b}\Big) \\ &= \mathbb{P}\Big(\sup_{1\le r\le 2} \sup_{|x|\le 2^{1/(\alpha-1)}/(\log r_n)^\mu} \big(L^x_{\tau(r)} - r\big) > \frac{1}{(\log r_n)^b}\Big) \\ &\le c_{17} \exp\Big(-\frac{(\log r_n)^{(\alpha-1)\mu-2b}}{486}\Big),\end{aligned}$$

which sums. Lemma 4.5 follows from an immediate application of the Borel–Cantelli lemma. □

Proof of Theorem 1.1. Fix $\gamma > 9/(\alpha - 1)$. It is possible to choose $b > 4$ and $\varepsilon > 0$ such that

$$b < \frac{\alpha - 1}{2}\Big(\gamma - \frac{1}{\alpha - 1} - \frac{\varepsilon}{\alpha}\Big).$$

Since $r \mapsto \tau(r)$ is a stable subordinator of index $(\alpha - 1)/\alpha$, some well-known theorems (see for example, Fristedt [8, Theorems 11.2 and 11.7]) confirm that almost surely for all sufficiently large r,

$$\tau(r) < r^{\alpha/(\alpha-1)} (\log r)^{\alpha/(\alpha-1)+\varepsilon}, \tag{4.9}$$

$$\tau(r) > r^{(1-\varepsilon)\alpha/(\alpha-1)}. \tag{4.10}$$

Let t be very large, say $t \in [\tau(r-), \tau(r)]$. By (4.1),

$$L_t^* \geq L_{\tau(r-)}^* > r + \frac{r}{(\log r)^b}. \tag{4.11}$$

On the other hand, by (4.9) and (4.10), and writing $c_{19} \stackrel{\text{def}}{=} ((\alpha-1)/(1-\varepsilon)\alpha)^\gamma$,

$$\begin{aligned}
\sup_{|x| \leq t^{1/\alpha}/(\log t)^\gamma} L_t^x &\leq \sup_{|x| \leq (\tau(r))^{1/\alpha}/(\log \tau(r))^\gamma} L_{\tau(r)}^x \\
&\leq \sup_{|x| \leq c_{19}\,(\tau(r))^{1/\alpha}/(\log r)^\gamma} L_{\tau(r)}^x \\
&\leq \sup_{|x| \leq c_{19}\, r^{1/(\alpha-1)}/(\log r)^{\gamma-1/(\alpha-1)-\varepsilon/\alpha}} L_{\tau(r)}^x .
\end{aligned}$$

Applying Lemma 4.5 gives that

$$\sup_{|x| \leq t^{1/\alpha}/(\log t)^\gamma} L_t^x < r + \frac{r}{(\log r)^b},$$

which, in view of (4.11), implies that $|V(t)| > t^{1/\alpha}/(\log t)^\gamma$, where $V(t)$ is as before the most visited site (see (1.2)). As a consequence, for $\gamma > 9/(\alpha-1)$,

$$\liminf_{t \to \infty} \frac{(\log t)^\gamma}{t^{1/\alpha}} |V(t)| \geq 1, \qquad \text{a.s.}$$

This completes the proof of Theorem 1.1. □

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