

# Favourite sites of simple random walk

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**Summary.** We survey the current status of the list of questions related to the favourite (or: most visited) sites of simple random walk on  $\mathbb{Z}$ , raised by Pál Erdős and Pál Révész in the early eighties.

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*Dedicated to Professor Endre Csáki on the occasion of his 65th birthday*

## 1 Introduction

Let  $(S(n), n \in \mathbb{Z}_+)$  be a simple symmetric random walk on  $\mathbb{Z}$  with  $S(0) = 0$ . Define

$$(1.1) \quad \xi(n, x) := \#\{0 \leq k \leq n : S(k) = x\},$$

which is referred to as the (site) local time of the random walk. For each  $n$ , consider

$$(1.2) \quad \mathbb{V}(n) := \left\{ x \in \mathbb{Z} : \xi(n, x) = \max_{y \in \mathbb{Z}} \xi(n, y) \right\},$$

which is the set of sites which are the “most visited” by the random walk at step  $n$ . Following Erdős and Révész [11], an element of  $\mathbb{V}(n)$  is called “favourite site”.

The study of  $\mathbb{V}(n)$  was initiated by Erdős and Révész [11], and Bass and Griffin [2], and received much research interest from other mathematicians since. Somewhat surprisingly, many of the easy-to-formulate and innocent-looking open questions raised by Erdős and Révész in [11] and [12] remain unanswered so far. The present paper aims not only to have an overview upon known results in this field, but also to insist on unsolved problems in the hope that they will meet the interest of, and find solutions from, the reader.

To illustrate these solved or unsolved problems, we mention the following question: what is the probability that  $0 \in \mathbb{V}(n)$  for infinitely many  $n$ ?

Since the random walk is symmetric, one would be tempted to think that this probability would be 1. However, the correct answer is “0”. In fact, Bass and Griffin [2] proved the following result:

$$(1.3) \quad \lim_{n \rightarrow \infty} \inf_{x \in \mathbb{V}(n)} |x| = +\infty, \quad \text{a.s.}$$

In words, the process of favourite sites is transient. Actually, Bass and Griffin [2] showed that, for any  $\varepsilon > 0$ , the distance of the set of favourite sites from the origin goes faster than  $\sqrt{n}/(\log n)^{1+\varepsilon}$  but slower than  $\sqrt{n}/(\log n)^{1-\varepsilon}$ , almost surely. We will discuss this result in more detail in Section 2.

Let us mention another innocent-looking question. It is trivial that  $\mathbb{P}\{\#\mathbb{V}(n) = 1, \text{ i.o.}\} = 1$  (where “i.o.” stands for infinitely often). A little more thinking reveals that  $\mathbb{P}\{\#\mathbb{V}(n) = 2, \text{ i.o.}\} = 1$ . Erdős and Révész [11] asked:

$$\mathbb{P}\{\#\mathbb{V}(n) \geq 3, \text{ i.o.}\} = ?$$

This problem is still open. However, the following was recently proved by Tóth [23]:  $\mathbb{P}\{\#\mathbb{V}(n) \geq 4, \text{ i.o.}\} = 0$ . For more details, see Subsection 3.2.

The rest of the paper is splitted into four sections according to the natures of the problems invoked. Section 2 concerns the problems of how far and how close the favourite sites can be to the origin. For each of these two problems, we have some useful but incomplete information. In Section 3, we mention ten questions raised by Erdős and Révész in [11] and [12], and quoted as Open problems 1–10 in the book of Révész [21] (pages 130–131). Only a few of these questions have found solutions. Section 4 is devoted to some related problems. More precisely, we will discuss problems for rarely visited sites, favourite edges,

and the location of favourite sites. Finally in Section 5, we briefly describe some known results for favourite sites of other stochastic processes such as Brownian motion and general Lévy processes.

## 2 Large and small values of the favourite sites

Throughout this section, we pick an arbitrary element  $V(n) \in \mathbb{V}(n)$ . According to (1.3),  $|V(n)|$  goes to  $\infty$  almost surely (when  $n \rightarrow \infty$ ). The question is to determine the rate with which  $|V(n)|$  goes to infinity. The answers are different for lower and upper limits.

### 2.1 Escape rates of favourite sites

As far as the lower limits are concerned, the best possible result available so far is due to Bass and Griffin [2].

**Theorem A** ([2]). *With probability one,*

$$(2.1) \quad \liminf_{n \rightarrow \infty} \frac{|V(n)|}{n^{1/2}(\log n)^{-\gamma}} = \begin{cases} 0 & \text{if } \gamma < 1, \\ \infty & \text{if } \gamma > 11. \end{cases}$$

Throughout the paper, when we state a limit result for  $V(n)$  as in (2.1), it is to be understood that the convergence holds uniformly in all  $V(n) \in \mathbb{V}(n)$ .

Bass and Griffin [2] also asked about the exact rate of escape of the transient process of the favourite sites. This seems to be a very challenging problem. Here we formulate it in a weaker form.

**Question 2.1** *Find the value of the constant  $\gamma_0$  such that with probability one,*

$$\liminf_{n \rightarrow \infty} \frac{|V(n)|}{n^{1/2}(\log n)^{-\gamma}} = \begin{cases} 0 & \text{if } \gamma < \gamma_0, \\ \infty & \text{if } \gamma > \gamma_0. \end{cases}$$

According to (2.1), we must have  $\gamma_0 \in [1, 11]$ . There is good reason to expect that  $\gamma_0$  would lie in  $[1, 2]$ .

## 2.2 Upper rates of favourite sites

If we are interested in the upper limits of  $|V(n)|$ , here is an answer in the form of the law of the iterated logarithm (LIL), which was discovered independently by Erdős and Révész [11] and Bass and Griffin [2].

**Theorem B** ([11], [2]). *With probability one,*

$$\limsup_{n \rightarrow \infty} \frac{V(n)}{(2n \log \log n)^{1/2}} = 1, \quad \text{a.s.}$$

Therefore, the favourite site  $V(n)$  and the random walk  $S(n)$  satisfy the same LIL. However, a closer inspection by Erdős and Révész [11] reveals that they have different Lévy upper functions:

**Theorem C** ([11]). *There exists a deterministic sequence  $(a_n)_{n \geq 1}$  satisfying*

$$\mathbb{P}\{S(n) \geq a_n, \text{ i.o.}\} = 1, \quad \mathbb{P}\{V(n) \geq a_n, \text{ i.o.}\} = 0.$$

The upper functions of  $S(n)$  are characterized by the Erdős–Feller–Kolmogorov–Petrowsky integral test (Révész [21], p. 35): if  $(a_n)_{n \geq 1}$  is a non-decreasing sequence of positive numbers, then

$$(2.2) \quad \mathbb{P}\{S(n) \geq n^{1/2} a_n, \text{ i.o.}\} = \begin{cases} 0 \\ 1 \end{cases} \iff \sum_n \frac{a_n}{n} \exp\left(-\frac{a_n^2}{2}\right) \begin{cases} < \infty, \\ = \infty. \end{cases}$$

Erdős and Révész [11] and Bass and Griffin [2] asked the following question:

**Question 2.2** *Find an integral test to decide whether  $\mathbb{P}\{V(n) \geq a_n, \text{ i.o.}\} = 0$ .*

We believe that a key to Question 2.2 would be to control the upper tail probability of the favourite site, formulated here for Brownian motion: let  $W$  be a standard Brownian motion whose local time process is denoted by  $(L(t, x); t \in \mathbb{R}_+, x \in \mathbb{R})$ , i.e., for any bounded Borel function  $f$ ,

$$\int_0^t f(W(s)) ds = \int_{\mathbb{R}} f(x) L(t, x) dx.$$

Let  $U$  denote the (almost surely) unique favourite site at time 1:  $L(1, U) = \sup_{x \in \mathbb{R}} L(1, x)$ .

We pose the following

**Conjecture 2.3** *There exists a constant  $\nu > 1$  such that*

$$(2.3) \quad 0 < \liminf_{x \rightarrow +\infty} x^\nu e^{x^2/2} \mathbb{P}(U > x) \leq \limsup_{x \rightarrow +\infty} x^\nu e^{x^2/2} \mathbb{P}(U > x) < +\infty.$$

If (2.3) holds, then we think that we should be able to obtain an integral test characterizing the upper functions of  $V(n)$ :

**Conjecture 2.4** *Let  $\nu > 1$  be the constant satisfying (2.3). For any non-decreasing sequence  $(a_n)_{n \geq 1}$  of positive numbers,*

$$(2.4) \quad \mathbb{P}\{V(n) \geq n^{1/2} a_n, \text{ i.o.}\} = \begin{cases} 0 \\ 1 \end{cases} \iff \sum_n \frac{a_n^{2-\nu}}{n} \exp\left(-\frac{a_n^2}{2}\right) \begin{cases} < \infty, \\ = \infty. \end{cases}$$

Let us explain why we conjecture  $\nu > 1$ . By the trivial inequality  $U \leq \sup_{0 \leq t \leq 1} W(t)$  and the usual estimate for Gaussian tails, it is easily seen that if the first inequality in (2.3) holds, then  $\nu \geq 1$ . On the other hand, if  $\nu$  were equal to 1, then (2.4) would be the same as the Erdős–Feller–Kolmogorov–Petrowsky test in (2.2) which would contradict Theorem C. This leads us to the conjecture  $\nu > 1$ .

We mention that there seems to be no unanimity about Conjectures 2.3 and 2.4.

More discussions upon the distribution of  $U$  can be found in Subsection 3.5 below.

### 3 Ten Erdős–Révész questions

Erdős and Révész in [11] and [12] raised many questions about favourite sites of random walk, which we quote below. They correspond exactly to Questions 1–10 on pp. 130–131 of the book of Révész [21]. Recall from (1.1) that  $\xi(n, x)$  is the local time of the random walk, and from (1.2) that  $\mathbb{V}(n)$  is the set of favourite sites. As before,  $V(n)$  denotes an arbitrary element of  $\mathbb{V}(n)$ .

#### 3.1 Joint behaviour of favourite site and maximum local time

Let  $\xi^*(n) := \sup_{x \in \mathbb{Z}} \xi(x, n)$ , which is the maximum local time. In their proof of Theorem B (cf. Subsection 2.2), Erdős and Révész [11] noticed that, for any  $\varepsilon > 0$ , almost surely there exist infinitely many  $n$  such that simultaneously,  $V(n) \geq (1 - \varepsilon)(2n \log \log n)^{1/2}$  and

$\xi^*(n) \geq c(2n \log \log n)^{1/2}$ , for some positive constant  $c$  depending on  $\varepsilon$ . This led them to ask the following question: what can be said about the joint asymptotics of  $V(n)$  and  $\xi^*(n)$ ? In particular, if  $V(n)$  is close to its maximal possible value, how large can  $\xi^*(n)$  be?

If  $V(n)$  and  $\xi^*(n)$  were asymptotically independent, then one would expect that the limit set of  $\{V(n)/(2n \log \log n)^{1/2}, \xi^*(n)/(2n \log \log n)^{1/2}\}$  should be the half-disc  $\{(x, y) : y \geq 0, x^2 + y^2 \leq 1\}$ . However, the correct answer provided by Csáki et al. [5] shows that things do not go exactly like this.

**Theorem D** ([5]). *With probability one, the random sequence*

$$\left( \frac{V(n)}{(2n \log \log n)^{1/2}}, \frac{\xi^*(n)}{(2n \log \log n)^{1/2}} \right)_{n \geq 3}$$

*is relatively compact, whose limit set is identical to the simplex  $\{(x, y) : y \geq 0, |x| + y \leq 1\}$ .*

In particular, Theorem D implies Theorem B, and also the LIL for the maximum local time of random walk which was originally proved by Kesten [15].

The proof of Theorem D relies on an invariance principle for local times due to Révész [20] (recalled in (3.3) below) and on the Ray–Knight theorem for Brownian local time.

### 3.2 Many favourite sites

At each step (say,  $n \rightarrow (n + 1)$ ) of the random walk exactly one of the following three possibilities occurs:

- (1) the currently occupied site is not favourite,  $S(n + 1) \notin \mathbb{V}(n + 1)$ , and thus  $\mathbb{V}(n + 1) = \mathbb{V}(n)$  remains unchanged;
- (2) the currently occupied site becomes a new favourite besides the favourites of the previous time  $n$ , thus  $\mathbb{V}(n) \subset \mathbb{V}(n + 1)$  and  $\mathbb{V}(n + 1) \setminus \mathbb{V}(n) = \{S(n + 1)\}$ ;
- (3) the random walk revisits a site which was already favourite in the previous time  $n$ , and so this new site becomes the only new favourite  $\mathbb{V}(n + 1) = \{S(n + 1)\} \subset \mathbb{V}(n)$ .

It follows that the number of favourite sites either remains unchanged, or increases by one, or drops down to 1. From the recurrence of the random walk it follows easily that  $\mathbb{P}\{\#\mathbb{V}(n) = 1, \text{ i.o.}\} = 1$  and  $\mathbb{P}\{\#\mathbb{V}(n) = 2, \text{ i.o.}\} = 1$ . The question is

$$(3.1) \quad \mathbb{P}\{\#\mathbb{V}(n) = r, \text{ i.o.}\} = ? \quad r = 3, 4, \dots$$

Erdős and Révész [11] conjectured that this probability should be 0, for any  $r \geq 3$ .

Recently, Tóth [23] proved the following

**Theorem E** ([23]). *We have,*

$$\mathbb{P}\{\#\mathbb{V}(n) \geq 4, \text{ i.o.}\} = 0.$$

The main argument of the proof is based on some (quite natural) rearrangements of sums and the Ray–Knight representation of the local time process stopped at inverse local times. On the technical level, the proof of Theorem E relies on controlling the probability distribution of the number of global maxima of given height  $h \gg 1$  of critical Galton–Watson processes.

### 3.3 Frequency of having many favourite sites

It is intuitively clear that in “most situations”, there is only one favourite site. Erdős and Révész [11] were interested in the question about how often there are at least two favourite sites. To formulate their question precisely, let

$$\nu_0 := 0, \quad \nu_{k+1} := \inf\{n > \nu_k : \#\mathbb{V}(n) > 1\},$$

that is:  $\nu_k$  is the  $k$ -th time when there are more than one favourite sites. Or, alternatively, we can define

$$\kappa_n := \#\{0 < j \leq n : \#\mathbb{V}(j) > 1\}.$$

Now, the question is:

Is it true that  $\lim_{k \rightarrow \infty} \frac{\nu_k}{k} = \infty$ , or, equivalently,  $\lim_{n \rightarrow \infty} \frac{\kappa_n}{n} = 0$ , with probability one?

The question remains open.

### 3.4 Big jumps of favourite sites

In order to formulate precisely questions 4, 8 and 9 of the Erdős–Révész list we define the *last visited favourite site*  $\ell(n) \in \mathbb{V}(n)$  inductively, as follows:

$$(3.2) \quad \ell(0) := 0, \quad \ell(n+1) := \begin{cases} \ell(n) & \text{if } S(n+1) \notin \mathbb{V}(n+1), \\ S(n+1) & \text{if } S(n+1) \in \mathbb{V}(n+1). \end{cases}$$

The fourth question concerns how large the jump sizes of favourite site can be. The answer is formulated in the following LIL.

**Theorem F** ([5]). *With probability one,*

$$\limsup_{n \rightarrow \infty} \frac{|\ell(n+1) - \ell(n)|}{(2n \log \log n)^{1/2}} = 1.$$

In particular, Theorem F tells us that the extraordinarily large jumps of favourite site are asymptotically comparable to the size of the range of the random walk.

The main ingredient in the proof of Theorem F is the Ray–Knight theorem.

### 3.5 Limit law of favourite sites

The question here is: what is the limit distribution of  $V(n)/n^{1/2}$  when  $n \rightarrow \infty$ ?

Here is an argument to show that  $V(n)/n^{1/2}$  has a non-degenerate limit distribution. Indeed, according to a theorem of Révész [20], possibly in an enlarged probability space, there exists a coupling for random walk  $(S(n), n \in \mathbb{Z}_+)$  and Brownian motion  $(W(t), t \in \mathbb{R}_+)$  such that for any  $\varepsilon > 0$ ,

$$(3.3) \quad \sup_{x \in \mathbb{Z}} |\xi(n, x) - L(n, x)| = o(n^{1/4+\varepsilon}), \quad \text{a.s.},$$

where  $\xi$  and  $L$  denote the local times of  $S(n)$  and  $W(t)$ , respectively. Thus, for any fixed  $a > 0$ ,

$$\begin{aligned} \mathbb{P}\left(\frac{V(n)}{n^{1/2}} > a\right) &\leq \mathbb{P}\left(\sup_{x > n^{1/2}a} \xi(n, x) \geq \sup_{x \leq n^{1/2}a} \xi(n, x)\right) \\ &\leq \mathbb{P}\left(\sup_{x > n^{1/2}a} L(n, x) \geq \sup_{x \leq n^{1/2}a} L(n, x) - n^{1/4+\varepsilon}\right) \end{aligned}$$

$$\begin{aligned}
&= \mathbb{P} \left( \sup_{y>a} L(1, y) \geq \sup_{y \leq a} L(1, x) - n^{-1/4+\varepsilon} \right) \\
&\rightarrow \mathbb{P} \left( \sup_{y>a} L(1, y) \geq \sup_{y \leq a} L(1, y) \right) = \mathbb{P}(U > a),
\end{aligned}$$

when  $n \rightarrow \infty$ , where  $U$  denotes as in Subsection 2.2 the location of the maximum (on  $\mathbb{R}$ ) of  $x \mapsto L(1, x)$ :  $L(1, U) = \sup_{x \in \mathbb{R}} L(1, x)$ . Similarly,

$$\begin{aligned}
\mathbb{P} \left( \frac{V(n)}{n^{1/2}} > a \right) &\geq \mathbb{P} \left( \sup_{x>n^{1/2}a} \xi(n, x) > \sup_{x \leq n^{1/2}a} \xi(n, x) \right) \\
&\geq \mathbb{P} \left( \sup_{x>n^{1/2}a} L(n, x) > \sup_{x \leq n^{1/2}a} L(n, x) + n^{1/4+\varepsilon} \right) \\
&\rightarrow \mathbb{P}(U > a),
\end{aligned}$$

so that,

$$\lim_{n \rightarrow \infty} \frac{V(n)}{n^{1/2}} = U, \quad \text{in distribution.}$$

We are grateful to Endre Csáki for having communicated to us this simple argument for the weak convergence.

The distribution of  $U$  was characterized by Theorem 6.2 of Borodin [4], where a double Laplace transform of the limit distribution was computed, namely, he got a close-form expression for  $\mathbb{E}(e^{-a\sqrt{A}|U|})$  for  $a > 0$  and an exponential variable  $A$  which is independent of  $U$ . However, the expression for  $\mathbb{E}(e^{-a\sqrt{A}|U|})$  found in [4] looks very complicated, involving ratios of Whittaker functions. We have not been able to invert the double Laplace transform, or even to get reasonably good information for the tail probability of  $U$  which would be useful for the upper functions of  $V(n)$ , see Conjectures 2.3 and 2.4 in Subsection 2.2.

### 3.6 Total number of favourite sites

Let  $\alpha(n)$  denote the number of all the different favourite sites up to step  $n$ , i.e.,  $\alpha(n) = \#(\bigcup_{k=0}^n \mathbb{V}(k))$ . Is it true that with probability one, for all large  $n$ ,  $\alpha(n) \leq (\log n)^c$  (for some constant  $c > 0$ )?

The question is still open. Omer Adelman (personal communication) has a lower bound for  $\alpha(n)$ , proving that with probability one, for all large  $n$ ,  $\alpha(n) \geq c^* \log n$ , for some constant  $c^* > 0$ .

### 3.7 Durations of favourite sites

The question is: how long a favourite site can stay favourite? More precisely, let  $\beta(n) := \max\{j - i : 0 \leq i \leq j \leq n, \bigcap_{k=i}^j \mathbb{V}(k) \neq \emptyset\}$ . In words,  $\beta(n)$  is the duration of the longest period (before  $n$ ) during which a favourite site stays favourite. What can be said about the asymptotic behaviour of  $\beta(n)$ ?

No answer available so far.

### 3.8 “Capricious” favourite sites

If  $x$  is a favourite site at some stage, can it happen that the favourite site moves away from  $x$  but later returns to  $x$ ? More precisely, do infinite random sequences  $\dots < c_{n-1} < a_n < b_n < c_n < a_{n+1} < \dots$  of positive integers exist such that  $\ell(a_n) = \ell(c_n) \neq \ell(b_n)$ , for  $n = 1, 2, \dots$ ? (Recall the definition of the last visited favourite site, (3.2).)

The question remains open.

### 3.9 Small jumps of favourite sites

Consider the random increasing sequence of times, when the last visited favourite site changes value

$$\lambda_0 := 0, \quad \lambda_{k+1} := \inf\{n > \lambda_k : \ell(n) \neq \ell(n-1)\},$$

and the jump sizes of  $\ell(n)$  at these times:

$$j_k := |\ell(\lambda_{k+1}) - \ell(\lambda_k)|.$$

It seems likely that  $j_k \rightarrow \infty$  almost surely, with  $k \rightarrow \infty$ . How to describe the limit behaviour of  $j_k$ ? This question is a companion to the one in Subsection 3.4. While in Subsection 3.4 the upper behaviour of the jump size was determined with satisfying precision, we have not been able to get any non-trivial information about the lower behaviour.

### 3.10 Occupation times and favourite sites

The arcsine law says that with big probability, the random walk spends a long time on one half of the line (say,  $\mathbb{Z}_+$ ) and only a short time on the other half (in this case  $\mathbb{Z}_-$ ). Is it true that the favourite site is located on the same side where the random walk spends the most

time? For example, if  $(\mu_k)_{k \geq 1}$  is a random sequence of integers satisfying  $\mu_k^{-1} \sum_{j=1}^{\mu_k} \mathbb{1}_{\{S_j \geq 0\}} \rightarrow 1$  ( $k \rightarrow \infty$ ), then Erdős and Révész [11] conjectured that  $V(\mu_k) \rightarrow +\infty$ , a.s.

The answer to the conjecture is no. Relying on the Ray–Knight theorem and careful analysis of the sample paths of Bessel processes, Csáki and Shi [7] prove the existence of a random sequence  $(\mu_k, k \geq 1)$  such that  $\mu_k^{-1} \sum_{j=1}^{\mu_k} \mathbb{1}_{\{S_j < 0\}} \leq c/(\log \log \mu_k)^2$  (for some constant  $c > 0$  and all  $k$ ), and yet  $V(\mu_k) < 0$  for all  $k$ .

## 4 Some related questions for simple random walk

### 4.1 Rarely visited sites

(1) *Rarely visited sites.* Let  $R(n) := \{S(0), S(1), \dots, S(n)\}$  be the range of the random walk up to step  $n$ . Erdős and Révész [11] conjectured that for any integer  $r \geq 3$ , the probability that there are infinitely many  $n$  such that each of the sites of  $R(n)$  has been visited at least  $r$  times up to step  $n$  is 0. This was disproved by Tóth [22], who showed that for any integer  $r$ , this probability actually equals 1. The proof of this result again relies on the Ray–Knight representation of local times. This time one has to control the probability of the event that the value of the critical Galton–Watson process drops down to zero from a given value  $r$ , uniformly in the initial condition.

Another aspect of rarely visited sites was studied by Major [18]. He proved that, if  $Z(n)$  denotes the number of sites of  $R(n)$  which have been visited exactly once, then with probability one,  $\limsup_{n \rightarrow \infty} Z(n)/(\log n)^2 = c$ , where  $c$  is a finite and positive constant.

### 4.2 Favourite edges

(2) *Favourite edges.* Instead of looking at favourite sites (which by definition maximize the *site* local time of the random walk), we look at favourite edges which maximize the edge local time. We can ask a similar question as Question 3.2: are there infinitely many  $n$  such that there are at least 3 favourite edges at time  $n$ ? Tóth and Werner [24] proved that with probability one, there are at most finitely many  $n$  such that there are at least 4 favourite edges at time  $n$ . This is a simpler predecessor of Theorem E, cited above.

### 4.3 Location of favourite sites

(3) *Location of favourite sites.* The question is whether it is possible for a favourite site to be close to the boundary of the range  $R(n)$ , i.e., close to  $\max_{0 \leq k \leq n} S(k)$  or to  $\min_{0 \leq k \leq n} S(k)$ . (The question was originally due to Omer Adelman). Csáki and Shi [6] proved that the answer is no: with probability one,  $\max_{0 \leq k \leq n} S(k) - V(n)$  (and by symmetry,  $V(n) - \min_{0 \leq k \leq n} S(k)$ ) goes to infinity, and the rate of escape was determined.

## 5 Favourite sites of other processes

(1) *Brownian motion.* Many questions and results mentioned above can be formulated for the favourite sites of Brownian motion in the obvious way. Some further discussions and questions can be found in Leuridan [17].

(2) *Symmetric stable processes.* In Bass et al. [1], it was proved that the favourite site of a symmetric stable process is also transient. Eisenbaum [9] proved a collection of interesting results for favourite sites of symmetric stable processes. For example, she showed that for each given  $t \geq 0$  there is almost surely a unique favourite site at time  $t$ , and that with probability one, for all  $t \geq 0$ , there are at most two favourite sites at time  $t$ .

(3) *Lévy and Markov processes.* Most of the results for symmetric stable processes can be extended to a larger class of Lévy and even symmetric Markov processes. See Marcus [19], Eisenbaum and Khoshnevisan [10].

(4) *Two-dimensional random walk.* Let  $V_2(n)$  denote a favourite site, at time  $n$ , of a simple symmetric random walk on  $\mathbb{Z}^2$ . Dembo et al. [8] proved that  $(\log \|V_2(n)\|)/\log n$  converges to  $1/2$  with probability one, where  $\|x\|$  denotes the Euclidean modulus in  $\mathbb{R}^2$ .

(5) *Transient processes.* If  $\{X(t), t \geq 0\}$  is a transient process, having local time at  $t = \infty$ , denoted by  $L(\infty, x)$ , then  $\{x \in [0, T] : L(\infty, x) = \sup_{y \in [0, T]} L(\infty, y)\}$  is the set of favourite sites of  $X$  in  $[0, T]$ . Bertoin and Marsalle [3] studied the case when  $X$  is a Brownian motion with a positive drift, and Hu and Shi [13] the case when  $X$  is the modulus of a  $d$ -dimensional Brownian motion ( $d > 2$ ). They obtained respective rates of escape (when  $T \rightarrow \infty$ ) of favourite sites.

(6) *Poisson process*. Khoshnevisan and Lewis [16] studied favourite sites of a Poisson process. They obtained several laws of the iterated logarithm.

(7) *Random walk in random environment*. The favourite sites can be defined for nearest neighbour random walk in random environment, exactly as for the usual random walk. Hu and Shi [14] considered the recurrent case and proved that the process of favourite sites is again transient, and the escape rate was characterized via an integral test. The latter question is still open for the usual simple random walk, see Question 2.1 above. The problem of escape rates of favourite sites is the only problem we are aware of, which is solved for random walk in random environment, but which is open for the usual random walk.

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