Small Deviations for Some Multi-Parameter Gaussian Processes

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Summary. We prove some general lower bounds for the probability that a multi-parameter Gaussian process has very small values. These results, when applied to a certain class of fractional Brownian sheets, yield the exact rate for their so-called small ball probability. We show by example how to use such results to compute the Hausdorff dimension of some exceptional sets determined by maximal increments.

Keywords. Gaussian random field, fractional Brownian sheet, small ball probability, Hausdorff dimension, exceptional set, random fractal.

1991 Mathematics Subject Classification. 60G60; 60G17.

1 Introduction

Let $X := \{X(\mathbf{u}),\ \mathbf{u} \in I^d\}$, $d \geq 1$, be a real-valued mean zero Gaussian process indexed by the $d-$dimensional unit cube I^d, where $I = [0, 1]$, and let

$$\|X\| := \sup_{\mathbf{u} \in I^d} |X(\mathbf{u})|.$$

While the study of the large deviations of $\|X\|$ has enjoyed much popularity, and has lead to a systematic theory (cf. Adler [1], Ledoux and Talagrand [12], Lifshits [16]), properties

[1]Research partly supported by an NSF/CNRS grant.

of its *small deviations* remain far from being well understood. More precisely, the problem is to investigate the asymptotics of

$$\log P\left(\|X\| \le \varepsilon\right), \tag{1.1}$$

when ε goes to 0. It turns out that evaluating the small ball probability in (1.1) is in general equivalent to solving a long open metric entropy problem in functional analysis (cf. Kuelbs and Li [9] and Li and Linde [14]), and it is for this reason that relatively little is known about the probability in (1.1), except for some special examples of X. For more details, we refer to the survey papers by Ledoux [11], Lifshits [17] and Li and Shao [15].

The aim of this paper is to provide explicit lower bounds (Proposition 2.1 in Section 2) for the probability in (1.1), under some regularity conditions upon the covariance function of X. Our featured example is the $\alpha-$fractional Brownian sheet.

We define an $\alpha-$fractional Brownian sheet on I^d to be a mean zero Gaussian process with the following covariance function:

$$E\left(X(\mathbf{s})X(\mathbf{t})\right) = a\prod_{i=1}^{d} \frac{s_i^{\alpha_i} + t_i^{\alpha_i} - |s_i - t_i|^{\alpha_i}}{2}, \tag{1.2}$$

where a is some positive constant and each $0 < \alpha_i < 2$. Our result in Theorem 2.1 (Section 2) says that whenever there is a unique minimum γ among $\alpha_1, ..., \alpha_d$, we can find a finite positive constant $\tau = \tau(a, \alpha)$ such that

$$-\lim_{\varepsilon \searrow 0} \varepsilon^{2/\gamma} \log P\{\|X\| \le \varepsilon\} = \tau. \tag{1.3}$$

Here we use the notation $\mathbf{s} = (s_1, ..., s_d)$, $\mathbf{t} = (t_1, ..., t_d)$ and $\alpha = (\alpha_1, ..., \alpha_d)$. This of course shows that γ, whenever it is unique, completely determines the small ball behavior of X. Li and Linde [13] and Shao [21], independently by different methods, showed that (1.3) holds in the $1-$parameter case. Our proof borrows some of the ideas in [13].

To obtain exact small ball probabilities for the $\alpha-$fractional Brownian sheet when more than one $\alpha_i = \gamma$ appears to be a challenging problem. Recall that in the case $d = 2$ and $\alpha_1 = \alpha_2 = 1$, the two parameter Brownian sheet, Talagrand [25] gave

$$\begin{aligned} -\infty &< \liminf_{\varepsilon \to 0} \frac{\varepsilon^2}{(\log(1/\varepsilon))^3} \log P\{\|W_{(1,1)}\| \le \varepsilon\} \\ &\le \limsup_{\varepsilon \to 0} \frac{\varepsilon^2}{(\log(1/\varepsilon))^3} \log P\{\|W_{(1,1)}\| \le \varepsilon\} < 0. \end{aligned}$$

He remarked that the case $d > 2$ and $\alpha_1 = ... = \alpha_d = 1$ is likely very difficult. Some partial results can be found in Dunker [5], Dunker *et al.* [6] and Shao and Wang [22]. For instance, for the fractional Brownian sheet in the case $\alpha_1 = ... = \alpha_d = \alpha$, with $d \ge 2$, the general results of Dunker [5] imply there exist constants $0 < K_1, K_2 < \infty$ such that for all $0 < \varepsilon \le 1$,

$$-K_1\varepsilon^{-2/\alpha} \log(1/\varepsilon)^{(1+\alpha)d/\alpha - 1} \le \log P\{\|W_{(\alpha,\dots,\alpha)}\| \le \varepsilon\} \le -K_2\varepsilon^{-2/\alpha} \log(1/\varepsilon)^{(1+\alpha)d/\alpha - 2}.$$

Refer to Theorem 5.7 in the survey paper of Li and Shao [15]. Roughly speaking, our result shows then that whenever γ is unique and $d \geq 2$ the small ball problem for the fractional Brownian sheet reduces to the 1−parameter case.

One of the motivations for the study of small ball probabilities is that they are essential tools to prove strong theorems for the lower limits of X. These are the so-called Chung-type laws of the iterated logarithm. There is also a close connection between small ball probabilities and the Hausdorff dimension of certain exceptional sets. As an example of how this connection works, we shall follow the general procedure detailed in Deheuvels and Mason [4] to show that our small ball results enable us to calculate the Hausdorff dimension of some exceptional sets determined by the maximal increments of X along a fixed coordinate.

The rest of the paper is organized as follows. In Section 2, we present our main results. For the sake of clarity, this section is divided into two subsections. The first is devoted to the statements of our small probability results and the second to those on Hausdorff dimension. We prove the corresponding results in Sections 3 and 4, respectively.

2 Statement of results

2.1 Small ball probabilities for Gaussian sheets

Let $X := \{X(\mathbf{u}),\ \mathbf{u} \in I^d\}$ be, as before, a real-valued mean zero Gaussian process and write

$$X([\mathbf{s}, \mathbf{t}]) = \int_{[\mathbf{s},\mathbf{t}]} X(d\mathbf{u}),$$

for any $[\mathbf{s}, \mathbf{t}] = [s_1, t_1] \times ... \times [s_d, t_d] \subset I^d$.

Proposition 2.1. *Let $X(\mathbf{0}) = 0$. Assume that for some constants $\kappa > 0$, $\eta > 0$ and $0 < \alpha_i < 2$, $i = 1, ..., d$,*

$$E[X([\mathbf{s}, \mathbf{s}+\mathbf{h}])^2] \leq \kappa \prod_{i=1}^{d} h_i^{\alpha_i}, \tag{2.1}$$

for all $[\mathbf{s}, \mathbf{s}+\mathbf{h}] = [s_1, s_1+h_1] \times ... \times [s_d, s_d+h_d] \subseteq I^d$ satisfying $0 < h_i < \eta$. Assume, furthermore, that

(2.2) *there is a unique minimum among $\alpha_1, ..., \alpha_d$.*

Then there exists a constant $\tau > 0$ depending on α, η and κ such that for all $\varepsilon > 0$ sufficiently small

$$\log P\{\|X\| \leq \varepsilon\} \geq -\tau \varepsilon^{-2/\gamma}$$

where

$$\gamma = \min_{1 \leq i \leq d} \alpha_i. \tag{2.3}$$

The following result confirms that the lower bound in Proposition 2.1 is sharp when X has a particular covariance structure.

Corollary 2.1. *Let X be a mean zero Gaussian process on I^d, where $d \geq 2$, with a covariance function of the form*

$$E[X(s_1, \mathbf{t}_1)X(s_2, \mathbf{t}_2)] = (s_1 \wedge s_2)\sigma(\mathbf{t}_1, \mathbf{t}_2), \tag{2.4}$$

for $(s_1, \mathbf{t}_1), (s_2, \mathbf{t}_2) \in I \times I^{d-1}$, where $\sigma(\mathbf{t}_1, \mathbf{t}_2)$ is the convariance function of a mean zero Gaussian process Y on I^{d-1}such that for some constants $\kappa > 0$, $\eta > 0$ and $1 < \beta_i < 2$, $i = 1, ..., d-1$,

$$E[Y([\mathbf{t}, \mathbf{t} + \mathbf{h}])^2] \leq \kappa \prod_{i=1}^{d-1} h_i^{\beta_i}, \tag{2.5}$$

for all $[\mathbf{t}, \mathbf{t} + \mathbf{h}] = [t_1, t_1 + h_1] \times ... \times [t_{d-1}, t_{d-1} + h_{d-1}] \subseteq I^{d-1}$ satisfying $0 < h_i < \eta$. Then there exists a finite positive constant τ_1 such that

$$- \lim_{\varepsilon \searrow 0} \varepsilon^2 \log P\{\|X\| \leq \varepsilon\} = \tau_1. \tag{2.6}$$

Our next result concerns the $\alpha-$fractional Brownian sheet. This process has the following integral representation.

For α satisfying $0 < \alpha_i < 2$ and $\mathbf{t} \in I^d$ let

$$W_\alpha(\mathbf{t}) = \sqrt{a} \int_{-\infty}^{t_1} ... \int_{-\infty}^{t_d} \prod_{i=1}^{d} g_{\alpha_i}(t_i, u_i) W(d\mathbf{u}),$$

where $W(\mathbf{u})$ is a standard Brownian sheet, a is a positive constant and

$$g_{\alpha_i}(s, u) = (s - u)^{(\alpha_i - 1)/2} - ((-u)_+)^{(\alpha_i - 1)/2}. \tag{2.7}$$

We will call the process W_α the $\alpha-$fractional Brownian sheet. It has covariance function

$$E[W_\alpha(\mathbf{s})W_\alpha(\mathbf{t})] = a \prod_{i=1}^{d} \sigma_{\alpha_i}(s_i, t_i), \tag{2.8}$$

where

$$\sigma_{\alpha_i}(s, t) = \frac{1}{2}[s^{\alpha_i} + t^{\alpha_i} - |s - t|^{\alpha_i}]. \tag{2.9}$$

Whenever (2.2) is satisfied, we get the exact rate function in the logarithmic scale for the small probability of W_α.

Theorem 2.1. *Whenever W_α is the $\alpha-$fractional Brownian sheet satisfying* (2.2), *there exists a finite positive constant τ_γ such that*

$$- \lim_{\varepsilon \searrow 0} \varepsilon^{2/\gamma} \log P\{\|W_\alpha\| \leq \varepsilon\} = \tau_\gamma, \tag{2.10}$$

where γ is as in (2.3).

For work closely related to our results, giving lower and upper bounds for the small ball probability under Hölder-type norms for mean zero Gaussian processes $X := \{X(\mathbf{u}),\ \mathbf{u} \in I^d\}$ taking values in $C[0,1]^d$ refer to [24].

2.2 Hausdorff dimension of exceptional sets

For any mean zero Gaussian process X on I^d, $d \geq 1$, define for $s \in [0,1)$ and $h > 0$ such that $s + h \in [0,1]$,

$$M(s, s+h) = \sup_{(u,\mathbf{v}) \in I \times I^{d-1}} |X(s+uh, \mathbf{v}) - X(s, \mathbf{v})|. \tag{2.11}$$

Let X satisfy the conditions of either Corollary 2.1 or Theorem 2.1, with $\alpha_1 = 1$, and for any $c \geq 1$ set

$$E_\gamma(c) = \{s \in [0,1) : \liminf_{h \searrow 0} \frac{M(s, s+h)}{h^{1/2}/(|\log h|/\tau_\gamma)^{\gamma/2}} \leq c\}, \tag{2.12}$$

where under the conditions of Corollary 2.1, $\gamma = 1$ and $\tau_\gamma = \tau_1$.

Recall (see e.g. Falconer [7]) that the Hausdorff dimension of a subset E of $[0,1]$ is defined by

$$\dim E = \inf \{\rho > 0 : s^\rho\text{-mes } E = 0\}, \tag{2.13}$$

where s^ρ-mes E denotes the s^ρ-measure of E equal to

$$s^\rho\text{-mes } E = \liminf_{\varepsilon \downarrow 0} \Big\{ \sum_{j \in J} |I_j|^\rho : E \subseteq \bigcup_{j \in J} I_j,\ |I_j| \leq \varepsilon,\ j \in J \Big\}. \tag{2.14}$$

In words, the infimum in (2.14) is taken over all collections $\{I_j :\ j \in J\}$ of closed intervals with lengths $|I_j| \leq \varepsilon$ for all $j \in J$, and such that $E \subseteq \bigcup_{j \in J} I_j$.

Our goal is to prove the following theorem which gives the Hausdorff dimension of the *random fractal* $E_\gamma(c)$.

Theorem 2.2. *Let X satisfy the conditions of Theorem* 2.1, *with $\alpha_1 = 1$, or Corollary* 2.1, *then for any $c > 1$, with probability* 1,

$$\dim E_\gamma(c) = 1 - c^{-2/\gamma}. \tag{2.15}$$

From Theorem 2.2 we shall derive the following two limit results for the infimum of the increments of X.

Corollary 2.2. *Under the conditions of Theorem* 2.1, *with $\alpha_1 = 1$, with probability* 1,

$$\lim_{h \searrow 0} \inf_{0 \leq s \leq 1-h} \frac{(|\log h|)^{\gamma/2} M(s, s+h)}{\tau_\gamma^{\gamma/2} h^{1/2}} = 1. \tag{2.16}$$

Corollary 2.3. *Under the conditions of Corollary* 2.1, *with probability* 1,

$$\lim_{h\searrow 0}\inf_{0\le s\le 1-h}\frac{(|\log h|)^{1/2}M(s,s+h)}{\tau_1^{1/2}\sqrt{h}}=1. \tag{2.17}$$

3 Proofs of small ball results

This section is devoted to the proofs of results in Subsection 2.1.

3.1 Proof of Proposition 2.1

We will first consider the case when $d=2$ and without loss of generality assume that $\alpha:=\alpha_1<\alpha_2:=\beta$.

Choose R so large that $1/R^{1/\alpha}<\eta$ and $1/R^{1/\beta}<\eta$. Now for $(s,t)\in I^2$,

$$|X(s,t)|\le\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\max_{1\le k\le R^{i/\alpha}}\max_{1\le l\le R^{j/\beta}}|X_{i,j,k,l}|, \tag{3.1}$$

where

$$X_{i,j,k,l}=X([\frac{k-1}{R^{i/\alpha}},\frac{k}{R^{i/\alpha}}]\times[\frac{l-1}{R^{j/\beta}},\frac{l}{R^{j/\beta}}])=_d N(0,\sigma^2_{i,j,k,l}),$$

with

$$\sigma^2_{i,j,k,l}=E(X^2_{i,j,k,l}).$$

Thus by (2.1) and the Khatri–Šidák inequality (Khatri [8], Šidák [23]), for any choice of $\varepsilon_{i,j}>0$,

$$\begin{aligned}P\{\|X\|<\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\varepsilon_{i,j}\}&\ge\prod_{i=1}^{\infty}\prod_{j=1}^{\infty}P\{|N(0,1)|<R^{(i+j)/2}\varepsilon_{i,j}\}^{(R^{i/\alpha}+1)(R^{j/\beta}+1)}\\&\ge\prod_{i=1}^{\infty}\prod_{j=1}^{\infty}P\{|N(0,1)|<R^{(i+j)/2}\varepsilon_{i,j}\}^{4R^{i/\alpha+j/\beta}}.\end{aligned}$$

where for notational convenience we will assume $\kappa=1$ in (2.1). (The $R^{i/\alpha}+1$ and $R^{j/\beta}+1$ come from the fact that $R^{i/\alpha}$ and $R^{j/\beta}$ are not necessarily integers.)

So writing

$$\mu=\mu(\alpha):=R^{1/\alpha}\text{ and }\delta=\delta(\alpha,\beta):=\alpha/\beta,$$

we get

$$P\{\|X\|<\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\varepsilon_{i,j}\}\ge\prod_{i=1}^{\infty}\prod_{j=1}^{\infty}P\{|N(0,1)|<R^{(i+j)/2}\varepsilon_{i,j}\}^{4\mu^{i+\delta j}}.$$

Notice that $\mu > 1$ and $0 < \delta < 1$. Since $\delta < 1$, it is possible to choose $b > 1$ and $1 < a < \sqrt{R}$ such that

$$\log b < (1-\delta)\log(\sqrt{R}/a).$$

Let $N \geq 1$ be an integer and let us take

$$\varepsilon_{i,j} = \sqrt{R^{-N-|i+j-N|/2} a^{N-i-j} b^{-j}}.$$

Observe that by the change of variables $k = i + j$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \varepsilon_{i,j} = R^{-N/2} \sum_{k=2}^{\infty} R^{-|k-N|/4} a^{(N-k)/2} \sum_{j=1}^{k-1} b^{-j/2}$$

$$\leq c_1 R^{-N/2} \sum_{k=2}^{\infty} R^{-|k-N|/4} a^{(N-k)/2} = c_1 R^{-N/2}\left[\sum_{k=2}^{N} (a/\sqrt{R})^{(N-k)/2} + \sum_{k=N+1}^{\infty} (a\sqrt{R})^{(N-k)/2}\right],$$

which since $a < \sqrt{R}$ and $a\sqrt{R} > 1$,

$$\leq c_2 R^{-N/2}.$$

Thus

$$P\{\|X\| < c_2 R^{-N/2}\} \geq \prod_{i=1}^{\infty}\prod_{j=1}^{\infty} P\{|N(0,1)| < R^{(i+j)/2}\varepsilon_{i,j}\}^{4\mu^{i+\delta j}}.$$

Let us examine the double infinite product term more closely. Now

$$R^{(i+j)/2}\varepsilon_{i,j} = \sqrt{(a/R)^{N-i-j} R^{-|i+j-N|/2} b^{-j}}.$$

Case I: $k := i + j \leq N$.

In this case

$$R^{(i+j)/2}\varepsilon_{i,j} = \sqrt{(R^{3/2}/a)^{-(N-k)} b^{-j}} < 1.$$

Thus by using the inequality

$$P\{|N(0,1)| < x\} \geq c_3 x \text{ for } x \in (0,1),$$

we get

$$P\{|N(0,1)| < R^{(i+j)/2}\varepsilon_{i,j}\} \geq c_3 \sqrt{(R^{3/2}/a)^{-(N-k)} b^{-j}} \geq \exp[-c_4((N-k)+j)].$$

Hence

$$\prod_{1\leq i,j<\infty, i+j\leq N} P\{|N(0,1)| < R^{(i+j)/2}\varepsilon_{i,j}\}^{4\mu^{i+\delta j}}$$

$$\geq \exp[-c_5 \sum_{k=2}^{N} \mu^k \sum_{j=1}^{k-1} \mu^{-(1-\delta)j}((N-k)+j)] \geq \exp[-c_6 \sum_{k=2}^{N} \mu^k (N-k+1)]$$

$$= \exp[-c_6\mu^N \sum_{l=0}^{N-2} \mu^{-l}(l+1)] \geq \exp[-c_7\mu^N].$$

Therefore

$$\prod_{1\leq i,j<\infty, i+j\leq N} P\{|N(0,1)| < R^{(i+j)/2}\varepsilon_{i,j}\}^{4\mu^{i+\delta j}} \geq \exp[-c_7\mu^N].$$

Case II: $k := i+j > N$.

Now

$$R^{(i+j)/2}\varepsilon_{i,j} = \sqrt{(\sqrt{R}/a)^{k-N} b^{-j}} \tag{3.2}$$

can be either < 1 or ≥ 1. In fact,

$$R^{(i+j)/2}\varepsilon_{i,j} < 1 \text{ if and only if } j > \{\log(\sqrt{R}/a)/\log b\}(k-N).$$

Subcase II (i). $k := i+j > N$, $R^{(i+j)/2}\varepsilon_{i,j} < 1$.

In this subcase

$$P\{|N(0,1)| < R^{(i+j)/2}\varepsilon_{i,j}\} \geq c_3\sqrt{(\sqrt{R}/a)^{k-N} b^{-j}}$$

$$\geq \exp\left[-c_8(j - \{\log(\sqrt{R}/a)/\log b\}(k-N)+1)\right],$$

where $c_8 > 0$ is chosen so that $\exp(-c_8) \leq c_3$.

This gives with $x := \log(\sqrt{R}/a)/\log b > 0$,

$$\prod_{i+j>N,\ R^{(i+j)/2}\varepsilon_{i,j}<1} P\{|N(0,1)| < R^{(i+j)/2}\varepsilon_{i,j}\}^{4\mu^{i+\delta j}}$$

$$\geq \exp[-c_9 \sum_{k=N+1}^{\infty} \mu^k \sum_{1\leq j\leq k-1, j>x(k-N)} (j - x(k-N)+1)\mu^{-(1-\delta)j}]$$

$$\geq \exp[-c_{10} \sum_{k=N+1}^{\infty} \mu^k \mu^{-(1-\delta)x(k-N)}] = \exp[-c_{10}\mu^N \sum_{k=N+1}^{\infty} (\mu/\mu^{(1-\delta)x})^{k-N}],$$

which since $(1-\delta)\log(\sqrt{R}/a) > \log b$, is

$$\geq \exp(-c_{11}\mu^N).$$

Thus

$$\prod_{i+j>N,\ R^{(i+j)/2}\varepsilon_{i,j}<1} P\{|N(0,1)| < R^{(i+j)/2}\varepsilon_{i,j}\}^{4\mu^{i+\delta j}} \geq \exp(-c_{11}\mu^N).$$

Subcase II (ii). $k := i+j > N$, $R^{(i+j)/2}\varepsilon_{i,j} \geq 1$. Recalling (3.2) we see that by using the inequality

$$P\{|N(0,1)| < x\} \geq \exp(-c_{12}\exp(-x^2/2)), \text{ for } x \geq 1,$$

we get

$$P\{|N(0,1)| < R^{(i+j)/2}\varepsilon_{i,j}\} \geq \exp\left[-c_{12}\exp(-\frac{1}{2}(\sqrt{R}/a)^{k-N}b^{-j})\right].$$

Therefore

$$\prod_{i+j>N,\ R^{(i+j)/2}\varepsilon_{i,j}\geq 1} P\{|N(0,1)| < R^{(i+j)/2}\varepsilon_{i,j}\}^{4\mu^{i+\delta j}}$$

$$\geq \exp[-c_{13}\sum_{k=N+1}^{\infty}\mu^k\sum_{j=1}^{k-1}\mu^{-(1-\delta)j}\exp(-\frac{1}{2}(\sqrt{R}/a)^{k-N}b^{-j})],$$

which by the change of variables $l = k - N$ is

$$\geq \exp[-c_{13}\mu^N\sum_{l=1}^{\infty}\sum_{j=1}^{\infty}\mu^{l-(1-\delta)j}\exp(-\frac{1}{2}(\sqrt{R}/a)^l b^{-j})]. \tag{3.3}$$

At this point we need a lemma.

Lemma 3.1. *Let* $p > 1$, $q > 1$, $r > 1$, $s > 1$ *and* $c > 0$. *If*

$$(\log p)(\log s) < (\log q)(\log r),$$

then

$$\sum_{l=1}^{\infty}\sum_{j=1}^{\infty}p^l q^{-j}\exp(-cr^l s^{-j}) < \infty.$$

Applying the lemma with

$$p = \mu,\ q = \mu^{1-\delta},\ r = \sqrt{R}/a,\ s = b \text{ and } c = 1/2$$

and noting that

$$\log b < (1-\delta)\log(\sqrt{R}/a),$$

we get that the right side of (3.3) is greater than or equal to

$$\exp[-c_{14}\mu^N],$$

which implies

$$\prod_{i+j>N,\ R^{(i+j)/2}\varepsilon_{i,j}\geq 1} P\{|N(0,1)| < R^{(i+j)/2}\varepsilon_{i,j}\}^{4\mu^{i+\delta j}} \geq \exp[-c_{14}\mu^N].$$

Putting Case I, Case II (i) and Case II (ii) together we get

$$\prod_{i=1}^{\infty}\prod_{j=1}^{\infty}P\{|N(0,1)| < R^{(i+j)/2}\varepsilon_{i,j}\}^{4\mu^{i+\delta j}} \geq \exp[-c_{15}\mu^N].$$

Hence

$$P\{\|X\| < c_2 R^{-N/2}\} \geq \exp[-c_{15}\mu^N],$$

where $\mu = R^{1/\alpha}$ and $\beta > \alpha$. This proves Proposition for the case when $d = 2$.

For general case of $d \geq 2$, assume again without loss of generality that

$$\alpha_1 < \min_{2 \leq i \leq d} \alpha_i.$$

As before, set $\mu = R^{1/\alpha}$, but now replace the use of δ in the above arguments by

$$\delta_k = \alpha_k/\alpha_1, \text{ for } 2 \leq k \leq d,$$

and choose $b > 1$ and $1 < a < \sqrt{R}$ such that

$$\log b < (1 - \delta_k) \log\left(\sqrt{R}/a\right), \text{ for } 2 \leq k \leq d.$$

Finally replace the $\varepsilon_{i,j}$ by

$$\sqrt{R^{-N-|i_1+\ldots+i_d-N|/2} a^{N-(i_1+\ldots+i_d)} b^{-(i_2+\ldots+i_d)}}.$$

With these notational changes the proof goes as before. □

Proof of Lemma 3.1. For $0 < v < 1$ write

$$\sum_{l=2}^{\infty} p^l \exp(-vr^l) \leq p \int_1^{\infty} p^x \exp(-vr^x) dx,$$

which, after the change of variables $x = \log(y/v)/\log r$, is equal to

$$\begin{aligned}
&= \frac{p}{\log r} \int_{vr}^{\infty} \frac{dy}{y} \exp[-y + \frac{\log p}{\log r} \log y + \frac{\log p}{\log r} \log(1/v)] \\
&\leq \frac{p}{\log r} \exp[\frac{\log p}{\log r} \log(1/v)] \int_0^{\infty} \frac{dy}{y} \exp[-y + \frac{\log p}{\log r} \log y] \\
&\leq C_{p,r} \exp[\frac{\log p}{\log r} \log(1/v)],
\end{aligned}$$

where $C_{p,r} < \infty$ is a constant depending only on (p, r). Let $j_0 < \infty$ be such that $cs^{-j_0} < 1$. We have

$$\begin{aligned}
&\sum_{l=2}^{\infty} \sum_{j=1}^{\infty} p^l q^{-j} \exp(-cr^l s^{-j}) \\
&= \sum_{l=2}^{\infty} \sum_{j=1}^{j_0} p^l q^{-j} \exp(-cr^l s^{-j}) + \sum_{l=2}^{\infty} \sum_{j=j_0+1}^{\infty} p^l q^{-j} \exp(-cr^l s^{-j}) \\
&\leq \sum_{l=2}^{\infty} j_0 q^{-1} p^l \exp(-cr^l s^{-j_0}) + C_{p,r} \sum_{j=j_0+1}^{\infty} q^{-j} \exp\left[\frac{\log p}{\log r}(j \log s - \log c)\right] < \infty,
\end{aligned}$$

as long as $\log q > \frac{\log p}{\log r} \log s$.

□

3.2 Proof of Corollary 2.1

First by Proposition 2.1 we have for some $\mu_1 > 0$

$$P\{\|X\| \le \varepsilon\} \ge \exp(-\mu_1 \varepsilon^{-2}). \tag{3.4}$$

Next observe that for all $x > 0$ and $n \ge 1$,

$$P\{\|X\| \le x\}$$

$$= P\left\{ \sup_{0 \le s \le 1/n, t \in I^{d-1}} |X(s,t)| \le x, \sup_{1/n \le s \le 1, t \in I^{d-1}} |X(s,t) - X(1/n,t) + X(1/n,t)| \le x \right\},$$

which by independence of the sets

$$\left\{X(s,t), 0 \le s \le 1/n, t \in I^{d-1}\right\} \text{ and } \left\{X(s,t) - X(1/n,t),\ 1/n \le s \le 1,\ t \in I^{d-1}\right\}$$

and an application of Anderson's inequality (Anderson [2]) is

$$\le P\left\{ \sup_{0 \le s \le 1/n, t \in I^{d-1}} |X(s,t)| \le x, \sup_{1/n \le s \le 1, t \in I^{d-1}} |X(s,t) - X(1/n,t)| \le x \right\}$$

$$= P\{\|X\| \le \sqrt{n}x\} P\left\{ \sup_{0 \le s \le 1-1/n, t \in I^{d-1}} |X(s,t)| \le x \right\}$$

$$\le P\{\|X\| \le \sqrt{n}x\}^2 P\left\{ \sup_{0 \le s \le 1-2/n, t \in I^{d-1}} |X(s,t)| \le x \right\} \le \left(P\{\|X\| \le \sqrt{n}x\}\right)^n. \tag{3.5}$$

Now using a well known subadditivity argument (see Kuelbs and Li [10]) based on inequalities (3.4) and (3.5), we can infer the existence of a constant $\tau_1 > 0$ such that (2.6) holds. □

3.3 Proof of Theorem 2.1

Without loss of generality we can assume that $\alpha_1 = \gamma$. Observe that we can write

$$W_\alpha(\mathbf{t}) = U_\alpha(\mathbf{t}) + Z_\alpha(\mathbf{t}), \tag{3.6}$$

where

$$U_\alpha(\mathbf{t}) = \int_0^{t_1} \int_{-\infty}^{t_2} \cdots \int_{-\infty}^{t_d} \prod_{i=1}^d g_{\alpha_i}(t_i, u_i) W(d\mathbf{u}),$$

$$Z_\alpha(\mathbf{t}) = \int_{-\infty}^{0} \int_{-\infty}^{t_2} \cdots \int_{-\infty}^{t_d} \prod_{i=1}^d g_{\alpha_i}(t_i, u_i) W(d\mathbf{u})$$

and the functions g_{α_i} are defined as in (2.7). Notice that the process $U_\alpha(\mathbf{t})$ has covariance function

$$E[U_\alpha(\mathbf{s})U_\alpha(\mathbf{t})] = a\tau_{\alpha_1}(s_1,t_1)\prod_{i=2}^{d}\sigma_{\alpha_i}(s_i,t_i),$$

where σ_{α_i} is as in (2.9) and

$$\tau_{\alpha_1}(s_1,t_1) = \int_0^{s_1\wedge t_1} g_{\alpha_1}(s_1,u)g_{\alpha_1}(t_1,u)du.$$

Further, the process $Z_\alpha(\mathbf{t})$ has covariance function

$$E[Z_\alpha(\mathbf{s})Z_\alpha(\mathbf{t})] = a\rho_{\alpha_1}(s_1,t_1)\prod_{i=2}^{d}\sigma_{\alpha_i}(s_i,t_i),$$

where

$$\rho_{\alpha_1}(s_1,t_1) = \int_{-\infty}^{0} g_{\alpha_1}(s_1,u)g_{\alpha_1}(t_1,u)du.$$

Let for $0<\alpha<2$

$$f_\alpha(s,h) = \rho_\alpha(s+h,s+h) - 2\rho_\alpha(s,s+h) + \rho_\alpha(s,s).$$

We can write

$$f_\alpha(s,h) = \int_0^\infty \left[\frac{1}{(u+s)^{(1-\alpha)/2}} - \frac{1}{(u+s+h)^{(1-\alpha)/2}}\right]^2 du.$$

Lemma 3.2. *For all* $0<\alpha<2$ *there exist constants* θ_α *and* π_α *such that*

(3.7) $$f_\alpha(s,h) \le \theta_\alpha \frac{h^2}{s^{2-\alpha}}$$

and

(3.8) $$f_\alpha(s,h) \le \pi_\alpha h^\alpha.$$

Proof. First consider (3.7). By the mean value theorem for some h^* between 0 and h

$$\frac{1}{(u+s)^{(1-\alpha)/2}} - \frac{1}{(u+s+h)^{(1-\alpha)/2}} = \frac{1-\alpha}{2}\frac{h}{(u+s+h^*)^{(3-\alpha)/2}},$$

so that

$$\left|\frac{1}{(u+s)^{(1-\alpha)/2}} - \frac{1}{(u+s+h)^{(1-\alpha)/2}}\right| \le \frac{|1-\alpha|}{2}\frac{h}{(u+s)^{(3-\alpha)/2}}.$$

Thus

$$f_\alpha(s,h) \le \left(\frac{1-\alpha}{2}\right)^2 h^2 \int_0^\infty \frac{1}{(u+s)^{3-\alpha}}du = \left(\frac{1-\alpha}{2}\right)^2 \frac{1}{2-\alpha}\frac{h^2}{s^{2-\alpha}} =: \theta_\alpha \frac{h^2}{s^{2-\alpha}}.$$

Next consider (3.8). Obviously by (3.7) we have $f_\alpha(s,h) \le \theta_\alpha h^\alpha$ if $h \le s$. Now if $h > s$, we get by the change of variables $u = (s+h)v$

$$f_\alpha(s,h) \le (s+h)^\alpha C_\alpha \le 2^\alpha C_\alpha h^\alpha,$$

where

$$C_\alpha = \int_0^\infty \left[\frac{1}{v^{(1-\alpha)/2}} - \frac{1}{(v+1)^{(1-\alpha)/2}}\right]^2 dv.$$

□

Lemma 3.3. *Assume condition* (2.2) *with* $\gamma = \alpha_1$. *Then there exists a* $c_\alpha > 0$ *such that*

$$\liminf_{\varepsilon \searrow 0} \varepsilon^{2/\gamma} \log P\left\{\sup_{\mathbf{t} \in I^d} |Z_\alpha(\mathbf{t})| \le \varepsilon\right\} \ge -c_\alpha. \tag{3.9}$$

Furthermore, for all $0 < \delta < 1$

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2/\gamma} \log P\left\{\sup_{\mathbf{t} \in [\delta,1] \times I^{d-1}} |Z_\alpha(\mathbf{t})| \le \varepsilon\right\} = 0. \tag{3.10}$$

Proof. Notice by Lemma 3.2 we can infer

$$E\left[Z_\alpha^2([\mathbf{s}, \mathbf{s}+\mathbf{h}])\right] = a f_{\alpha_1}(s_1, h_1) \prod_{i=2}^d h_i^{\alpha_i} \le a\pi_{\alpha_1} h_1^{\alpha_1} \prod_{i=2}^d h_i^{\alpha_i}.$$

Thus by Proposition 2.1 we have (3.9).

Choose any $0 < \delta < 1$ and define the process

$$\overline{Z}(\mathbf{t}) = Z_\alpha(t_1 + \delta, t_2, ..., t_d), \qquad \mathbf{t} \in I^d.$$

Note that $\overline{Z}(\mathbf{0}) = Z_\alpha(\delta, 0, ..., 0) = 0$. We have by Lemma 3.2 for some $\kappa > 0$ and all b satisfying $\alpha_1 < b < \min_{2 \le i \le d} \alpha_i < 2$,

$$E\left[\overline{Z}^2([\mathbf{s}, \mathbf{s}+\mathbf{h}])\right] = a f_{\alpha_1}(s_1 + \delta, h_1) \prod_{i=2}^d h_i^{\alpha_i} \le a\theta_{\alpha_1} \delta^{\alpha_1 - 2} h_1^2 \prod_{i=2}^d h_i^{\alpha_i} \le \kappa h_1^b \prod_{i=2}^d h_i^{\alpha_i},$$

where $\kappa = a\theta_{\alpha_1}\delta^{\alpha_1 - 2}$. By applying Proposition 2.1 again we get

$$\liminf_{\varepsilon \searrow 0} \varepsilon^{2/b} \log P\left\{\sup_{\mathbf{t} \in I^d} |\overline{Z}(\mathbf{t})| \le \varepsilon\right\} > -\infty,$$

which since $\gamma = \alpha_1 < b$ implies that

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2/\gamma} \log P\left\{\sup_{\mathbf{t} \in I^d} |\overline{Z}(\mathbf{t})| \le \varepsilon\right\} = 0.$$

Since

$$\sup_{\mathbf{t}\in[\delta,1]\times I^{d-1}} |Z_\alpha(\mathbf{t})| \le \sup_{\mathbf{t}\in I^d} |\overline{Z}(\mathbf{t})|,$$

we have (3.10). $\square$

For our next lemma we shall need the following fact due to Schechtman *et al.* [20]:

Fact 3.1. *For any centered Gaussian measure μ on a separable Banach space E,*

$$\mu(A\cap B) \ge \mu(A/\sqrt{2})\mu(B/\sqrt{2})$$

for all symmetric convex subsets A and B of E.

Lemma 3.4. *Assume condition* (2.2) *with $\gamma = \alpha_1$. Then*

$$\lim_{\varepsilon\searrow 0} \varepsilon^{2/\gamma} \log P\left\{\sup_{\mathbf{t}\in I^d} |Z_\alpha(\mathbf{t})| \le \varepsilon\right\} = 0. \tag{3.11}$$

Proof. We see that for any $0<\delta<1$ by Fact 3.1

$$P\left\{\sup_{\mathbf{t}\in I^d} |Z_\alpha(\mathbf{t})| \le \varepsilon\right\}$$

$$\ge P\left\{\sup_{\mathbf{t}\in I^d} |Z_\alpha(\mathbf{t})| \le \frac{\varepsilon}{\delta^{\alpha_1/2}\sqrt{2}}\right\} P\left\{\sup_{\mathbf{t}\in[\delta,1]\times I^{d-1}} |Z_\alpha(\mathbf{t})| \le \frac{\varepsilon}{\sqrt{2}}\right\}.$$

Now by Lemma 3.3 we can conclude from this last inequality that

$$\liminf_{\varepsilon\searrow 0} \varepsilon^{2/\gamma} \log P\left\{\sup_{\mathbf{t}\in I^d} |Z_\alpha(\mathbf{t})| \le \varepsilon\right\} \ge -2^{1/\alpha_1}\delta c_{\alpha_1}.$$

Letting $\delta \searrow 0$ completes the proof of Lemma 3.4. $\square$

Lemma 3.5. *Assume condition* (2.2) *with $\gamma = \alpha_1$. There exists a finite positive constant τ_γ such that*

$$-\lim_{\varepsilon\searrow 0} \varepsilon^{2/\gamma} \log P\{\|U_\alpha\| \le \varepsilon\} = \tau_\gamma. \tag{3.12}$$

Proof. Straightforward computations show that U_α satisfies (2.1). Thus using Proposition 2.1 we get

$$\liminf_{\varepsilon\searrow 0} \varepsilon^{2/\gamma} \log P\{\|U_\alpha\| \le \varepsilon\} > -\infty. \tag{3.13}$$

Thus a straightforward change of notation in the proof of Theorem 2.1 of Li and Linde [13] shows that for all $n \geq 1$ and $x > 0$

$$P\{\|U_\alpha\| \leq x\} \leq \left(P\{\|U_\alpha\| \leq n^{\gamma/2}x\}\right)^n. \tag{3.14}$$

(It suffices to replace, respectively, their

$$\sup_{0\leq t\leq\lambda} |\widehat{W}_\beta(t)| \text{ and } \sup_{\lambda\leq t\leq 1} |Y_{\beta,\lambda}(t) + \int_0^t (t-s)^{(\beta-1)/2} dB(s)|$$

by

$$\sup_{\mathbf{t}\in[0,\lambda]\times I^{d-1}} |U_\alpha(\mathbf{t})| \text{ and}$$

$$\sup_{\mathbf{t}\in[\lambda,1]\times I^{d-1}} \left|\int_0^\lambda \int_{-\infty}^{t_2} \cdots \int_{-\infty}^{t_d} \prod_{i=1}^d g_{\alpha_i}(t_i,u_i) W(d\mathbf{u}) + \int_\lambda^s \int_{-\infty}^{t_2} \cdots \int_{-\infty}^{t_d} \prod_{i=1}^d g_{\alpha_i}(t_i,u_i) W(d\mathbf{u})\right|,$$

and we get (3.14)). Now (3.13) and (3.14) allow us to use a standard subadditivity argument to conclude (3.12). □

We are now ready to finish the proof of Theorem 2.1. First by Anderson's inequality (cf. [2]) using the representation (3.6) and the independence of U_α and Z_α we get

$$P\{\|W_\alpha\| \leq \varepsilon\} \leq P\{\|U_\alpha\| \leq \varepsilon\}. \tag{3.15}$$

To go the other way, note that by the independence U_α and Z_α for any $0 < \delta < 1$

$$P\{\|W_\alpha\| \leq \varepsilon\} \geq P\{\|U_\alpha\| \leq \varepsilon(1-\delta)\} P\{\|Z_\alpha\| \leq \varepsilon\delta\}.$$

Since $\delta > 0$ can be made as small as desired, the proof clearly follows from Lemmas 3.4 and 3.5, in conjunction with (3.15). □

4 Proofs of increment results

In this section, we prove Theorem 2.2 and Corollaries 2.2 and 2.3, separately.

4.1 Proof of Theorem 2.2

4.1.1 Upper bound case

First we shall prove that with probability 1

$$\dim E_\gamma(c) \leq 1 - c^{-2/\gamma}. \tag{4.1}$$

To accomplish this we require some lemmas, which we shall prove in a little more generality than we actually need.

Lemma 4.1. *Let X be a mean zero Gaussian process on I^d, $d \geq 2$, with covariance function of the form*

$$E[X(\mathbf{s})X(\mathbf{t})] = \frac{1}{2}(s_1^\alpha + t_1^\alpha - |s_1 - t_1|^\alpha)\sigma(\widetilde{\mathbf{s}}, \widetilde{\mathbf{t}}),$$

where $0 < \alpha < 2$ and $\widetilde{\mathbf{s}} = (s_2, ..., s_d)$ and $\widetilde{\mathbf{t}} = (t_2, ..., t_d)$. *Assume that*

$$\sigma_0^2 := \sup_{\mathbf{t} \in I^d} t_1^\alpha \sigma(\widetilde{\mathbf{t}}, \widetilde{\mathbf{t}}) < \infty.$$

Then we have with probability 1,

$$\limsup_{h \searrow 0} \sup_{0 \leq s \leq 1-h} \frac{M(s, s+h)}{h^{\alpha/2}\sqrt{|\log h|}} \leq 2^{\alpha+3/2}\sigma_0, \tag{4.2}$$

where M is as in (2.11).

Proof. First by a result of Marcus and Shepp [18] we have for all $0 < \eta < 1$ and z sufficiently large

$$-\frac{z^2}{2\sigma_0^2}(1+\eta) \leq \log P\{\|X\| > z\} \leq -\frac{z^2}{2\sigma_0^2}(1-\eta). \tag{4.3}$$

Choose any $\varepsilon > 0$. Clearly for $2^{-(k+1)} < h \leq 2^{-k}$ and $(i-1)2^{-k} \leq s \leq i2^{-k}$, we have

$$(i-1)2^{-k} \leq s + h \leq (i+1)2^{-k},$$

which gives by the triangle inequality that

$$\sup_{0 \leq s \leq 1-h} \frac{M(s, s+h)}{h^{\alpha/2}\sqrt{|\log h|}} \leq 2 \max_{1 \leq i \leq 2^k} \frac{M((i-1)2^{-k}, (i+1)2^{-k})}{2^{-(k+1)\alpha/2}\sqrt{\log 2^k}}$$

$$=: 2 \max_{1 \leq i \leq 2^k} \frac{M_{i,k}}{2^{-(k+1)\alpha/2}\sqrt{\log 2^k}}.$$

Now since

$$\frac{M_{i,k}}{\sqrt{2^{-(k-1)\alpha}}} =_d \|X\|,$$

we see by (4.3) that for every $\varepsilon > 0$ there is a $k_0 \geq 1$ such that for all $k \geq k_0$

$$P\left\{\frac{M_{i,k}}{\sqrt{2^{-(k-1)\alpha}\log 2^k}} \geq \sqrt{2}\sqrt{1+\varepsilon}(1-\eta)^{-1/2}\sigma_0\right\} \leq \exp(-k(1+\varepsilon)\log 2) = 2^{-k(1+\varepsilon)}.$$

Thus

$$P\left\{\max_{1\le i\le 2^k}\frac{M_{i,k}}{\sqrt{2^{-(k-1)\alpha}}}\ge\sqrt{2}\sqrt{1+\varepsilon}(1-\eta)^{-1/2}\sigma_0\sqrt{\log 2^k}\right\}\le 2^{-k\varepsilon}.$$

Since both ε and η can be made arbitrarily small we conclude (4.2) by a simple application of the Borel–Cantelli theorem. $\square$

We shall make use of the following discretization scheme. Let $\lambda > 0$ be an arbitrary constant whose value will be chosen later on. For each $n \ge 1$, set $h_n = n^{-\lambda}$. Denote by $\lfloor u\rfloor \le u < \lfloor u\rfloor + 1$ the integer part of u. We set for each $n \ge 1$,

$$M_n := \lfloor 1/(h_n(|\log h_n|)^{-K})\rfloor, \tag{4.4}$$

and for each $i = 0, 1, \ldots, M_n$,

$$s_{i,n} = ih_n(|\log h_n|)^{-K}, \tag{4.5}$$

where $K > (1+\gamma)/\alpha$ is arbitrary but fixed. We note for further use that, for any $s \in [0,1]$, there exists an $i \in \{1, \ldots, M_n\}$ such that $|s - s_{i,n}| \le h_n(|\log h_n|)^{-K}$. Now set

$$U_n = \max_{1\le i\le M_n}\ \sup_{|s-s_{i,n}|\le h_n(|\log h_n|)^{-K}}\ \sup_{h_{n+1}<h\le h_n}\ \sup_{(u,\mathbf{v})\in I\times I^{d-1}}|\{X(s+uh,\mathbf{v})-X(s,\mathbf{v})\}$$
$$-\{X(s_{i,n}+uh_n,\mathbf{v})-X(s_{i,n},\mathbf{v})\}|.$$

Lemma 4.2. *We have with probability* 1

$$\lim_{n\to\infty} h_{n+1}^{-\alpha/2}(|\log h_{n+1}|)^{\gamma/2}U_n = 0. \tag{4.6}$$

Proof. We observe, via the triangle inequality, that

$$U_n \le 2\sup_{0\le s\le 1-a'_n} M(s, s+a'_n),$$

where

$$a'_n := h_n(|\log h_n|)^{-K} + (h_n - h_{n+1}).$$

Now our definition of $h_n = n^{-\lambda}$ ensures that as $n \to \infty$

$$h_n - h_{n+1} = (1+o(1))\lambda n^{-\lambda-1} = o\left(h_n(|\log h_n|)^{-K}\right).$$

Therefore, by setting $a''_n = 2h_n(|\log h_n|)^{-K}$ we see that for all n sufficiently large

$$U_n \le 2\sup_{0\le s\le 1-a''_n} M(s, s+a''_n).$$

Applying Lemma 4.1, we infer from this inequality and $K \geq 3/\alpha$ that with probability 1,

$$U_n = O\left((a''_n)^{\alpha/2}\sqrt{|\log(a''_n)|}\right) = o\left(h_n^{\alpha/2}(|\log h_n|)^{-\gamma/2}\right),$$

which gives (4.6). □

We shall be applying Lemma 4.2 for case $\alpha = 1$.

The method of proof of the upper bound part is based upon computing moments as in the proof of Theorem 1.1 in Deheuvels and Mason [3]. In fact, we shall follow very closely the general procedure for calculating the Hausdorff dimension of exceptional sets of this type described in their paper.

Fix an arbitrary $\varepsilon > 0$, and choose any $\lambda > 1/\varepsilon$. We define $Y_{i,n}$, for $i = 1, \ldots, M_n$, to be 1 or 0 according as the random variable

$$\frac{M(s_{i,n}, s_{i,n} + h_n)}{(h_n)^{1/2}/(|\log h_n|/\tau_\gamma)^{\gamma/2}} \leq (1+\varepsilon)c$$

is or not. Making use of (2.10), we have uniformly over $i = 1, \ldots, M_n$, as $n \to \infty$,

$$P(Y_{i,n} = 1) = P(Y_{1,n} = 1) = \exp\left[(1+o(1))(\log h_n)((1+\varepsilon)c)^{-2/\gamma}\right]. \tag{4.7}$$

Consider now the (possibly empty and at most countably infinite) collection $\{I_j : j \in J\}$ of closed intervals of the form $[s_{i,n} - h_n, s_{i,n} + h_n]$ for which $Y_{i,n} = 1$, where $n \geq 1$ and $1 \leq i \leq M_n$. Set

$$E = \cup \{I_j : j \in J\},$$

and

$$\delta = 1 + 2\varepsilon - ((1+\varepsilon)c)^{-2/\gamma}.$$

Introduce the (possibly infinite valued) random variable

$$Z = \sum_{j \in J} |I_j|^\delta,$$

where $\sum_{j\in\emptyset} (\cdot)$ is defined to be 0 whenever $J = \emptyset$. Obviously, we have

$$EZ = \sum_{n \geq 1} M_n (2h_n)^\delta P(Y_{1,n} = 1) =: \sum_{n \geq 1} u_n.$$

Note that, as $n \to \infty$,

$$M_n = \lfloor 1/(h_n(|\log h_n|)^K)\rfloor = \exp((1+o(1))|\log h_n|). \tag{4.8}$$

Thus we infer from (4.8) and (4.7) that for all large n

$$u_n = \exp\left(\{2\varepsilon + o(1)\}\log h_n\right) \leq h_n^\varepsilon = n^{-\varepsilon\lambda}.$$

Since our choice of $\lambda > 1/\varepsilon$ and $h_n = n^{-\lambda}$ entails that

$$\sum_{n=1}^{\infty} u_n < \infty,$$

we see that $EZ < \infty$, which, in turn, implies that $Z < \infty$ with probability 1. In view of (2.13) and (2.14), it follows that, with probability 1, the measure $s^\delta - \dim E < \infty$, and hence

$$\dim E \leq \delta. \tag{4.9}$$

We finish by comparing the sets E and $E_\gamma(c)$. By Lemma 4.2, with $\alpha = 1$, there exists almost surely an $n_0 < \infty$ such that for all $n \geq n_0$

$$h_{n+1}^{-1/2}(|\log h_{n+1}|/\tau_\gamma)^{\gamma/2} U_n \leq \frac{\varepsilon c}{3}.$$

Hence, whenever $h_{n+1} < h \leq h_n$, for some $n \geq n_0$, and $0 \leq s \leq 1$, we have

$$\frac{M(s, s+h)}{h^{1/2}/(|\log h|/\tau_\gamma)^{\gamma/2}} \leq c(1 + \frac{\varepsilon}{3}), \tag{4.10}$$

then there exists an $i \in \{1, \ldots, M_n\}$ such that both $Y_{i,n} = 1$ and $s \in [s_{i,n} - h_n, s_{i,n} + h_n]$.

Since $E_\gamma(c)$ is a subset of the set of all points s such that (4.10) holds for some $h_{n+1} < h \leq h_n$ for infinitely many n, it follows that, with probability 1, $E_\gamma(c) \subseteq E$, which, via (4.9), implies

$$\dim E_\gamma(c) \leq 1 + 2\varepsilon - ((1+\varepsilon)c)^{-2/\gamma}.$$

We conclude (4.1) by observing that $\varepsilon > 0$ may be chosen arbitrarily small.

4.1.2 Lower bound case

In the second part of the proof we shall prove that

$$\dim E_\gamma(c) \geq 1 - c^{-2/\gamma}. \tag{4.11}$$

We shall use the following result in Deheuvels and Mason [4]. Also see Orey and Taylor [19]. We begin by introducing some notation.

Let $\{H_n : n \geq 1\}$ denote a sequence of constants satisfying the following conditions:

$$H_n \downarrow, \ 0 < H_n < 1 \text{ for all large } n \geq 1. \tag{4.12}$$

$$\sum_{n \geq 1} \exp\left(-H_n^{-\varepsilon}\right) < \infty \text{ for any } \varepsilon > 0. \tag{4.13}$$

Assume that, for each $n \geq 1, Z_{i,n}$, $i = 1, \ldots, N_n := \lfloor 1/H_n \rfloor$, is a sequence of independent and identically distributed Bernoulli random variables. Set

$$p_n = P\left(Z_{1,n} = 1\right) = 1 - P\left(Z_{1,n} = 0\right).$$

Further assume that, for some $0 < \delta < 1$, as $n \to \infty$

$$p_n = H_n^{\delta+o(1)}. \tag{4.14}$$

For each $n \geq 1$, set $s_{i,n} = iH_n,\ i = 1, \ldots, N_n$, and introduce the disjoint closed intervals

$$I_{i,n} = \begin{cases} [s_{i,n} - H_n/2, s_{i,n}] \text{ when } Z_{i,n} = 1, \\ \\ \emptyset \text{ when } Z_{i,n} = 0. \end{cases}$$

Our proof of the lower bound is based upon the following fact, which is proved in Deheuvels and Mason (1998).

Fact 4.1. *Under* (4.12), (4.13) *and* (4.14), *for any* $\varepsilon > 0$, *there exist almost surely a sequence of integers* $1 \leq q_1 < q_2 < \ldots$, *and sets* E_1, E_2, ..., *such that*

$$\dim E \geq 1 - \delta - \varepsilon,$$

where $E = \bigcap_{j=1}^{\infty} E_j$ *and for each* $j \geq 1$, E_j *is a union of some intervals taken from the set* $\left\{I_{i,q_j} : 1 \leq i \leq N_{q_j}\right\}$.

We are now ready to complete the proof of Theorem 2.2. We shall prove that almost surely for each $c > 1$ and $\varepsilon > 0$, chosen so that the right hand side of the inequality below is strictly positive

$$\dim E_\gamma(c) \geq 1 - c^{-2/\gamma} - \varepsilon. \tag{4.15}$$

Since $\varepsilon > 0$ may be chosen arbitrarily small, the proof of (4.11) will follow from (4.15).

To establish (4.15), we apply Fact 4.1, with the following special choices of $\{H_n : n \geq 1\}$ and $\{Z_{1,n} : 1 \leq i \leq N_n\}$ fulfilling (4.12), (4.13) and (4.14). Choose a constant $\lambda > 0$ and set $h_n = n^{-\lambda}$ and

$$H_n = h_n(|\log h_n|)^{-2(1+\gamma)}$$

for $n \geq 1$. Now let $Z_{i,n} = 1$ or 0 according as the random variable

$$\frac{M(s_{i,n}, s_{i,n} + h_n)}{h_n^{1/2}/(|\log h_n|/\tau_\gamma)^{\gamma/2}} \leq (1-\varepsilon)c$$

or not. Notice that from the covariance structure of X, we readily see that for each $n \geq 1$ the $Z_{1,n} : 1 \leq i \leq N_n$ are i.i.d. Bernoulli random variables.

Applying (2.10), we get that uniformly over $i = 1, \ldots, N_n$, as $n \to \infty$

$$P\left(Z_{i,n} = 1\right) = \exp\left[(\log h_n)\{((1-\varepsilon)c)^{-2/\gamma} + o(1)\}\right]$$

which shows that (4.14) holds with $\delta = ((1-\varepsilon)c)^{-2/\gamma}$. Further note that assumptions (4.12) and (4.13) clearly hold. Hence we may apply Fact 4.1 to establish the existence of a set E such that

$$\dim E \geq 1 - \delta - \varepsilon = 1 - ((1-\varepsilon)c)^{-2/\gamma} - \varepsilon.$$

To conclude, we observe from the definition of $Z_{i,n}$ and Lemma 4.2, noting that $N_n = M_n$ of (4.4), with $K = 2(1+\gamma)$, that with probability 1, for all large n, whenever $Z_{i,n} = 1$ we have

$$\frac{M(s, s+h_n)}{h_n^{1/2}/(|\log h_n|/\tau_\gamma)^{\gamma/2}} \leq c$$

for all $s \in I_{i,n} = [s_{i,n} - H_n/2, s_{i,n}]$. This readily implies that $E \subseteq E_\gamma(c)$, which yields (4.15). This completes the proof of Theorem 2.2. □

4.2 Proofs of Corollaries 2.2 and 2.3

We will only prove Corollary 2.2. Corollary 2.3 is proved similarly. From Theorem 2.2 we can immediately conclude that with probability 1

$$\limsup_{h \searrow 0} \inf_{0 \leq s \leq 1-h} \frac{(|\log h|)^{\gamma/2} M(s, s+h)}{\tau_\gamma^{\gamma/2} h^{1/2}} \leq 1. \tag{4.16}$$

Therefore we need to only verify that, with probability 1,

$$\liminf_{h \searrow 0} \inf_{0 \leq s \leq 1-h} \frac{(|\log h|)^{\gamma/2} M(s, s+h)}{\tau_\gamma^{\gamma/2} h^{1/2}} \geq 1. \tag{4.17}$$

Choose any $\lambda > 1$. Set $h_k = \lambda^{-k}$. For $\lambda^{-(k+1)} \leq h \leq \lambda^{-k}$, we clearly have

$$\inf_{0 \leq s \leq 1-h} \frac{(|\log h|)^{\gamma/2} M(s, s+h)}{\tau_\gamma^{\gamma/2} h^{1/2}} \geq \inf_{0 \leq s \leq 1-h_{k+1}} \frac{(|\log h_k|)^{\gamma/2} M(s, s+h_{k+1})}{\lambda^{1/2} \tau_\gamma^{\gamma/2} h_{k+1}^{1/2}}.$$

We shall show that with probability 1

$$\liminf_{k \to \infty} \inf_{0 \leq s \leq 1-h_{k+1}} \frac{(|\log h_k|)^{\gamma/2} M(s, s+h_{k+1})}{\tau_\gamma^{\gamma/2} h_{k+1}^{1/2}} \geq 1. \tag{4.18}$$

For $s_k(i-1) = (i-1)\lambda^{-k}/(\log k)^\varrho \leq s \leq i\lambda^{-k}/(\log k)^\varrho = s_k(i)$, with $\varrho > \gamma + 1$, we have

$$M(s, s+h_{k+1}) \geq M(s_k(i), s_k(i) + h_{k+1}) - 2M_k,$$

where

$$M_k = \sup_{0 \leq s \leq 1-a_k} M(s, s+a_k),$$

with $a_k = \lambda^{-k}/(\log k)^{\varrho}$. Now by Lemma 4.1 in combination with $\varrho > \gamma + 1$, we have, with probability 1,

$$M_k = O\left(a_k^{1/2}\sqrt{\log(1/a_k)}\right) = o\left(h_k^{1/2}(|\log h_k|)^{-\gamma/2}\right).$$

Set $m_k = \left\lceil \lambda^k (\log k)^{\varrho} \right\rceil$. Thus to establish (4.18) it is enough to prove that with probability 1

$$\liminf_{k\to\infty} \min_{1\le i\le m_k} \frac{(|\log h_k|)^{\gamma/2} M(s_k(i), s_k(i) + h_{k+1})}{\tau_\gamma^{\gamma/2} h_{k+1}^{1/2}} \ge 1. \tag{4.19}$$

Now for any $\varepsilon > 0$

$$P\left\{\min_{1\le i\le m_k} \frac{(|\log h_k|)^{\gamma/2} M(s_k(i), s_k(i) + h_{k+1})}{\tau_\gamma^{\gamma/2} h_{k+1}^{1/2}} \le \frac{1}{(1+\varepsilon)^{\gamma/2}}\right\}$$

$$\le m_k P\left\{\frac{(|\log h_k|)^{\gamma/2} M(s_k(1), s_k(1) + h_{k+1})}{\tau_\gamma^{\gamma/2} h_{k+1}^{1/2}} \le \frac{1}{(1+\varepsilon)^{\gamma/2}}\right\},$$

which by Theorem 2.1 is equal to

$$m_k \exp(-(1+\varepsilon)k(\log\lambda + o(1))).$$

This, in turn, is for all large enough k less than or equal to $\lambda^{-\varepsilon k/2}$. An application of the Borel–Cantelli lemma now shows (4.19), thus (4.18). Finally since $\lambda > 1$ can be chosen arbitrarily close to 1, we have (4.17). $\square$

Acknowledgment The authors thank Wenbo Li for useful discussions while this paper was being written, and an anonymous referee for a careful reading of the manuscript.

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