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SMALL DEVIATIONS OF RIEMANN–LIOUVILLE PROCESSES IN L_q–SPACES WITH RESPECT TO FRACTAL MEASURES

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Abstract

We investigate Riemann–Liouville processes $R_H, H > 0$, and fractional Brownian motions B_H, $0 < H < 1$, and study their small deviation properties in the spaces $L_q([0,1],\mu)$. Of special interest are hereby thin (fractal) measures μ, i.e., those which are singular with respect to the Lebesgue measure. We describe the behavior of small deviation probabilities by numerical quantities of μ, called mixed entropy numbers, characterizing size and regularity of the underlying measure. For the particularly interesting case of self–similar measures the asymptotic behavior of the mixed entropy is evaluated explicitly. We also provide two–sided estimates for this quantity in the case of random measures generated by subordinators.

While the upper asymptotic bound for the small deviation probability is proved by purely probabilistic methods, the lower bound is verified by analytic tools concerning entropy and Kolmogorov numbers of Riemann–Liouville operators.

1. *Introduction*

1.1. *Main Result*

The aim of the present paper is the investigation of the small deviation behavior of Riemann–Liouville processes (RL–processes) and of fractional Brownian motions (fBm) with respect to the L_q–norm of a given (finite, continuous) measure μ on $[0,1]$.

Recall that the RL–process R_H of index $H > 0$ is defined by

$$R_H(t) := \int_0^t (t-u)^{H-1/2}\, \mathrm{d}W(u)\ , \quad 0 \le t < \infty\ ,$$

where W denotes a Wiener process on $[0,\infty)$, while the fBm B_H with Hurst index $H \in (0,1)$ is the centered Gaussian process on $[0,\infty)$ with a.s. continuous paths and covariance

$$\mathbb{E}\, B_H(t)\, B_H(s) = \frac{1}{2}\left\{ t^{2H} + s^{2H} - |t-s|^{2H} \right\}\ , \quad t,s \ge 0\ . \tag{1.1}$$

Thus, if X_H denotes either R_H or B_H, respectively, our objective is to describe the behavior of the small deviation function

$$\varphi_{q,\mu}(\varepsilon) : -\log \mathbb{P}\left(\int_0^1 |X_H(t)|^q\ \mathrm{d}\mu(t) < \varepsilon^q \right) := -\log \mathbb{P}\left(\|X_H\|_{q,\mu} < \varepsilon \right) \tag{1.2}$$

as $\varepsilon \to 0$ in terms of certain quantitative properties of the underlying measure μ.

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Here and later on we use the notation $L_q(\mu) := L_q([0,1],\mu)$ and

$$\|X\|_{q,\mu} := \|X\|_{L_q([0,1],\mu)} \ .$$

General small deviation problems attracted much attention during the last years due to their deep relations to various mathematical topics like operator theory, quantization, strong limit laws in Statistics, etc, see the surveys [**15**, **17**]. A more specific motivation for this work comes from [**22**], where the behavior of small deviations of non-Gaussian Lévy processes in $L_q([0,1],\lambda_1)$ was studied for the Lebesgue measure λ_1. This problem was solved by reducing it to the study of the deviation of the Wiener process in $L_q([0,1],\mu)$ with a random and singular measure μ.

In the sequel we write $f \sim g$ if $\lim_{\varepsilon\to 0}\frac{f(\varepsilon)}{g(\varepsilon)} = 1$ while $f \preceq g$ (or $g \succeq f$) means that $\limsup_{\varepsilon\to 0}\frac{f(\varepsilon)}{g(\varepsilon)} < \infty$. Finally, $f \approx g$ says that $f \succeq g$ as well as $g \succeq f$.

In the case $\mu = \lambda_1$, the behavior of $\varphi_{q,\mu}(\varepsilon)$ is well known, namely, $\varphi_{q,\mu}(\varepsilon) \sim c_{q,H}\,\varepsilon^{-1/H}$, as $\varepsilon \to 0$. The exact value of the finite and positive constant $c_{q,H}$ is known only in few cases; sometimes a variational representation for $c_{q,H}$ is available. See more details in [**15**] and [**20**].

If μ is absolutely continuous with respect to λ_1, the behavior of $\varphi_{q,\mu}(\varepsilon)$ was investigated in [**13**], [**18**] and [**19**]. Under mild assumptions, the order $\varepsilon^{-1/H}$ remains unchanged, only an extra factor depending on the density of μ (with respect to λ_1) appears. The situation is completely different for measures μ being singular to λ_1. This question was recently investigated in [**21**] for $q = \infty$ (here only the size of the support of μ is of importance) and in [**24**] for self–similar measures and $q = 2$. When passing from $q = \infty$ to finite q's, the problem becomes more involved because in the latter case the distribution of the mass of μ becomes important. Consequently, one has to introduce some kind of entropy of μ taking into account the size of its support as well as the distribution of the mass on $[0,1]$. This is done in the following way.

Let μ be a measure on $[0,1]$, let $H > 0$ and $q \in [1,\infty)$. Then the following numbers will play an important role later on.

Definition 1.1 *We define a number $r > 0$ by*

$$1/r := H + 1/q \ ,$$

and set

$$\sigma_\mu(m) = \sigma_\mu^{(H,q)}(m) : \inf\left\{ \left(\sum_{j=1}^{m} |\Delta_j|^{H\,r}\,\mu(\Delta_j)^{r/q}\right)^{1/r} \ : \ [0,1] \subseteq \bigcup_{j=1}^{m} \Delta_j \right\} \tag{1.3}$$

where the Δ_j's are supposed to be real intervals. The numbers $\sigma_\mu(\,\cdot\,)$ are called outer mixed entropy numbers of μ.

Here "mixed" means that we take into account the measure as well as the length of an interval. Note that a very similar expression already appeared in [**18**], section 6.2, for absolutely continuous measures μ.

The main result of this paper shows a very tight relation between the behavior of $\sigma_\mu(m)$, as $m \to \infty$, and of the small deviation function (1.2). More precisely, the following will be proved.

Theorem 1.2 *Let μ be a finite continuous measure on $[0,1]$ and let R_H be the*

Riemann–Liouville process of index $H > 0$. For $q \in [1, \infty)$ define $\sigma_\mu(m)$ as in (1.3). Then the following facts are true.

(a) *If*

$$\sigma_\mu(m) \succeq m^{-\nu} (\log m)^\beta$$

for certain $\nu \geq 0$ and $\beta \in \mathbb{R}$, then

$$-\log \mathbb{P}\left(\|R_H\|_{q,\mu} < \varepsilon\right) \succeq \varepsilon^{-1/(H+\nu)} \cdot \log(1/\varepsilon)^{\beta/(H+\nu)} . \tag{1.4}$$

(b) *On the other hand, if*

$$\sigma_\mu(m) \preceq m^{-\nu} (\log m)^\beta$$

then

$$-\log \mathbb{P}\left(\|R_H\|_{q,\mu} < \varepsilon\right) \preceq \varepsilon^{-1/(H+\nu)} \cdot \log(1/\varepsilon)^{\beta/(H+\nu)} . \tag{1.5}$$

For $0 < H < 1$, both assertions also hold for the fractional Brownian motion B_H.

1.2. *An Example: Self–Similar Fractal Measures*

We show now how our result applies to a particularly interesting class of singular measures, the class of so–called self–similar measures. First, let us recall their definition. Take an integer $N \geq 2$, some positive weights $\rho_1, \dots, \rho_N$ such that $\sum_{k=1}^N \rho_k = 1$, and N intervals with disjoint interiors $[a_1, b_1], \dots, [a_N, b_N]$ in $[0, 1]$. For every $k = 1, \dots, N$ let S_k be one of two possible affine mappings from $[0, 1]$ onto $[a_k, b_k]$. The corresponding self–similar measure μ is now uniquely defined by the equation

$$\mu = \sum_{k=1}^N \rho_k \ [\mu \circ S_k^{-1}]. \tag{1.6}$$

We refer to [**10**] for further information on self–similar measures.

The following result can be derived from Theorem 1.2 in the case of self–similar measures. We state it for the RL–process, yet here and later on all assertions are also valid for fractional Brownian motions B_H, $0 < H < 1$.

Theorem 1.3 *Let $\lambda_k := b_k - a_k$ and let $\gamma > 0$ be the unique solution of the equation*

$$\sum_{k=1}^N \lambda_k^{H\gamma} \rho_k^{\gamma/q} = 1. \tag{1.7}$$

Then, $\gamma \leq r$, and if μ satisfies (1.6), it follows that

$$\sigma_\mu(m) \approx m^{-(1/\gamma - 1/r)} , \tag{1.8}$$

hence

$$-\log \mathbb{P}\left(\|R_H\|_{q,\mu} < \varepsilon\right) \approx \varepsilon^{-1/(1/\gamma - 1/q)} . \tag{1.9}$$

We see that the order of the small deviations does not depend on the special choice of the mappings S_k (recall that there are two possibilities to map $[0, 1]$ onto $[a_k, b_k]$ by an affine mapping).

In view of the tight connection between small deviation results and compactness properties of related operators, Theorem 1.3 may also be formulated as follows.

Theorem 1.4 *Let the self–similar measure μ be as above. If $\alpha > 1/2$ and $\gamma > 0$ is the unique solution of the equation*

$$\sum_{k=1}^{N} \lambda_k^{(\alpha-1/2)\gamma} \rho_k^{\gamma/q} = 1,$$

then the entropy numbers of the Riemann–Liouville operator $\mathcal{R}_\alpha$, (cf. (4.1) for its definition) regarded as operator from $L_2[0,1]$ into $L_q(\mu)$, $1 \le q < \infty$, behave like $n^{-(1/\gamma+1/2-1/q)}$.

The following subclass of self–similar measures is of particular interest. Assume that for some $s \ge 1$

$$\lambda_k = \rho_k^s \, , \qquad 1 \le k \le N. \tag{1.10}$$

Then we have $\gamma = (sH + 1/q)^{-1}$ and

$$-\log \mathbb{P}\left(\|R_H\|_{q,\mu} < \varepsilon \right) \approx \varepsilon^{-1/(sH)} \, .$$

We see that under condition (1.10) the order of the small deviation does not depend on the number $q \ge 1$. Yet for general self–similar measures one should expect a dependence on q.

To investigate this question let us examine some special classes of self–similar measures.

1) If $\rho_k = \lambda_k$, then the intervals $[a_k, b_k]$ have to cover $[0,1]$ and μ is the Lebesgue measure. Thus relation (1.10) holds with $s = 1$ and we obtain

$$-\log \mathbb{P}\left(\|R_H\|_{q,\mu} < \varepsilon \right) \approx \varepsilon^{-1/H},$$

in accordance with known results.

2) For $N = 2$ let $[a_1, b_1] = [0, 1/3]$, $[a_2, b_2] = [2/3, 1]$ and $\rho_1 = \rho_2 = 1/2$. Then μ is the classical Cantor measure, hence relation (1.10) holds with $s = \frac{\log 3}{\log 2}$ and we obtain

$$-\log \mathbb{P}\left(\|R_H\|_{q,\mu} < \varepsilon \right) \approx \varepsilon^{-\frac{\log 2}{H \log 3}}.$$

This order of small deviation was known for the special case $q = \infty$ (see the example after Theorem 6.2 in [**21**]).

3) Let the intervals be as in the previous example, yet this time we choose $\rho_1 = \theta$ and $\rho_2 = 1 - \theta$ for some $\theta \in (0,1)$, $\theta \ne 1/2$. The corresponding measure μ could be called skewed Cantor measure. Here equation (1.7) becomes

$$\theta^{\gamma/q} + (1-\theta)^{\gamma/q} = 3^{H\gamma} \tag{1.11}$$

and the relation (1.10) does not hold anymore. No explicit expression for γ is available, but we can examine what is going on in the case of extreme skewness, i.e., for $\theta \to 0$. For any given $H > 0$ and $q < \infty$ we obviously have $\gamma = \gamma(\theta) \to 0$. For the left hand side of (1.11) we have

$$\theta^{\gamma/q} + (1-\theta)^{\gamma/q} - 1 \sim \theta^{\gamma/q} \, , \quad \theta \to 0 \, ,$$

while for the right hand side of (1.11)

$$3^{H\gamma} - 1 \sim \log 3 \, H \, \gamma \, , \quad \theta \to 0 \, .$$

Hence,

$$\theta^{\gamma/q} \sim \log 3\, H\, \gamma\,, \quad \theta \to 0\,.$$

Taking the logarithms yields

$$\frac{\gamma \log\theta}{q} \sim \log\gamma\,, \quad \theta \to 0\,.$$

Thus for any given $H > 0$ and $q < \infty$ it follows that

$$\gamma = \gamma(\theta) \sim \frac{q \log\log(1/\theta)}{\log(1/\theta)}\,, \qquad \theta \to 0\,,$$

and

$$-\log \mathbb{P}\left(\|R_H\|_{q,\mu} < \varepsilon\right) \approx \varepsilon^{-\kappa(\theta)},$$

where

$$\kappa(\theta) \sim \gamma(\theta) \sim \frac{q \log\log(1/\theta)}{\log(1/\theta)}\,.$$

Consequently, the order of the small deviation decreases to zero when the measure becomes more and more degenerated. Moreover, we see that in general the small deviation behavior does depend on the parameter q.

The results for self–similar measures presented in this section were greatly inspired by some previously known results for the Hilbert space case $q = 2$. Actually, in a Hilbert space one can study the small deviation behavior in the language of spectral theory. The corresponding results on the eigenvalues of appropriate operators were obtained by Fujita [**9**] and Solomyak and Verbitsky [**28**] (for $q = 2$ and $H = 1/2$). The latter work provides even more information than one needs for our level of precision. In [**28**], the so–called arithmetic and non-arithmetic cases are distinguished, according to the properties of the ρ_k's and λ_k's. Following the ideas of [**28**], Nazarov [**24**] was recently able to provide the *exact* logarithmic asymptotic behavior of small deviations. He proved (for special H's) that in the non–arithmetic case

$$-\log \mathbb{P}\left(\|R_H\|_{2,\mu} \le \varepsilon\right) \sim C\ \varepsilon^{-1/(1/\gamma-1/2)}$$

for some $C \in (0,\infty)$, while in the arithmetic case

$$-\log \mathbb{P}\left(\|R_H\|_{2,\mu} \le \varepsilon\right) \sim \Psi(\log(1/\varepsilon))\ \varepsilon^{-1/(1/\gamma-1/2)}\,,$$

with some *periodic* function Ψ bounded away from zero and infinity.

These results are of course more precise than our (1.9) and they are actually valid for a wider class of processes, being however restricted to the particular case $q = 2$ and (less important) to $H = m + 1/2$ with $m = 0, 1, 2, \ldots$.

1.3. *Random Fractal Measures*

Other interesting singular measures are those generated by subordinators. Recall that a Lévy process $A = (A(t))_{t\ge 0}$ is called subordinator provided it is non–decreasing. Its Laplace exponent Φ is then defined by

$$\mathbb{E}\, e^{-A(t)\cdot x} = e^{-t\cdot\Phi(x)}\,, \qquad t, x \ge 0\,.$$

We refer to [**3**] for more information about subordinators.

Every subordinator A generates random measures $\mu = \mu_\omega$ by $\mu := \lambda_1 \circ A^{-1}$ where λ_1 denotes the Lebesgue measure on $[0,1]$, i.e.,

$$\mu([0,s]) = \mu_\omega([0,s]) = \lambda_1\left(\{t \in [0,1] : A(t,\omega) \le s\}\right) , \quad s \ge 0 . \tag{1.12}$$

Our basic result about these random measures is as follows.

Theorem 1.5 *For any $H > 0$ and any $q \in [1,\infty)$ there exist constants $c_1, c_2 > 0$ such that the following is valid. For all strictly increasing subordinators A with Laplace exponent Φ and almost all measures μ defined by* (1.12) *it is true that $\sigma_\mu(m) = \sigma_\mu^{(H,q)}(m)$ satisfies*

$$\left(\frac{m}{\Phi^{-1}(c_1 m)}\right)^H \preceq \sigma_\mu(m) \preceq \left(\frac{m}{\Phi^{-1}(c_2 m)}\right)^H . \tag{1.13}$$

When combining Theorem 1.5 with Theorem 1.2 we obtain the following corollary about an important class of so called subordinated processes.

Corollary 1.6 *Let A be a strictly increasing subordinator such that its Laplace exponent satisfies*

$$\Phi(x) \approx x^\beta \,(\log x)^\kappa , \qquad x \to \infty ,$$

for certain $\beta \in (0,1]$ and $\kappa \in \mathbb{R}$. If R_H is an RL–process, $H > 0$, independent of A, then for almost all ω and each $q \in [1,\infty)$ we have

$$-\log \mathbb{P}\left(\int_0^1 |R_H(A(t,\omega))|^q \,\mathrm{d}t < \varepsilon^q\right) \approx \varepsilon^{-\beta/H} \,\log(1/\varepsilon)^\kappa .$$

Remark 1.7 *In the case $0 < H < 1$ the preceding corollary is slightly weaker than Theorem* 2.1 *in* [**22**]*. Yet observe that the methods used in* [**22**] *are no longer applicable for $H \ge 1$.*

The organization of the paper is as follows. In Section 2 we investigate $\sigma_\mu(m)$ and some related quantities needed later on. Section 3 is devoted to the essential part of the (purely probabilistic) proof of assertion (a) in Theorem 1.2, yet an important time inversion argument is left for special consideration in Section 7. The proof of part (b), in contrast, employs several quite delicate analytic methods. This is carried out in Section 4. A particular result in this section, which may be of interest by itself, is an estimate of the Kolmogorov numbers d_n of the Riemann–Liouville fractional integration operator from $L_2[0,1]$ into $L_q(\mu)$ in terms of $\sigma_\mu(m)$. The case of self–similar measures and, especially, the proof of Theorem 1.3 is the subject of Section 5 while random fractal measures are investigated in Section 6.

2. *Mixed Entropy of Measures*

Let μ be a finite and continuous measure on $[0,1]$. Given an interval $\Delta \subseteq [0,1]$, we set

$$J_\mu(\Delta) = J_\mu^{(H,q)}(\Delta) := |\Delta|^H \cdot \mu(\Delta)^{1/q} .$$

With this notation the outer mixed entropy numbers $\sigma_\mu(m) = \sigma_\mu^{(H,q)}(m)$ introduced in (1.3) may be written as

$$\sigma_\mu(m) = \inf\left\{ \left(\sum_{j=1}^m J_\mu(\Delta_j)^r \right)^{1/r} : [0,1] \subseteq \bigcup_{j=1}^m \Delta_j \right\} .$$

We mention that by Hölder's inequality, $m \mapsto \sigma_\mu(m)$ is non-increasing.

For later purposes we also need an other kind of entropy. This one is defined as follows.

Definition 2.1 *Given μ as before for each $\delta > 0$ we set*

$$\begin{aligned} M_\mu(\delta) &= M_\mu^{(H,q)}(\delta) \\ &:= \max\{m : \exists\, \Delta_1, \ldots, \Delta_m \text{ in } [0,1] \text{ with disjoint interiors, } J_\mu(\Delta_j) \geq \delta\} . \end{aligned} \tag{2.1}$$

The function $M_\mu(\,\cdot\,)$ is called inner mixed entropy of μ.

Note that for continuous measures μ we may replace in the above definition $J_\mu(\Delta) \geq \delta$ by $J_\mu(\Delta) = \delta$.

Let us consider the inverse of M_μ.

Definition 2.2 *The numbers*

$$\delta_\mu(m) = \delta_\mu^{(H,q)}(m) : \inf\{\delta > 0 : M_\mu(\delta) \leq m\} \tag{2.2}$$

are called inner mixed entropy numbers of μ.

The following relations between inner and outer mixed entropy numbers are valid

Proposition 2.3 *Let μ be a finite continuous measure on $[0,1]$. Then for each integer $m \geq 1$, we have*

$$\sigma_\mu(2m+1) \leq (2m+1)^{1/r}\, \delta_\mu(m) \quad \text{and} \quad m^{1/r}\, \delta_\mu(2m) \leq \sigma_\mu(m) . \tag{2.3}$$

Proof. Let us start with the proof of the first estimate. We choose an arbitrary $\delta > \delta_\mu(m)$, whence $M := M_\mu(\delta) \leq m$. By the definition of $M_\mu(\delta)$ we find M intervals with disjoint interiors $\Delta_1, \ldots, \Delta_M$ in $[0,1]$ satisfying $J_\mu(\Delta_j) = \delta$. There are at most $M+1$ disjoint intervals $\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_{M+1}$ complementary to $\Delta_1, \ldots, \Delta_M$ such that the $\widetilde{\Delta}_k$'s and the Δ_j's together cover $[0,1]$. By the maximality of $M = M_\mu(\delta)$ we obtain $J_\mu(\widetilde{\Delta}_k) \leq \delta$, hence

$$\sigma_\mu(2m+1) \leq \sigma_\mu(2M+1) \leq (2M+1)^{1/r} \cdot \delta \leq (2m+1)^{1/r} \cdot \delta$$

proving the first estimate in (2.3) by letting $\delta \to \delta_\mu(m)$.

To verify the second estimate, let us fix $m \in \mathbb{N}$, take an arbitrary covering $\Delta_1, \ldots, \Delta_m$ of $[0,1]$, and assume that the intervals have disjoint interiors. Next we set $\delta := m^{-1/r}\, \sigma_\mu(m)$ and assume that there are n intervals with disjoint interiors $\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_n$ in $[0,1]$ such that $J_\mu(\widetilde{\Delta}_k) \geq \delta$, $1 \leq k \leq n$. Let

$$L_j := \left\{ k \leq n : \widetilde{\Delta}_k \subseteq \Delta_j \right\} , \quad 1 \leq j \leq m .$$

An application of Hölder's inequality then leads to

$$J_\mu(\Delta_j)^r \geq \sum_{k \in L_j} J_\mu(\widetilde{\Delta}_k)^r \geq \#(L_j) \cdot \delta^r .$$

Thus

$$\sum_{j=1}^{m} J_\mu(\Delta_j)^r \geq \delta^r \cdot \sum_{j=1}^{m} \#(L_j) \ .$$

Now observe that at most $m-1$ of the $\widetilde{\Delta}_k$'s may not belong to a certain Δ_j, i.e., we have $\sum_{j=1}^{m} \#(L_j) \geq n - m + 1$. This being true for any choice of the Δ_j's implies

$$\sigma_\mu(m) \geq \delta \cdot (n-m+1)^{1/r} \ ,$$

hence, by the definition of δ, we necessarily have $n < 2m$. Thus $M_\mu(\delta) < 2m$, whence

$$\delta_\mu(2m) \leq \delta = m^{-1/r} \cdot \sigma_\mu(m)$$

completing the proof. □

For later needs, we now slightly modify J_μ, M_μ and $\delta_\mu(m)$. Namely, we use a quantity which (in the absolutely continuous case) already appeared in [**7**, **8**]. Given an interval $\Delta = [a, b] \subseteq [0, 1]$ we set

$$\overline{J}_\mu(\Delta) = \overline{J}_\mu^{(H,q)}(\Delta) : \sup_{x \in (a,b)} (x-a)^H \mu([x, b])^{1/q} \ .$$

Then we define $\overline{M}_\mu$ and $\overline{\delta}_m$ as in (2.1) and (2.2), respectively, yet this time taken with respect to $\overline{J}_\mu$.

Of course, we have $\overline{J}_\mu(\Delta) \leq J_\mu(\Delta)$, hence $M_\mu(\delta) \geq \overline{M}_\mu(\delta)$. In general these expressions are not equivalent. However, some kind of inverse inequality still holds. To make this more precise, let us introduce the time inversion of a measure μ as its image with respect to the mapping $t \mapsto 1-t$, i.e., we set

$$\mu^-([a, b]) := \mu([1-b, 1-a]) \ .$$

With this notation the following is valid.

Lemma 2.4 *Let μ be a finite measure on $[0,1]$ and $\delta > 0$. Then we have*

$$\max\left\{\overline{M}_\mu(\delta), \overline{M}_{\mu^-}(\delta)\right\} \geq \frac{M_\mu(\delta)-1}{2} \ .$$

Proof. For $m \in \mathbb{N}$ assume that there are $2m$ intervals with disjoint interiors $\Delta_1, \ldots, \Delta_{2m}$ in $[0,1]$ such that $J_\mu(\Delta_j) \geq \delta$. Of course, we may suppose that $\bigcup_{j=1}^{2m} \Delta_j = [0,1]$ and that the intervals are numbered according to their position in $[0,1]$. For $1 \leq j \leq m$ define $\widetilde{\Delta}_j := \Delta_{2j-1} \cup \Delta_{2j}$. If $|\Delta_{2j-1}| \geq |\Delta_{2j}|$, then

$$|\Delta_{2j-1}|^H \mu(\Delta_{2j})^{1/q} \geq |\Delta_{2j}|^H \mu(\Delta_{2j})^{1/q} J_\mu(\Delta_{2j}) \geq \delta \ ,$$

hence by definition, $\overline{J}_\mu(\widetilde{\Delta}_j) \geq \delta$. If, conversely, $|\Delta_{2j-1}| < |\Delta_{2j}|$, then it follows by the same arguments that $\overline{J}_{\mu^-}(1-\widetilde{\Delta}_j) \geq \delta$. Thus in any case there exist at least m intervals with disjoint interiors having a $\overline{J}_\mu$–value larger than δ or such m intervals exist for $\overline{J}_{\mu^-}$. This completes the proof. □

The following property of $\overline{J}_\mu$ will be crucial later on.

Lemma 2.5 *Let μ be a finite continuous measure on $[0,1]$. Let $H>0$, $q\in[1,\infty)$ and let $\Delta:=[a,b]\subseteq[0,1]$. If $x_*\in(a,b)$ satisfies*

$$(x_*-a)^H\mu[x_*,b]^{1/q}\geq\frac{1}{2}\,\overline{J}_\mu(\Delta)\ , \tag{2.4}$$

then there exists a $y_*\in(x_*,b]$ such that the following inequalities hold:

$$(x_*-a)^H\mu[x_*,y_*]^{1/q}\geq\frac{1}{6}\,\overline{J}_\mu(\Delta), \tag{2.5}$$

$$y_*-x_*\leq 3^{1/H}(x_*-a). \tag{2.6}$$

Proof. Let $y_*:=\min\{b,\ a+3^{1/H}(x_*-a)\}\in(x_*,b]$. Clearly (2.6) is satisfied, thus it remains to prove (2.5). We distinguish two possible cases.
First case: $b\leq a+3^{1/H}(x_*-a)$.
Then $y^*=b$, and it follows from (2.4) that $\overline{J}_\mu([x_*,y_*])\geq\frac{1}{2}\,\overline{J}_\mu(\Delta)$, implying (2.5).
Second case: $b>a+3^{1/H}(x_*-a)$.
In this case, $y_*a+3^{1/H}(x_*-a)\in(a,b)$, and by the definition of $\overline{J}_\mu(\Delta)$, we have

$$(3^{1/H}(x_*-a))^H\mu[y_*,b]^{1/q}\leq\overline{J}_\mu(\Delta)\ ,$$

i.e., $\mu[y_*,b]\leq\overline{J}_\mu(\Delta)^q/(3^q(x_*-a)^{qH})$. On the other hand, by (2.4),

$$\mu[x^*,b]\geq\overline{J}_\mu(\Delta)^q/(2^q(x_*-a)^{qH})\ .$$

Therefore,

$$\mu[x_*,y_*]=\mu[x^*,b]-\mu[y_*,b]\geq(2^{-q}-3^{-q})\,\frac{\overline{J}_\mu(\Delta)^q}{(x_*-a)^{qH}},$$

from which it follows that

$$(x_*-a)^H\mu[x_*,y_*]^{1/q}\geq(2^{-q}-3^{-q})^{1/q}\overline{J}_\mu(\Delta)\geq\frac{1}{6}\,\overline{J}_\mu(\Delta)\ ,$$

as desired. □

3. *Proof of Part (a) in Theorem 1.2*

The following proposition provides the crucial estimate for the proof of (1.4).

Proposition 3.1 *Let $H>0$ and $q\in[1,\infty)$ and suppose that for some $\delta>0$ there are intervals $\Delta_1,\ldots,\Delta_m$ in $[0,1]$ with disjoint interiors such that*

$$\overline{J}_\mu(\Delta_j)\geq\delta\,,\quad 1\leq j\leq m\,. \tag{3.1}$$

Then

$$\log\mathbb{P}\left(\|R_H\|_{q,\mu}^q<c_1\cdot\delta^q\,m\right)\leq-c_2\cdot m \tag{3.2}$$

with $c_1,c_2>0$ only depending on H and q.

Proof. Let us assume that the intervals $\Delta_i:=[a_i,b_i]$ are numbered according to their position. We observe that

$$\|R_H\|_{q,\mu}^q\geq\sum_{i=1}^m\int_{a_i}^{b_i}|R_H(t)|^q\,\mathrm{d}\mu(t):=\sum_{i=1}^m I_i\ , \tag{3.3}$$

with obvious notation. Note that, for any i, I_i is measurable with respect to $\mathcal{F}_{b_i}$, where $(\mathcal{F}_t)_{t\geq 0}$ denotes the canonical filtration of the underlying Wiener process W.

Let us now study the random variables I_i more thoroughly. For any $t \in \Delta_i$,

$$R_H(t) = \int_0^{a_i} (t-s)^{H-1/2}\,\mathrm{d}W(s) + \int_{a_i}^t (t-u)^{H-1/2}\,\mathrm{d}W(u).$$

The first term of this decomposition is $\mathcal{F}_{a_i}$-measurable, whereas the second is independent of $\mathcal{F}_{a_i}$.

By Anderson's inequality (see [**1**] or [**16**], Chapter 11), and in view of (3.3), this leads to the following: for any $\varepsilon > 0$,

$$\mathbb{P}\left(\|R_H\|_{q,\mu}^q < \varepsilon\right) \leq \mathbb{P}\left(\sum_{i=1}^{m-1} I_i + \int_{a_m}^{b_m} \left|\int_{a_m}^t (t-u)^{H-1/2}\,\mathrm{d}W(u)\right|^q \mathrm{d}\mu(t) < \varepsilon\right).$$

Iterating the procedure, we obtain that

$$\mathbb{P}\left(\|R_H\|_{q,\mu}^q < \varepsilon\right) \leq \mathbb{P}\left(\sum_{i=1}^{m} X_i < \varepsilon\right), \tag{3.4}$$

where

$$X_i := \int_{a_i}^{b_i} \left|\int_{a_i}^t (t-u)^{H-1/2}\,\mathrm{d}W(u)\right|^q \mathrm{d}\mu(t).$$

In a next step, we use Lemma 2.5 jointly with (3.1) to find $[x_i,\, y_i] \subset \Delta_i$ such that

$$(x_i - a_i)^H \mu[x_i, y_i]^{1/q} \geq \frac{\delta}{6} \qquad \text{and} \qquad y_i - x_i \leq 3^{1/H}(x_i - a_i)\,. \tag{3.5}$$

Of course, $X_i \geq \int_{x_i}^{y_i} |\int_{a_i}^t (t-u)^{H-1/2}\,\mathrm{d}W(u)|^q\,\mathrm{d}\mu(t)$. Hence, by Hölder's inequality,

$$X_i \geq \mu[x_i, y_i]^{1-q} \left|\int_{x_i}^{y_i}\int_{a_i}^t (t-u)^{H-1/2}\,\mathrm{d}W(u)\,\mathrm{d}\mu(t)\right|^q : \mu[x_i, y_i]^{1-q}\,|\eta_i|^q\,.$$

Clearly, η_i is a zero–mean Gaussian random variable with variance

$$\sigma_i^2 = \int_{x_i}^{y_i} \mathrm{d}\mu(t) \int_{x_i}^{y_i} \mathrm{d}\mu(s) \int_{a_i}^{\min\{s,t\}} (s-u)^{H-1/2}(t-u)^{H-1/2}\,\mathrm{d}u.$$

Therefore, writing $(\xi_i)_{i\geq 1}$ for a sequence of i.i.d. $\mathcal{N}(0,1)$ random variables, we have, for any $\varepsilon > 0$,

$$\mathbb{P}\left(\|R_H\|_{q,\mu}^q < \varepsilon\right) \leq \mathbb{P}\left(\sum_{i=1}^{m} \mu[x_i, y_i]^{1-q}\sigma_i^q |\xi_i|^q < \varepsilon\right). \tag{3.6}$$

We now give a lower bound for σ_i by distinguishing two possible situations depending on the value of H.

When $H \geq 1/2$, it is clear that

$$\begin{aligned}\int_{a_i}^{\min\{s,t\}} (s-u)^{H-1/2}(t-u)^{H-1/2}\,\mathrm{d}u &\geq \int_{a_i}^{\min\{s,t\}} (\min\{s,t\} - u)^{2H-1}\,\mathrm{d}u \\ &= \frac{1}{2H}(\min\{s,t\} - a_i)^{2H},\end{aligned}$$

which implies

$$\sigma_i^2 \geq \frac{1}{2H}(x_i - a_i)^{2H}\mu[x_i, y_i]^2.$$

When $H < 1/2$, we can argue that

$$\int_{a_i}^{\min\{s,t\}} (s-u)^{H-1/2}(t-u)^{H-1/2}\,\mathrm{d}u \geq (y_i - a_i)^{2H-1}(x_i - a_i)\ ,$$

for s, $t \in [x_i, y_i]$, from which we deduce that

$$\sigma_i^2 \geq (y_i - a_i)^{2H-1}(x_i - a_i)\,\mu[x_i, y_i]^2\ .$$

By (3.5), we have $(y_i - a_i)^{2H-1} \geq (1 + 3^{1/H})^{2H-1}(x_i - a_i)^{2H-1}$. Therefore,

$$\sigma_i^2 \geq (1 + 3^{1/H})^{2H-1}(x_i - a_i)^{2H}\mu[x_i, y_i]^2.$$

As a consequence, for any value of H, there exists a constant $c_0 \in (0,\infty)$ (whose value only depends on H), such that

$$\sigma_i \geq c_0\,(x_i - a_i)^H\mu[x_i, y_i] \geq \frac{c_0\,\delta}{6}\,\mu[x_i, y_i]^{1-(1/q)},$$

the last inequality being a consequence of (3.5). Plugging this into (3.6), we see that, for any $\varepsilon > 0$,

$$\mathbb{P}\left(\|R_H\|_{q,\mu}^q < \varepsilon\right) \leq \mathbb{P}\left(\sum_{i=1}^m |\xi_i|^q < (6/c_0)^q\,\frac{\varepsilon}{\delta^q}\right).$$

We now choose $\varepsilon := c_1\,\delta^q\,m$, where the constant $c_1 > 0$ is so small that it satisfies $(6/c_0)^q(c_1)^q < \mathbb{E}\,(|\xi_1|^q)$. By the exponential Chebyshev inequality, there exists a constant $c_2 > 0$ such that

$$\mathbb{P}\left(\|R_H\|_{q,\mu}^q < c_1\,\delta^q\,m\right) \leq e^{-c_2\,m}.$$

This completes the proof. □

Corollary 3.2 *There are constants $C_1, C_2 > 0$ (only depending on H and q) as well as a non–increasing function $g : (0,1) \to [0,\infty)$ with $\lim_{\varepsilon\to 0}\varepsilon^\theta g(\varepsilon) = 0$ for any $\theta > 0$ such that*

$$\log\mathbb{P}\left(\|R_H\|_{q,\mu}^q < C_1\cdot\delta^q\,M_\mu(\delta)\right) \leq -C_2\cdot M_\mu(\delta) + g\left(\delta^q\,M_\mu(\delta)\right) \qquad (3.7)$$

for all $\delta > 0$ satisfying $M_\mu(\delta) \geq 2$.

Proof. Let $\delta > 0$ be such that $M_\mu(\delta) \geq 2$. By Lemma 2.4, at least one of the following inequalities is true:

$$\overline{M}_\mu(\delta) \geq \frac{M_\mu(\delta)}{3} \quad \text{or} \qquad (3.8)$$

$$\overline{M}_{\mu^-}(\delta) \geq \frac{M_\mu(\delta)}{3}\ . \qquad (3.9)$$

If (3.8) holds, then the assertion follows directly from Proposition 3.1 with $g = 0$, $C_1 = c_1/3$ and $C_2 = c_2/3$, where c_1 and c_2 are the constants from (3.2). Indeed, we obtain from (3.2) and (3.8) that

$$\begin{aligned}\log\mathbb{P}\left(\|R_H\|_{q,\mu}^q < C_1\,\delta^q\,M_\mu(\delta)\right) &= \log\mathbb{P}\left(\|R_H\|_{q,\mu}^q < \frac{c_1}{3}\ \delta^q\,M_\mu(\delta)\right)\\ &\leq \log\mathbb{P}\left(\|R_H\|_{q,\mu}^q < c_1\,\delta^q\,\overline{M}_\mu(\delta)\right)\\ &\leq -c_2\,\overline{M}_\mu(\delta) \leq -C_2\,M_\mu(\delta)\ ,\end{aligned}$$

as asserted.

Next, suppose that (3.9) is valid. Now take $C_1 = c_1/(3 \cdot 2^q)$ and $C_2 = c_2/3$, where c_1 and c_2 are the constants from (3.2), and use the inversion argument from Theorem 7.2 below by choosing $g(\cdot) = h\left((\frac{c_1}{3}\cdot)^{1/q}\right)$ where h is the function from (7.5). We apply Theorem 7.2 and obtain that

$$\begin{aligned}
&\log \mathbb{P}\left(\|R_H\|_{q,\mu}^q < C_1\, \delta^q\, M_\mu(\delta)\right) \\
&= \log \mathbb{P}\left(\|R_H\|_{q,\mu} < C_1^{1/q}\, \delta\, M_\mu(\delta)^{1/q}\right) \\
&\le \log \mathbb{P}\left(\|R_H\|_{q,\mu^-} < 2\, C_1^{1/q}\, \delta\, M_\mu(\delta)^{1/q}\right) + h\left(2\, C_1^{1/q}\, \delta\, M_\mu(\delta)^{1/q}\right) \\
&= \log \mathbb{P}\left(\|R_H\|_{q,\mu^-}^q < \frac{c_1}{3}\, \delta^q\, M_\mu(\delta)\right) + g\left(\delta^q\, M_\mu(\delta)\right) \\
&\le \log \mathbb{P}\left(\|R_H\|_{q,\mu^-}^q < c_1\, \delta^q\, \overline{M}_{\mu^-}(\delta)\right) + g\left(\delta^q\, M_\mu(\delta)\right) \ .
\end{aligned} \tag{3.10}$$

In a final step we apply Proposition 3.1 to μ^-. Thus (3.10) can be estimated by

$$-c_2\, \overline{M}_{\mu^-}(\delta) + g\left(\delta^q M_\mu(\delta)\right) \le -C_2\, M_\mu(\delta) + g\left(\delta^q M_\mu(\delta)\right)$$

leading to (3.7) as asserted. □

Now we are in the position to complete the proof of assertion (a) in Theorem 1.2.

Proof of part (a) in Theorem 1.2: We suppose $\sigma_\mu(m) \succeq m^{-\nu}(\log m)^\beta$ for some $\nu \ge 0$ and $\beta \in \mathbb{R}$. In view of Proposition 2.3, there is a constant $c > 0$ such that for all $m \in \mathbb{N}$,

$$\delta_\mu(m) \ge c \cdot m^{-\nu-1/r}\, (\log m)^\beta \ .$$

Given a positive integer $m \ge 2$, we set $\delta_m := \frac{c}{2} \cdot m^{-\nu-1/r}\, (\log m)^\beta$, hence $M_\mu(\delta_m) \ge m$. Plugging this δ_m into (3.7) yields

$$\begin{aligned}
\log \mathbb{P}\left(\|R_H\|_{q,\mu}^q < C_1 \cdot \delta_m^q\, m\right) &\le \log \mathbb{P}\left(\|R_H\|_{q,\mu}^q < C_1 \cdot \delta_m^q M_\mu(\delta_m)\right) \\
&\le -C_2\, M_\mu(\delta_m) + g(\delta_m^q M_\mu(\delta_m)) \\
&\le -C_2\, m + g(\delta_m^q m) \ .
\end{aligned}$$

Since $\delta_m^q m = (\frac{c}{2})^q m^{-\nu q - q/r+1} (\log m)^{\beta q} = (\frac{c}{2})^q m^{-\nu q - Hq} (\log m)^{\beta q}$, it follows from the main property of the function g that $g(\delta_m^q m) = o(m)$, $m \to \infty$. Therefore,

$$-\log \mathbb{P}\left(\|R_H\|_{q,\mu} < C_1^{1/q}\, \left(\frac{c}{2}\right)\, m^{-\nu-H} (\log m)^\beta\right) \succeq m \ ,$$

implying (1.4) in the case of RL–processes.

Thus it remains to verify (1.4) for fractional Brownian motions B_H as defined in (1.1). It is well known (cf. [**23**] or [**14**]) that there are a stochastic process $(Z_H(t))_{t\ge 0}$ and a constant $c_H > 0$ such that

$$B_H = c_H\, (R_H + Z_H) \tag{3.11}$$

with R_H and Z_H being independent. Hence, by Anderson's inequality ([**1**] or [**16**]) it is true that

$$-\log \mathbb{P}\left(\|B_H\|_{q,\mu} < c_H\, \varepsilon\right) \ge -\log \mathbb{P}\left(\|R_H\|_{q,\mu} < \varepsilon\right) \ , \tag{3.12}$$

hence (1.4) for B_H follows from (3.12) and from the corresponding estimate for R_H. □

4. *Proof of Part (b) in Theorem 1.2*

The aim of this section is to prove part (b) of Theorem 1.2 by an analytic approach. Let us introduce the necessary notation.

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces, and let $S : E \to F$ be a compact operator. The Kolmogorov numbers of S, denoted by $d_n(S)$, are defined as

$$d_n(S) = d_n(S : E \to F) : \inf \left\{ \sup_{\|x\|_E \le 1} d_F(Sx, F_n) \ : \ F_n \subseteq F \ , \ \dim(F_n) < n \right\}$$

where, as usual,

$$d_F(y, F_n) := \inf \{ \|y - z\|_F : z \in F_n \}$$

denotes the distance of $y \in F$ to the subspace F_n (w.r.t. the norm in F).

The (dyadic) entropy numbers of S are given by

$$e_n(S) = e_n(S : E \to F) := \inf \left\{ \varepsilon > 0 : S(B_E) \subseteq \bigcup_{j=1}^{2^{n-1}} (y_j + \varepsilon B_F) \right\} .$$

Here B_E and B_F denote the (closed) unit balls in E and F, respectively. In other words, $e_n(S)$ is the infimum over all $\varepsilon > 0$ such that $S(B_E)$ can be covered by at most 2^{n-1} balls of radius $\varepsilon > 0$ in F. We refer to [**5**], [**26**] and [**27**] for more information about Kolmogorov and entropy numbers.

Given $\alpha > 0$, the Riemann–Liouville operator of fractional integration is defined by

$$(\mathcal{R}_\alpha f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u)\, \mathrm{d}u \,, \quad 0 \le t \le 1 \, . \tag{4.1}$$

When $\alpha > 1/2$, this operator maps $L_2[0,1]$ into $L_\infty[0,1]$, hence also into $L_q(\mu)$ for $q \ge 1$ and any finite measure μ. As is well known, $\mathcal{R}_\alpha$ is a compact operator into $L_\infty[0,1]$ (cf. [**14**], Proposition 6.1, where even the degree of its compactness was evaluated), hence also into $L_q(\mu)$, and the question arises how its degree of compactness depends on μ and/or on $q \ge 1$, respectively.

Here we shall prove the following result.

Theorem 4.1 *There are constants $\kappa \in \mathbb{N}$ and $c > 0$ only depending on $\alpha > 1/2$ and $1 \le q < \infty$ such that for any finite continuous measure μ on $[0,1]$ and all $n, m \in \mathbb{N}$,*

$$d_{n+\kappa m}\big(\mathcal{R}_\alpha : L_2[0,1] \to L_q(\mu)\big) \le c \cdot n^{-\alpha}\, \sigma_\mu^{(\alpha-1/2\,,q)}(m) \,, \tag{4.2}$$

where $\sigma_\mu^{(\alpha-1/2\,,q)}(m)$ is defined in (1.3).

By admitting Theorem 4.1 for the moment, we are ready to prove part (b) in Theorem 1.2.

Proof of (b) in Theorem 1.2: Given $H > 0$ set $\alpha := H + 1/2$. Then $\mathcal{R}_\alpha$ generates the RL–process R_H in the following way:

$$R_H(t) = \Gamma(\alpha) \cdot \sum_{k=1}^{\infty} \xi_k (\mathcal{R}_\alpha f_k)(t) \,, \quad 0 \le t \le 1 \,, \tag{4.3}$$

where $(f_k)_{k\geq 1}$ is any fixed ONB in $L_2[0,1]$ and the ξ_k's are i.i.d. $\mathcal{N}(0,1)$.

By assumption we have $\sigma_\mu^{(H,q)}(m) \preceq m^{-\nu}(\log m)^\beta$, hence an application of (4.2) for $n = m$ gives

$$d_n\big(\mathcal{R}_\alpha : L_2[0,1] \to L_q(\mu)\big) \leq c \cdot n^{-\alpha-\nu}(\log n)^\beta \,. \tag{4.4}$$

Next we use Carl's inequality (cf. [**4**], Theorem 1.3) that states a relation between entropy and Kolmogorov numbers. By combining this inequality with (4.4) we obtain that

$$e_n\big(\mathcal{R}_\alpha : L_2[0,1] \to L_q(\mu)\big) \leq c' \cdot n^{-\alpha-\nu}(\log n)^\beta \,.$$

In view of (4.3) we may apply the well known relation between entropy numbers and small deviations (see Theorem 5.1 in [**14**]) and this finally leads to (recall that $\alpha = H + 1/2$) the desired estimate

$$-\log \mathbb{P}\left(\|R_H\|_{q,\mu} \leq \varepsilon\right) \preceq \varepsilon^{-1/(H+\nu)} \cdot \log(1/\varepsilon)^{\beta/(H+\nu)} \tag{4.5}$$

and completes the proof of (b) for RL–processes.

To verify (b) for fractional Brownian motions, we use representation (3.11). It was proved in [**2**] that the process Z_H appearing there satisfies

$$\lim_{\varepsilon\to 0} \varepsilon^\theta \log \mathbb{P}\left(\sup_{0\leq t\leq 1} |Z_H(t)| < \varepsilon\right) = 0 \tag{4.6}$$

for any $\theta > 0$. An application of the partial correlation inequality (7.3) to the representation $B_H = c_H\,(R_H + Z_H)$ leads to

$$\log \mathbb{P}\left(\|B_H\|_{q,\mu} < \varepsilon\right) \geq \log \mathbb{P}\left(\|R_H\|_{q,\mu} < \frac{\varepsilon}{c_H\sqrt{2}}\right) + \log \mathbb{P}\left(\|Z_H\|_{q,\mu} < \frac{\varepsilon}{c_H\,K}\right) \,.$$

Combining the last estimate with (4.5) and (4.6) we obtain (1.5) for B_H as well and this completes the proof. □

The rest of the section is devoted to the proof of Theorem 4.1, which is quite involved and needs some preparation. For example, given m intervals $\Delta_1, \ldots, \Delta_m$ in $[0,1]$ with disjoint interiors, a basic ingredient of the proof of Theorem 4.1 is to split $\mathcal{R}_\alpha$ into m operators V_j mapping $L_2[0,1]$ into $L_\infty(\Delta_j)$. Suppose now we are able to provide suitable estimates for the Kolmogorov numbers $d_n(V_j)$, $1 \leq j \leq m$. Then it is necessary to derive from these estimates a good estimate for the Kolmogorov numbers of the product operator. In the case of entropy numbers, this question was investigated in [**19**]; unfortunately it does not suffice for our later purposes and we need to prove a similar assertion for Kolmogorov numbers.

To be more precise, let us introduce the following notation. Let E_1, …, E_m be Banach spaces and let $q \in [1,\infty)$. We define the Banach space $E^{(q)}$ as $E_1 \times \ldots \times E_m$ endowed with the l_q–norm

$$\|x\| := \left(\sum_{j=1}^m \|x_j\|^q\right)^{1/q} \,, \quad x = (x_j)_{j=1}^m \in E_1 \times \ldots \times E_m \,.$$

For each $j \leq m$ let V_j be a (linear bounded) operator from a Hilbert space $\mathcal{H}$ into the Banach space E_j . Then the product operator $V = V_1 \times \ldots \times V_m$ from $\mathcal{H}$ into $E^{(q)}$ is defined in a canonical way by

$$Vh := (V_j h)_{j=1}^m \,, \quad h \in \mathcal{H} \,.$$

Now we will prove an estimate of the Kolmogorov numbers $d_n(V)$. In our opinion, this result is of independent interest, and may have applications in other problems.

Proposition 4.2 *Let $V_j : \mathcal{H} \to E_j$, $1 \le j \le m$, be operators on a separable Hilbert space $\mathcal{H}$, and let $V = V_1 \times \ldots \times V_m$. Assume there exist $\lambda > 1/2$ and $\rho_j > 0$, $1 \le j \le m$, such that*

$$d_n(V_j) \le \rho_j \cdot n^{-\lambda}, \qquad 1 \le j \le m, \;\; n \in \mathbb{N}. \tag{4.7}$$

Then we have

$$d_n(V : \mathcal{H} \to E^{(q)}) \le c \cdot \left(\sum_{j=1}^{m} \rho_j^r \right)^{1/r} \cdot n^{-\lambda}, \tag{4.8}$$

with $1/r = \lambda - 1/2 + 1/q$.

Proof. The proof relies heavily upon properties of the so–called average Kolmogorov numbers defined as follows: let V be an operator from a separable Hilbert space $\mathcal{H}$ into a Banach space E such that for some (i.e., any) ONB $(e_j)_{j\ge1}$ in $\mathcal{H}$, the sum $\sum_{j\ge1} \xi_j V(e_j)$ converges a.s. in E (as before, $(\xi_j)_{j\ge1}$ denotes a sequence of independent standard normal random variables). Then the n–th average Kolmogorov number $g_n(V)$ of V is defined by

$$g_n(V) := \inf \left\{ \left(\mathbb{E} \inf_{z \in N} \left\| \sum_{j=1}^{\infty} \xi_j V(e_j) - z \right\|^2 \right)^{1/2} \; : \; N \subseteq E, \; \dim(N) < n \right\}$$

where $(e_j)_{j\ge1}$ is some fixed ONB in $\mathcal{H}$. From Theorem 1.1 (ii) in [**6**] we get the following important fact about the relation between average and usual Kolmogorov numbers. Suppose that an operator S from $\mathcal{H}$ into some Banach space E satisfies

$$\Lambda := \sup_{n\ge1} n^{\lambda} d_n(S) < \infty$$

for some $\lambda > 1/2$. Then

$$\sup_{n\ge1} n^{\lambda-1/2} g_n(S) \le c \cdot \Lambda \,,$$

with a universal $c > 0$. An application of this to (4.7) leads to

$$g_n(V_j) \le c \cdot \rho_j \cdot n^{-(\lambda-1/2)}, \qquad 1 \le j \le m, \;\; n \ge 1. \tag{4.9}$$

Now we are in the situation of Theorem 5.2 in [**19**], hence (4.9) implies

$$g_n(V : \mathcal{H} \to E^{(q)}) \le c' \cdot \left(\sum_{j=1}^{m} \rho_j^r \right)^{1/r} \cdot n^{-(\lambda-1/2)} \tag{4.10}$$

where $1/r = \lambda - 1/2 + 1/q$. The estimate

$$d_{2n-1}(S) \le c \cdot n^{-1/2} \cdot g_n(S) \tag{4.11}$$

valid for any operator S from $\mathcal{H}$ into a Banach space E (cf. [**6**], Theorem 1.1 (i)) will be our last argument. The proof is completed by combining (4.10) and (4.11). □

Remark 4.3
(1) *The essential point (and particular reason to switch from* d_n *to* g_n*) is to obtain estimate* (4.8) *with* r *defined by* $1/r = \lambda - 1/2 + 1/q$*. A direct approach using Kolmogorov numbers gives* (4.8) *only with* r *defined by* $1/r = \lambda + 1/q$*.*
(2) *Of course, Proposition* 4.2 *can easily be extended (with some natural additional assumption on the existence of the product operator) to an infinite number of spaces and operators.*

In Section 3, a basic step in the proof of the upper estimate (1.4) was the application of Anderson's inequality which finally led to (3.4). Applying this procedure we were able to replace R_H by a process being the sum of m independent RL–processes $R_H^{\Delta_1}, \ldots, R_H^{\Delta_m}$ defined on the intervals $\Delta_1, \ldots, \Delta_m$, respectively. A similar idea will also play an important role in the proof of Theorem 4.1. To be more precise, let $\Delta_1, \ldots, \Delta_m$ be some intervals in $[0,1]$ with disjoint interiors. If $\Delta := \bigcup_{j=1}^m \Delta_j$, then we may regard on the one hand $\mathcal{R}_\alpha$ as usual as operator into $L_q(\Delta, \mu)$, i.e.,

$$(\mathcal{R}_\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u)\,\mathrm{d}u\,, \quad t \in \Delta\,,$$

and, on the other hand, we may define an "independent" version of $\mathcal{R}_\alpha$ as follows:

$$\mathcal{R}_\alpha^\Delta = \sum_{j=1}^m \mathcal{R}_\alpha^{\Delta_j}$$

where

$$(\mathcal{R}_\alpha^{\Delta_j} f)(t) := \frac{1}{\Gamma(\alpha)} \int_{a_j}^t (t-u)^{\alpha-1} f(u)\,\mathrm{d}u\,, \quad t \in \Delta_j\,, \tag{4.12}$$

with a_j being the left endpoint of Δ_j. In order to replace $\mathcal{R}_\alpha$ by $\mathcal{R}_\alpha^\Delta$, one has to control the Kolmogorov numbers of their difference

$$S_\alpha^\Delta := \mathcal{R}_\alpha - \mathcal{R}_\alpha^\Delta \tag{4.13}$$

by some expression which may depend on the size of the Δ_j's, $1 \le j \le m$, but has to be independent of their positions in $[0,1]$. The dependence on the positions will be put into some finite rank operator. The presence of this extra operator is reflected by the parameter κ in (4.2) and explains why we have to use Kolmogorov numbers instead of entropy numbers.

We start our investigations for a single interval $\Delta = [a,b]$ in $[0,1]$. Then the operator S_α^Δ defined in (4.13) acts on $L_2[0,a]$ as follows :

$$(S_\alpha^\Delta f)(t) := \int_0^a (t-u)^{\alpha-1} f(u)\,\mathrm{d}u\,, \quad t \in \Delta\,.$$

The following representation of S_α^Δ will be crucial for proving Theorem 4.1.

Proposition 4.4 *Suppose* $\alpha > 1/2$ *and regard* S_α^Δ *as operator from* $L_2[0,a]$ *into* $L_\infty(\Delta)$ *(both spaces are taken with respect to the Lebesgue measure). Then it admits a representation*

$$S_\alpha^\Delta = S_\alpha^0 + F_\alpha$$

where $\mathrm{rk}(F_\alpha) \le [\alpha] + 1$ *while* S_α^0 *is small in the sense that for all* $n \ge 1$,

$$d_n(S_\alpha^0) \le c_1 \cdot e^{-c_2\, n^{1/2}} \cdot |\Delta|^{\alpha - 1/2}\,.$$

Proof. We consider three cases.
First case: $\alpha \in \mathbb{N}$.
In this case, S_α^Δ itself is known to be of rank α (cf. Lemma 2.1 in [**21**]), hence the assertion holds trivially.
Second case: $\alpha > 1$ and $\alpha \notin \mathbb{N}$.
Let $k := [\alpha]$ be the integer part of α and $\beta \in (0,1)$ the remainder, i.e., $\alpha = k + \beta$. Then this implies (cf. Lemma 3.2 in [**21**])

$$S_\alpha^\Delta = R_k^\Delta \circ S_\beta^\Delta + F_k \tag{4.14}$$

where F_k has rank less than k, S_β^Δ maps $L_2[0,a]$ into $L_2(\Delta)$, and R_k^Δ from $L_2(\Delta)$ to $L_\infty(\Delta)$ is defined by (4.12) with $\Delta_j = \Delta$. In formula (3.17) in the proof of Lemma 3.4 in [**21**] the operator S_β^Δ from $L_2[0,a]$ into $L_2(\Delta)$ was represented as

$$S_\beta^\Delta = S_\beta^0 + F \tag{4.15}$$

where F is an operator of rank 1 and S_β^0 satisfies (cf. formula (3.18) and (3.19) in [**21**])

$$d_n(S_\beta^0) \le c_1 \cdot e^{-c_2\, n^{1/2}} \cdot |\Delta|^\beta , \qquad n \ge 1,$$

with $c_1 > 0$ and $c_2 > 0$ being independent of Δ. Consequently, (4.14) and (4.15) lead to

$$S_\alpha^\Delta = R_k^\Delta \circ S_\beta^0 + F_\alpha$$

where $F_\alpha := F_k + R_k^\Delta \circ F$. By this construction, it follows that $\mathrm{rk}(F_\alpha) \le k+1 = [\alpha]+1$, thus it remains to estimate $d_n(R_k^\Delta \circ S_\beta^0)$. But this is an immediate consequence of (note that by the scaling properties of R_k we have $\left\|R_k^\Delta\right\| \le c \cdot |\Delta|^{k-1/2}$)

$$\begin{aligned} d_n(R_k^\Delta \circ S_\beta^0) &\le \left\|R_k^\Delta\right\| \cdot d_n(S_\beta^0) \\ &\le c \cdot |\Delta|^{k-1/2} \cdot c_1 \cdot e^{-c_2\, n^{1/2}} \cdot |\Delta|^\beta \\ &= c' \cdot e^{-c_2\, n^{1/2}} \cdot |\Delta|^{\alpha-1/2} \quad . \end{aligned}$$

Third case: $1/2 < \alpha < 1$.
Set $d := |\Delta|$. By changing the variables we get isometries J_1 from $L_\infty[0,1]$ onto $L_\infty(\Delta)$ by setting

$$(J_1 f)(t) := f\big(d^{-1}\,(t-a)\big)\,, \quad a \le t \le b\,,$$

and J_2 from $L_2[0,a]$ onto $L_2[0,a/d]$ by letting

$$(J_2 f)(t) := d^{1/2}\, f(d\,t)\,, \quad 0 \le t \le a/d\,,$$

such that

$$S_\alpha^\Delta = d^{\alpha-\,1/2} \cdot J_1 \circ V_\alpha \circ J_2 \tag{4.16}$$

with $V_\alpha : L_2[0,a/d] \to L_\infty[0,1]$ defined by

$$(V_\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^{a/d} (t+u)^{\alpha-1} f(u)\ \mathrm{d}u\ , \quad 0 \le t \le 1\ .$$

Define now the rank 1 operator F_α by

$$(F_\alpha f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^{a/d} u^{\alpha-1} f(u)\ \mathrm{d}u\ .$$

We observe that

$$d_n(V_\alpha - F_\alpha) \le d_n(B_\alpha) \tag{4.17}$$

where $B_\alpha : L_2(0,\infty) \to L_\infty[0,1]$ is given by

$$(B_\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \left[(t+u)^{\alpha-1} - u^{\alpha-1}\right] f(u)\, \mathrm{d}u\ , \quad 0 \le t \le 1\ .$$

On the other hand, it is known (cf. [**2**], Proposition 3.5) that for $1/2 < \alpha < 3/2$

$$d_n(B_\alpha) \le c_1 \cdot e^{-c_2\, n^{1/2}}\ .$$

Combining this with (4.17) and (4.16) completes the proof in this case as well. □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1: Let $\Delta_1, \ldots, \Delta_m$ be an arbitrary covering of $[0,1]$ by intervals with disjoint interiors. By applying Proposition 4.4 to each interval Δ_j, we get operators S_j and F_j, $1 \le j \le m$, such that $S_\alpha^{\Delta_j} = S_j + F_j$, $\mathrm{rk}(F_j) \le [\alpha] + 1$ and

$$d_n\big(S_j : L_2[0,1] \to L_\infty(\Delta_j)\big) \le c_1 \cdot e^{-c_2\, n^{1/2}} \cdot |\Delta_j|^{\alpha - 1/2}\ . \tag{4.18}$$

Consequently, by the definition of S_α^Δ given in (4.13) it is true that

$$\mathcal{R}_\alpha = \sum_{j=1}^m \big(\mathcal{R}_\alpha^{\Delta_j} + S_j\big) + F$$

where $\mathrm{rk}(F) \le m([\alpha]+1)$ and the $\mathcal{R}_\alpha^{\Delta_j}$'s are as in (4.12). A theorem of Kashin (cf. [**11**]) implies

$$d_n\big(R_\alpha^{\Delta_j} : L_2([0,1]) \to L_\infty(\Delta_j)\big) \le c \cdot |\Delta_j|^{\alpha-1/2} \cdot n^{-\alpha}\ ,$$

hence, in view of (4.18) (using that the d_n's are additive s–numbers in the sense of [**26**], property (S_2) in 2.2.1) we derive

$$d_n\big(R_\alpha^{\Delta_j} + S_j : L_2([0,1]) \to L_\infty(\Delta_j)\big) \le c \cdot |\Delta_j|^{\alpha-1/2} \cdot n^{-\alpha}\ . \tag{4.19}$$

In a final step define $V_j : L_2[0,1] \to L_\infty(\Delta_j)$ by

$$V_j := \mu(\Delta_j)^{1/q} \cdot (R_\alpha^{\Delta_j} + S_j)\,, \quad 1 \le j \le m\,,$$

and set $E_\infty := L_\infty(\Delta_1) \times \ldots \times L_\infty(\Delta_m)$. If the operator $\Phi : E_\infty^{(q)} \to L_q(\mu)$ is given by

$$\Phi\big((f_j)_{j=1}^m\big) := \sum_{j=1}^m f_j \cdot \frac{\mathbf{1}_{\Delta_j}}{\mu(\Delta_j)^{1/q}}\ ,$$

then, of course,

$$\big\|\Phi\big((f_j)_{j=1}^m\big)\big\|_{q,\mu} \le \left(\sum_{j=1}^m \|f_j\|_{L_\infty(\Delta_j)}^q\right)^{1/q} = \big\|(f_j)_{j=1}^m\big\|_{E_\infty^{(q)}}$$

for all $(f_j)_{j=1}^m \in E_\infty^{(q)}$ and, moreover,

$$\mathcal{R}_\alpha = \Phi \circ V + F$$

where as in Proposition 4.2 the operator V denotes the product of the V_j's, i.e., we have $V = V_1 \times \ldots \times V_m$. It follows from (4.19) that

$$d_n(V_j) \le c\,|\Delta_j|^{\alpha-1/2} \cdot \mu(\Delta_j)^{1/q} \cdot n^{-\alpha} \; .$$

Hence, by Proposition 4.2, we obtain

$$d_n(V) \le c \cdot \left(\sum_{j=1}^{m} |\Delta_j|^{r(\alpha-1/2)} \; \mu(\Delta_j)^{r/q} \right)^{1/r} \cdot n^{-\alpha} \tag{4.20}$$

where

$$1/r = \alpha - 1/2 + 1/q \; .$$

Setting $\kappa := [\alpha] + 1$, we get

$$d_{n+\kappa m}(\mathcal{R}_a) = d_{n+\kappa m}(\Phi \circ V + F) \le d_n(\Phi \circ V) \le d_n(V) \; .$$

In view of (4.20) this completes the proof of Theorem 4.1 by taking the infimum over all coverings $\Delta_1, \ldots, \Delta_m$ of $[0,1]$. □

Finally we state a consequence of Theorem 1.2, which is interesting in its own right. It relates the degree of compactness of $\mathcal{R}_\alpha$ as operator into $L_q(\mu)$, with properties of the underlying measure μ. Note that such an estimate already appeared in the proof of (b) in Theorem 1.2.

Theorem 4.5 *Let μ be a finite continuous measure on $[0,1]$. Suppose $\alpha > 1/2$ and $1 \le q < \infty$. Then*

$$\sigma_\mu^{\alpha-1/2,q}(m) \approx m^{-\nu} \, (\log m)^\beta$$

for some $\nu \ge 0$ and $\beta \in \mathbb{R}$ implies

$$e_n\big(\mathcal{R}_\alpha : L_2[0,1] \to L_q(\mu)\big) \approx d_n\big(\mathcal{R}_\alpha : L_2[0,1] \to L_q(\mu)\big) \approx n^{-\nu-\alpha} \, (\log n)^\beta \; .$$

Proof. For the entropy numbers this is an immediate consequence of Theorem 1.2 and Theorem 5.1 in [**14**], while the assertion for the Kolmogorov numbers follows from [**25**]. □

Question: It would be interesting to know whether or not conversely the behavior of $e_n(\mathcal{R}_\alpha : L_2[0,1] \to L_q(\mu))$ describes that of $\sigma_\mu(m)$.

5. *Self–Similar Fractal Measures*

The aim of this section is to prove Theorem 1.3. The following statement allows us to find the asymptotic behavior of $M_\mu(\delta)$, when $\delta \to 0$, and $\sigma_\mu(m)$, when $m \to \infty$, for self–similar measures.

Lemma 5.1 *Let $N \ge 2$ be an integer, let $C > 0$, and let $(\alpha_k)_{1\le k\le N}$ be (strictly) positive weights such that*

$$\sum_{k=1}^{N} \alpha_k = 1.$$

If $F:(0,\infty)\to[0,\infty)$ is a non–decreasing function satisfying, for all $x>0$,

$$F(x)\geq\sum_{k=1}^{N}F(\alpha_k x)\geq F(x)-C, \tag{5.1}$$

then for any $x_0>0$

$$\inf_{x\geq x_0}\frac{F(x)}{x}\geq\alpha_{**}\cdot\frac{F(x_0)}{x_0}$$

and

$$\sup_{y\geq x_0}\frac{F(y)}{y}\leq C\ \alpha_{**}(1-\alpha_*)^{-1}x_0^{-1}+\frac{F(x_0/\alpha_{**})}{x_0}\ ,$$

where $\alpha_*:=\max_k\alpha_k<1$ and $\alpha_{**}:=\min_k\alpha_k>0$.

Proof.
a) The lower bound.
Since $\sum_{k=1}^N\alpha_k=1$, we have, by (5.1), for any $x\geq\alpha_{**}^{-1}x_0$,

$$\frac{F(x)}{x}\geq\sum_{k=1}^{N}\alpha_k\ \frac{F(\alpha_k x)}{\alpha_k x}\geq\min_{1\leq k\leq N}\ \frac{F(\alpha_k x)}{\alpha_k x}\geq\inf_{x_0\leq y\leq\alpha_* x}\ \frac{F(y)}{y}\ .$$

Hence, for any integer $n\geq0$,

$$\inf_{x_0\leq y\leq\alpha_{**}^{-1}\alpha_*^{-n-1}x_0}\ \frac{F(y)}{y}=\inf_{x_0\leq y\leq\alpha_{**}^{-1}\alpha_*^{-n}x_0}\ \frac{F(y)}{y}\ ,$$

and thus

$$\inf_{y\geq x_0}\frac{F(y)}{y}\ \inf_{x_0\leq y\leq\alpha_{**}^{-1}x_0}\ \frac{F(y)}{y}\geq\frac{\alpha_{**}F(x_0)}{x_0}\ ,$$

which yields the asserted lower bound.

b) The upper bound.
By (5.1), for any $x>0$,

$$\frac{F(x)}{x}\leq\frac{C}{x}+\sum_{k=1}^{N}\alpha_k\ \frac{F(\alpha_k x)}{\alpha_k x}\leq\frac{C}{x}+\max_{1\leq k\leq N}\ \frac{F(\alpha_k x)}{\alpha_k x}\ . \tag{5.2}$$

It follows from (5.2) that for any integer $n\geq0$,

$$\sup_{x_0\leq y\leq\alpha_{**}^{-1}\alpha_*^{-n-1}x_0}\ \frac{F(y)}{y}\leq C\alpha_{**}\alpha_*^n x_0^{-1}+\sup_{x_0\leq y\leq\alpha_{**}^{-1}\alpha_*^{-n}x_0}\ \frac{F(y)}{y}\ .$$

Hence,

$$\begin{aligned}\sup_{y\geq x_0}\frac{F(y)}{y}&\leq C\ \alpha_{**}(1-\alpha_*)^{-1}x_0^{-1}+\sup_{x_0\leq y\leq\alpha_{**}^{-1}x_0}\ \frac{F(y)}{y}\\&\leq C\ \alpha_{**}(1-\alpha_*)^{-1}x_0^{-1}+\frac{F(x_0/\alpha_{**})}{x_0}\,,\end{aligned}$$

and we are done. □

Corollary 5.2 *Let $N\geq2$ be an integer. Suppose that $M:(0,\infty)\to[0,\infty)$ is a* *non–increasing function satisfying for some* $\beta_k > 1$, $1 \le k \le N$, *and some* $C > 0$

$$M(\delta) \ge \sum_{k=1}^{N} M(\beta_k\,\delta) \ge M(\delta) - C$$

for all $\delta > 0$. *Then there exists a unique* $\gamma > 0$ *solving the equation*

$$\sum_{k=1}^{N} \beta_k^{-\gamma} = 1, \tag{5.3}$$

and we have for any $\delta_0 > 0$

$$0 < \inf_{0<\delta\le\delta_0} \frac{M(\delta)}{\delta^{-\gamma}} \le \sup_{0<\delta\le\delta_0} \frac{M(\delta)}{\delta^{-\gamma}} < \infty.$$

Proof. The function $\psi(\gamma) = \sum_k \beta_k^{-\gamma}$ is (strictly) decreasing, with $\psi(0) = N$ and with $\psi(\gamma) \to 0$, $\gamma \to \infty$. Hence, the solution of (5.3) exists and is unique. Setting $\alpha_k = \beta_k^{-\gamma}$ and $F(x) = M(x^{-1/\gamma})$, Corollary 5.2 follows from Lemma 5.1. □

We have now all ingredients for the proof of Theorem 1.3.

Proof of Theorem 1.3: Let

$$g(u) := \sum_{k=1}^{N} \lambda_k^{Hu}\,\rho_k^{u/q}\ , \quad u \ge 0\,,$$

then g is strictly decreasing. Moreover, $g(0) = N > 1$ and by Hölder's inequality

$$g(r) = \sum_{k=1}^{N} \lambda_k^{Hr}\,\rho_k^{r/q} \le \left(\sum_{k=1}^{N} \lambda_k\right)^{Hr} \left(\sum_{k=1}^{N} \rho_k\right)^{r/q} \le 1\ ,$$

hence the solution γ of equation (1.7) belongs to the interval $(0, r]$.

It follows from (1.6) that

$$M_\mu(\delta) \ge \sum_{k=1}^{N} M_{\mu_k}(\delta) \ge M_\mu(\delta) - (N-1)\ ,$$

where

$$\mu_k = \rho_k \cdot (\mu \circ S_k^{-1}). \tag{5.4}$$

By (5.4) we have $M_{\mu_k}(\delta) = M_\mu(\lambda_k^{-H}\rho_k^{-1/q}\delta)$. Hence,

$$M_\mu(\delta) \ge \sum_{k=1}^{N} M_\mu(\lambda_k^{-H}\rho_k^{-1/q}\delta) \ge M_\mu(\delta) - (N-1)$$

for all $\delta > 0$.
By applying Corollary 5.2 to $\beta_k = \lambda_k^{-H}\rho_k^{-1/q}$, we obtain

$$0 < \inf_{\delta\le 1} \frac{M_\mu(\delta)}{\delta^{-\gamma}} \le \sup_{\delta\le 1} \frac{M_\mu(\delta)}{\delta^{-\gamma}} < \infty\ ,$$

where γ solves the equation (1.7). Now (1.8) follows from Proposition 2.3, whereas (1.9) is a consequence of (1.8) and Theorem 1.2. □

Remark 5.3 *There exists an asymptotic representation of the function* $M_\mu(\cdot)$ *via* *equations from renewal theory (cf. [**24**]). This may also be used to derive* (1.8). *But since we do not need this deep result in its full strength, we preferred to give here an elementary and self–contained proof for the asymptotic behavior of* $M_\mu(\cdot)$.

6. *Proof of Theorem 1.5*

Throughout this section $A = (A(t))_{t\geq 0}$ denotes a strictly increasing subordinator with Laplace exponent Φ.

Proof of Theorem 1.5: We start with the proof of the right hand estimate in (1.13). To this end, for each fixed integer $m \geq 1$, set $s_m := \frac{2}{\log 2}\, m^{-1}$. Choose $\gamma > 0$ so small that

$$e^{-\gamma} \geq \frac{1}{2} + e^{-1} \tag{6.1}$$

and define $\delta_m > 0$ by

$$\delta_m := \frac{s_m^{1/q}}{\left(\Phi^{-1}(\gamma/s_m)\right)^H} \ .$$

In other words, we have

$$s_m\, \Phi\left(s_m^{1/(qH)}\, \delta_m^{-1/H}\right) = \gamma \ . \tag{6.2}$$

For a given $\delta > 0$ we now introduce the stopping time $T(\delta) \geq 0$ as

$$T(\delta) := \inf\left\{t \geq 0 : A(t)^H\, t^{1/q} \geq \delta\right\} \tag{6.3}$$

and notice that for all $\delta, s > 0$

$$1 - e^{-T(\delta)/s} \geq \exp(-s^{1/(qH)}\, \delta^{-1/H}\, A(s)) - e^{-1} \ . \tag{6.4}$$

Indeed, either $T(\delta) \geq s$, then the left–hand side of (6.4) is larger than $1 - e^{-1}$, thus (6.4) holds obviously, or $T(\delta) < s$ and then, by the definition of T, we obtain $A(s)^H\, s^{1/q} \geq \delta$. In this latter case the right hand side of (6.4) is negative, hence (6.4) is valid as well.
Now we take the expectation on both sides in (6.4), choosing $s = s_m$ and $\delta = \delta_m$ as defined above. Recalling the definition of Φ and using (6.2) this leads to

$$1 - \mathbb{E}\, e^{-T(\delta_m)/s_m} \geq e^{-s_m \cdot \Phi(s_m^{1/(qH)}\, \delta_m^{-1/H})} - e^{-1} = e^{-\gamma} - e^{-1} \ ,$$

hence by (6.1) we end up with

$$\mathbb{E}\, e^{-T(\delta_m)/s_m} \leq 1 - e^{-\gamma} + e^{-1} \leq \frac{1}{2} \ . \tag{6.5}$$

Next, we introduce a series of stopping times defined inductively as follows: Given $\delta > 0$ let $T_0 = T_0(\delta) := 0$ and, if $j \geq 1$, set

$$T_j = T_j(\delta) := \inf\left\{t > T_{j-1} : (A(t) - A(T_{j-1}))^H\, (t - T_{j-1})^{1/q} \geq \delta\right\} \ . \tag{6.6}$$

We note that by the strong Markov property the random times $T_j - T_{j-1}$, $1 \leq j < \infty$, are independent and distributed as T defined in (6.3). Thus by the exponential Chebyshev inequality and by (6.5), for each $m \geq 1$, (the numbers s_m and δ_m are as before) we have

$$\mathbb{P}\left(T_m(\delta_m) \leq 1\right) \leq e^{1/s_m} \cdot (\mathbb{E}\; e^{-T(\delta_m)/s_m})^m \leq e^{-(m/2)\,\log 2} = 2^{-m/2} \ .$$

By the Borel–Cantelli lemma, almost surely it is true that

$$T_m(\delta_m) \geq 1 \tag{6.7}$$

for all m greater than a certain (random) m_0.

Let the random measure μ be defined by (1.12) and suppose that (6.7) holds. Then consider the covering of the interval $[0, A(1)]$ by the $2m$ intervals $\Delta_j := [A(T_{j-1}), A(T_j-)]$, $\Delta'_j := [A(T_j-), A(T_j)]$, $1 \leq j \leq m$, where the stopping times are taken at level δ_m. Notice that by the construction $J_\mu(\Delta_j) \leq \delta_m$ and $J_\mu(\Delta'_j) = 0$. Hence, by the definition of $\sigma_\mu(m)$ and of δ_m we conclude

$$\sigma_\mu(2m) \leq m^{1/r}\, \delta_m m^{1/r-1/q} \left(\frac{2}{\log 2}\right)^{1/q} \cdot \left(\Phi^{-1}(\tilde{c}_2\, m)\right)^{-H} \preceq m^H \, \left(\Phi^{-1}(\tilde{c}_2\, m)\right)^{-H}$$

with $\tilde{c}_2 = \gamma \cdot \frac{\log 2}{2}$. This completes the proof of the right hand estimate in (1.13).

We now turn to the left hand estimate in (1.13). For an integer $m \geq 1$ we define this time the numbers s_m and δ_m by $s_m := \frac{1}{2 \log 7}\, m^{-1}$ and

$$\delta_m := \frac{s_m^{1/q}}{(\Phi^{-1}(2/s_m))^H} \ ,$$

i.e.,

$$s_m\, \Phi\left(s_m^{1/(qH)} \delta_m^{-1/H}\right) = 2 \ . \tag{6.8}$$

Let the stopping time T be given by (6.3). Then for any $s, \delta > 0$ and $T = T(\delta)$ we obtain by the exponential Chebyshev inequality that

$$\begin{aligned}
\mathbb{E}\, e^{T/s} &= \int_1^\infty \mathbb{P}\left(e^{T/s} \geq v\right) \mathrm{d}v + 1 = \int_0^\infty \mathbb{P}\left(T \geq su\right) e^u \,\mathrm{d}u + 1 \\
&= \int_0^\infty \mathbb{P}\left(A(su)^H \, (su)^{1/q} \leq \delta\right) e^u \,\mathrm{d}u + 1 \\
&= \int_0^\infty \mathbb{P}\left((su)^{1/(qH)}\, \delta^{-1/H}\, A(su) \leq 1\right) e^u \,\mathrm{d}u + 1 \\
&\leq e \int_0^\infty \mathbb{E}\, \exp\left(-(su)^{1/(qH)}\, \delta^{-1/H}\, A(su)\right) e^u \,\mathrm{d}u + 1 \\
&= e \int_0^\infty \exp\left(-su\, \Phi((su)^{1/(qH)} \delta^{-1/H}) + u\right) \mathrm{d}u + 1 \ .
\end{aligned}$$

Now we specify this to $s = s_m$ and $\delta = \delta_m$ defined above. If $u \geq 1$, then it follows from (6.8) that

$$s_m\, \Phi((s_m u)^{1/(qH)}\, \delta_m^{-1/H}) \geq s_m\, \Phi((s_m)^{1/(qH)}\, \delta_m^{-1/H}) = 2 \ ,$$

hence the preceding estimates lead to

$$\mathbb{E}\, e^{T(\delta_m)/s_m} \leq e \left(\int_0^1 e^u \,\mathrm{d}u + \int_1^\infty e^{-u} \,\mathrm{d}u\right) + 1 = e^2 - e + 2 \leq 7 \ . \tag{6.9}$$

Let the stopping times T_j be as in (6.6). Then by the exponential Chebyshev inequality, by the definition of s_m and by (6.9) we conclude that

$$\mathbb{P}\left(T_m(\delta_m) \geq 1\right) \leq e^{-1/s_m} \left(\mathbb{E}\, e^{T(\delta_m)/s_m}\right)^m \leq e^{-m \,\log 7} = 7^{-m} \ .$$

As before, this implies that $T_m(\delta_m) \leq 1$ almost surely for all m greater than a certain (random) m_0.

Define the random measure μ as before by (1.12) and take the stopping times T_j at level δ_m. If $T_m(\delta_m) \le 1$, then we have $\Delta_j := [A(T_{j-1}), A(T_j)] \subseteq [0, A(1)]$ and $J_\mu(\Delta_j) \ge \delta_m$ for all $1 \le j \le m$. Therefore, by the definition of $\delta_\mu(m)$ (cf. (2.2)) we get the estimate

$$\delta_\mu(m) \ge \delta_m = s_m^{1/q} \left(\Phi^{-1}(2/s_m)\right)^{-H} = (2 \log 7\, m)^{-1/q} \left(\Phi^{-1}(c_1\, m)\right)^{-H}$$

with $c_1 = 4 \cdot \log 7$.
Hence, by Proposition 2.3 we finally arrive at

$$\sigma_\mu(m) \succeq m^{1/r}\, \delta_\mu(m) \succeq m^H \, \left(\Phi^{-1}(c_1\, m)\right)^{-H},$$

as asserted. This completes the proof. □

7. *Time Inversion of RL–Processes*

For a given stochastic process $X = (X(t))_{0 \le t \le 1}$, the time inverted process X^- is defined by

$$X^-(t) := X(1-t)\,, \quad 0 \le t \le 1\,.$$

The aim of the present section is to compare the small deviation behavior of the RL–process R_H with that of R_H^-.

Let μ be a finite measure on $[0, 1]$. At the first glance, there is no obvious evidence for a similar behavior of $\mathbb{P}\left(\|R_H\|_{q,\mu} < \varepsilon\right)$ and $\mathbb{P}\left(\left\|R_H^-\right\|_{q,\mu} < \varepsilon\right)$ as $\varepsilon \to 0$. Note that the latter expression may also be written as $\mathbb{P}\left(\|R_H\|_{q,\mu^-} < \varepsilon\right)$; thus if the mass of μ is distributed in a very non–symmetric way around $1/2$, the two probabilities could be very different. However, the small deviation of a process is determined by its local increments rather than by its global properties. Since the processes R_H and R_H^- are *locally* very similar, the asymptotic behavior of their small deviations turns out to be similar. More precisely, we show that there exists a stationary Gaussian process $Y_H = (Y_H(t))_{0 \le t \le 1}$ which has similar small deviation behavior as R_H. Since any stationary processes is invariant with respect to time inversion, i.e., $Y_H \overset{d}{=} Y_H^-$, it is not a surprise that time inversion does not bring any essential change to the small deviation behavior of R_H either.

To make this precise, let us introduce Weyl operators of fractional integration. They are defined on the (complex) Hilbert space

$$L_2^0[0,1] := \left\{ f \in L_2[0,1] : \int_0^1 f(u)du = 0 \right\}$$

as follows. For $\alpha > 1/2$ the Weyl operator I_α from $L_2^0[0,1]$ into $L_\infty[0,1]$ is given on exponential functions by

$$I_\alpha(e^{2\pi i k \cdot})(x) : \frac{e^{2\pi i k x}}{(2\pi i k)^\alpha}\,, \quad k \in \mathbb{Z} \setminus \{0\}\,,$$

then on the whole $L_2^0[0,1]$ by linear and continuous extension. Note that I_α maps real valued functions into real ones. As in (4.3) for $\mathcal{R}_{H+1/2}$, for each $H > 0$ the operator $I_{H+1/2}$ generates a real Gaussian process Y_H by

$$Y_H(t) := \Gamma(H+1/2) \cdot \sum_{k=1}^{\infty} \xi_k\, (I_{H+1/2} f_k)(t)\,, \quad 0 \le t \le 1\,,$$

where ξ_k are i.i.d. standard normal random variables and $(f_k)_{k\geq 1}$ denotes an ONB in (the real) $L_2^0[0,1]$. We call Y_H the Weyl process of index $H > 0$. Elementary calculations (take the orthonormal basis generated by $x \mapsto \sqrt{2}\cos(2k\pi x)$ and $x \mapsto \sqrt{2}\sin(2k\pi x)$) give the covariance function

$$\mathbb{E}\, Y_H(t)\, Y_H(s) = 2 \cdot \Gamma(H+1/2)^2 \cdot \sum_{k=1}^{\infty} \frac{\cos\big(2\pi k(t-s)\big)}{(2\pi k)^{2H+1}}$$

for all $t, s \in [0,1]$. In particular, Y_H is stationary and it follows from this stationarity that

$$(Y_H(t))_{0\leq t\leq 1} \stackrel{d}{=} (Y_H^-(t))_{0\leq t\leq 1}\ .$$

As already mentioned, Y_H is tightly related to R_H in the following sense.

Proposition 7.1 *Given an RL–process R_H, $H > 0$, there exist a Weyl process Y_H and a centered Gaussian process Z_H on $[0,1]$ (not necessarily independent of Y_H) such that*

$$R_H(t) = Y_H(t) + Z_H(t)\ , \quad 0 \leq t \leq 1\,,$$

and, moreover, for any $\theta > 0$ it is true that

$$\lim_{\varepsilon\to 0} \varepsilon^{\theta} \cdot \log \mathbb{P}\left(\sup_{0\leq t\leq 1} |Z_H(t)| < \varepsilon \right) = 0\ . \tag{7.1}$$

Proof. This easily follows from the results proved in [**2**]. Namely, if $\alpha > 1/2$, the operator $S_\alpha := \mathcal{R}_\alpha - I_\alpha$ satisfies

$$e_n(S_\alpha : L_2[0,1] \to L_\infty[0,1]) \preceq 2^{-c_\alpha n^{1/3}}\ . \tag{7.2}$$

Consequently, defining Z_H by

$$Z_H(t) := \Gamma(H+1/2) \cdot \sum_{k=1}^{\infty} \xi_k (S_{H+1/2}\, f_k)(t)\ , \quad 0 \leq t \leq 1\,,$$

we have $R_H = Y_H + Z_H$ (when representing R_H and Y_H with the same ONB $(f_k)_{k\geq 1}$ and with the same ξ_k's). Finally, (7.1) follows from (7.2) by virtue of Theorem 5.1 in [**14**]. □

Before proceeding further, let us recall an important special case of the *partial correlation inequality* (cf. [**12**], Theorem 1.1).

Let X and Y be random elements of a Banach space $(E, \|\cdot\|)$ such that (X, Y) is a centered Gaussian vector. There exists a *universal* constant $K > 0$ such that for all $\varepsilon > 0$ we have

$$\mathbb{P}\left(\|X+Y\| < \varepsilon\right) \geq \mathbb{P}\left(\|X\| < \frac{\varepsilon}{\sqrt{2}}\right) \cdot \mathbb{P}\left(\|Y\| < \frac{\varepsilon}{K}\right)\ . \tag{7.3}$$

We may now state and prove the announced relation between the small deviation behaviors of R_H and R_H^-. We do it in a quite general form, not only for L_q–norms.

Theorem 7.2 *Let $(E, \|\cdot\|_E)$ be a Banach space of functions on $[0,1]$ such that for all $f \in E$ we have $\|f\|_E \leq c \cdot \|f\|_\infty$. Then there exists a non–increasing function*

$h : (0,1) \to [0,\infty)$ with $\lim_{\varepsilon\to 0} \varepsilon^\theta h(\varepsilon) = 0$ for any $\theta > 0$ such that for $\varepsilon > 0$

$$\log \mathbb{P}\left(\|R_H\|_E < \varepsilon\right) \geq \log \mathbb{P}\left(\left\|R_H^-\right\|_E < \frac{\varepsilon}{2}\right) - h(\varepsilon) \tag{7.4}$$

as well as

$$\log \mathbb{P}\left(\left\|R_H^-\right\|_E < \varepsilon\right) \geq \log \mathbb{P}\left(\|R_H\|_E < \frac{\varepsilon}{2}\right) - h(\varepsilon)\ . \tag{7.5}$$

Proof. Let us use the same notation as in Proposition 7.1. By (7.3), there is a constant $K > 0$ such that

$$\mathbb{P}\left(\|R_H\|_E < \varepsilon\right) \geq \mathbb{P}\left(\|Y_H\|_E < \frac{\varepsilon}{\sqrt{2}}\right) \cdot \mathbb{P}\left(\|Z_H\|_E < \frac{\varepsilon}{K}\right)\ . \tag{7.6}$$

Another application of (7.3) (this time to $Y_H^- = R_H^- - Z_H^-$) yields

$$\mathbb{P}\left(\left\|Y_H^-\right\|_E < \frac{\varepsilon}{\sqrt{2}}\right) \geq \mathbb{P}\left(\left\|R_H^-\right\|_E < \frac{\varepsilon}{2}\right) \cdot \mathbb{P}\left(\left\|Z_H^-\right\|_E < \frac{\varepsilon}{\sqrt{2}\,K}\right)\ . \tag{7.7}$$

Next, recall that the processes Y_H and Y_H^- possess the same distribution, thus (7.6) and (7.7) lead to

$$\mathbb{P}\left(\|R_H\|_E < \varepsilon\right) \geq \mathbb{P}\left(\left\|R_H^-\right\|_E < \frac{\varepsilon}{2}\right) \cdot \mathbb{P}\left(\left\|Z_H^-\right\|_E < \frac{\varepsilon}{\sqrt{2}\,K}\right) \cdot \mathbb{P}\left(\|Z_H\|_E < \frac{\varepsilon}{K}\right)\ . \tag{7.8}$$

Let

$$h(\varepsilon) := -\log \mathbb{P}\left(\left\|Z_H^-\right\|_E < \frac{\varepsilon}{\sqrt{2}\,K}\right) - \log \mathbb{P}\left(\|Z_H\|_E < \frac{\varepsilon}{K}\right)\ .$$

This function tends to infinity slower than any power of ε^{-1} (as $\varepsilon \to 0$). Indeed, Z_H^- satisfies (7.1) as well, and, moreover we assumed $\|f\|_E \leq c \cdot \|f\|_\infty$ for any $f \in E$. This observation combined with (7.8) completes the proof of (7.4). Property (7.5) is proved similarly. □

Remark 7.3 *It may be shown, either directly, or via results in* [**2**]*, that estimate* (7.4) *is also valid for the fractional Brownian motion B_H (defined on $[0,1]$). More precisely, for $0 < H < 1$ we have*

$$\log \mathbb{P}\left(\|B_H\|_E < \varepsilon\right) \geq \log \mathbb{P}\left(\left\|B_H^-\right\|_E < \varepsilon/2\right) - h(\varepsilon)$$

with h as in Theorem 7.2.

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