The first exit time of Brownian motion from a parabolic domain

by

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Summary. Consider a planar Brownian motion starting from an interior point of the parabolic domain $D = \{(x, y) : \ y > x^2\}$, and let τ_D denote the first time the Brownian motion exits from D. The tail behaviour [or equivalently, the integrability property] of τ_D is somewhat exotic since it arises from an interference of large deviation and small deviation events. Our main result implies that the limit of $T^{-1/3} \log \mathbb{P}\{\tau_D > T\}$ [as $T \to \infty$] exists and equals $-3\pi^2/8$, thus improving previous estimates by Bañuelos et al. (2001+) and Li (2001+). The existence of the limit is proved by applying the classical Schilder large deviation theorem. The identification of the limit leads to a variational problem, which is solved by exploiting a theorem of Biane and Yor (1987) relating different additive functionals of Bessel processes. Our result actually applies to more general parabolic domains in any [finite] dimensions.

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1 Introduction

Let $(\mathbf{B}(t),\ t \geq 0)$ be a Brownian motion taking values in $\mathbb{R}^{d+1}$, and let D be an unbounded Borel subset of $\mathbb{R}^{d+1}$. We assume that $\mathbf{B}$ starts from a point in the interior of D, and we are interested in

$$\tau_D := \inf \{t \geq 0 : \ \mathbf{B}(t) \notin D\},$$

the first exit time of the Brownian motion from D.

Of course, the distribution of τ_D strongly depends on the form of D. Apart from trivial situations, the example which has attracted the most research attention is when D is a [possibly generalized] cone. In this case, the exact distribution of τ_D is known, from which it can be deduced that

$$\mathbb{P}\{\tau_D > T\} \sim c\,T^{-\kappa}, \qquad T \to \infty,$$

where $c = c(D) > 0$ and $\kappa = \kappa(D) > 0$ are constants whose values can be explicitly formulated in terms of the eigenvalues and eigenfunctions of the Laplacian in D [see Bañuelos and Smits (1997) for a detailed account of the problem]. Throughout the paper, we adopt the usual notation $a(T) \sim b(T)$ $(T \to T_0)$ to denote $\lim_{T\to T_0} a(T)/b(T) = 1$.

The cone in dimension 2 can be thought of as the domain above the graph of a function of the form $y = a|x|$. As pointed out in Bañuelos et al. (2001+), it is highly non-trivial to find other unbounded domains above graphs of functions for which one can say something deep about the [tail] distribution of τ_D. They studied the natural example of a parabola in dimension two:

$$D := \left\{(x, y) \in \mathbb{R}^2 : \ y > x^2\right\}. \tag{1.1}$$

Their main result says that τ_D has a sub-exponential tail. More precisely, they proved the following

Theorem A [Bañuelos et al. (2001+)]. *Let $d = 1$ and let D be as in* (1.1). *There are two constants $A_1 > 0$ and $A_2 > 0$ such that*

$$-A_1 \le \liminf_{T\to\infty} T^{-1/3} \log \mathbb{P}\{\tau_D > T\} \le \limsup_{T\to\infty} T^{-1/3} \log \mathbb{P}\{\tau_D > T\} \le -A_2. \tag{1.2}$$

Therefore, the tail behaviour of τ_D in the case of a two-dimensional parabola differs very much from that in the case of a cone.

Recently, Li (2001+) has refined the result of Bañuelos et al. (2001+) by showing that (1.2) holds with

$$A_1 = (2^{-7/3}3^{4/3})\pi^{4/3}, \qquad A_2 = (2^{-7/3}3)\pi^{4/3}. \tag{1.3}$$

It is our aim to prove that $T^{-1/3} \log \mathbb{P}\{\tau_D > t\}$ has a limit and to determine its value. Our Theorem 1.1 below will imply that

$$\lim_{T\to\infty} T^{-1/3} \log \mathbb{P}\{\tau_D > T\} = -\frac{3\pi^2}{8}. \tag{1.4}$$

[It is easily checked that $(2^{-7/3}3)\pi^{4/3} < \frac{3\pi^2}{8} < (2^{-7/3}3^{4/3})\pi^{4/3}$. Thus (1.4) is in agreement with (1.3). These inequalities also indicate that neither of numerical estimates in Li (2001+) is optimal.]

Actually, one can consider generalized parabolic domains of arbitrary dimension. It was the idea of Li (2001+) to study the generalized parabolic shape in $\mathbb{R}^{d+1}$:

$$D = D_{d,p,a} := \left\{ (\mathbf{x}, y) := (x_1, \cdots, x_d, y) \in \mathbb{R}^{d+1} \; : \; y > a \, \|\mathbf{x}\|^p \right\} , \tag{1.5}$$

where $p > 1$, and $\|\mathbf{x}\| := [\sum_{i=1}^d x_i^2]^{1/2}$ is the Euclidean norm of $\mathbf{x} := (x_1, \cdots, x_d) \in \mathbb{R}^d$. Of course, if $a = d = 1$ and $p = 2$, then we are back to the case in (1.1). We mention that the presence of a is superfluous; it can be easily removed with appropriate changes by means of the scaling property of Brownian motion.

Theorem B [Li (2001+)]. *For $d \geq 1$, $a > 0$, $p > 1$ and $D = D_{d,p,a}$ as in* (1.5),

$$\begin{aligned} -A_3 &\leq \liminf_{T\to\infty} T^{-(p-1)/(p+1)} \log \mathbb{P}\{\tau_D > T\} \\ &\leq \limsup_{T\to\infty} T^{-(p-1)/(p+1)} \log \mathbb{P}\{\tau_D > T\} \leq -A_4 \, , \end{aligned}$$

where

$$\begin{aligned} A_3 &:= \frac{\left(a^2 (p+1)^{2p} p^{2-p} j_{(d-2)/2}^{2p}\right)^{1/(p+1)}}{2(p-1)} , \\ A_4 &:= \frac{p+1}{2} \left(\frac{a^2 j_{(d-2)/2}^{2p}}{(p-1)^p} \right)^{1/(p+1)} , \end{aligned}$$

and $j_{(d-2)/2}$ is the smallest positive zero of the Bessel function $J_{(d-2)/2}(\cdot)$.

Here is our main result.

Theorem 1.1. *Let $d \geq 1$, $a > 0$ and $p > 1$. Let $D = D_{d,p,a}$ be as in* (1.5)*. We have*

$$\lim_{T\to\infty} T^{-(p-1)/(p+1)} \log \mathbb{P}\{\tau_D > T\} = -(p+1) \left(\frac{\pi \, j_{(d-2)/2}^{2p} a^2}{2^{p+3} \, (p-1)^{p-1}} \, \frac{\Gamma^2(\frac{p-1}{2})}{\Gamma^2(\frac{p}{2})} \right)^{1/(p+1)} , \tag{1.6}$$

where $j_{(d-2)/2}$ is as before the smallest positive zero of the Bessel function $J_{(d-2)/2}(\cdot)$, and $\Gamma(\cdot)$ denotes the usual Gamma function.

Since $j_{-1/2} = \pi/2$, Theorem 1.1 immediately yields (1.4) by taking $a = d = 1$ and $p = 2$.

The rest of the paper is as follows. In Section 2, we give (an outline of) the proof of Theorem 1.1. Further technical details are given in Section 3. In Section 4 we give an extension of Theorem 1.1 by studying exit times from non-polynomial shapes.

Throughout the paper, the letter c with subscripts denotes some constants which are finite and positive.

2 Proof of Theorem 1.1

It is easily seen that the asymptotic behaviour of $\mathbb{P}\{\tau_D > T\}$ [for $T \to \infty$] does not depend on the starting point of the Brownian motion, as long as it is in the interior of D. Without loss of generality, we assume that the $a = 1$ and that our Brownian motion starts from $(0, \cdots, 0, 1) \in \mathbb{R}^{d+1}$. By definition, for any $T > 0$,

$$\{\tau_D > T\} = \{\mathbf{B}(t) \in D,\ \forall\, t \in [0, T]\}\,.$$

Therefore, if we write

$$\gamma_{d,p}(T) := \mathbb{P}\{\tau_D > T\}\,,$$

then

$$\gamma_{d,p}(T) = \mathbb{P}\left\{\|\mathbf{W}(t)\|^p < \widetilde{W}(t) + 1,\ \forall\, t \in [0, T]\right\}, \tag{2.1}$$

where $\mathbf{W} := (W_1, \cdots, W_d)$ is a d-dimensional Brownian motion starting from $\mathbf{0} \in \mathbb{R}^d$, and $\widetilde{W}$ is a one-dimensional Brownian motion starting from 0, such that $\mathbf{W}$ and $\widetilde{W}$ are independent.

A few simple lines of heuristics about the asymptotic order of $-\log \gamma_{d,p}(T)$ for large T. [These heuristics were already known to Li (2001+), where the correct rate $T^{(p-1)/(p+1)}$ was proved rigorously.] Indeed, the event on the right hand side of (2.1) is of very small probability: it is hard for the independent Brownian motions $\mathbf{W}$ and $\widetilde{W}$ to satisfy the condition $\|\mathbf{W}(t)\|^p < \widetilde{W}(t) + 1$ for all $t \in [0, T]$. A sufficiently economical way to meet such a condition is that both $\|\mathbf{W}(t)\|^p$ and $\widetilde{W}(t)$ should behave like $T^\alpha f(\frac{t}{T})$ for some $\alpha > 0$ and $f : [0, 1] \to \mathbb{R}_+$. An easy optimization of polynomial's degree yields $\alpha = \frac{p}{p+1}$ while the right choice of the profile function f boils down to a functional optimization problem. Summarizing the argument, one would expect that

$$\begin{aligned} &\gamma_{d,p}(T) \\ (2.2)\quad &\approx \mathbb{P}\left\{\|\mathbf{W}(t)\| \le T^{\frac{1}{p+1}} f^{1/p}(\frac{t}{T}), 1 \le t \le T\right\} \times \mathbb{P}\left\{\widetilde{W}(t) \ge T^{\frac{p}{p+1}} f(\frac{t}{T}), 1 \le t \le T\right\} \\ (2.3)\quad &\approx \exp\left(-c_1 \int_0^1 f^{-2/p}(s) ds\ T^{(p-1)/(p+1)}\right) \times \exp\left(-\frac{1}{2}\int_0^1 \dot{f}^2(s) ds\ T^{(p-1)/(p+1)}\right) \\ &= \exp\left(-c_2(f)\, T^{(p-1)/(p+1)}\right). \end{aligned}$$

The function f providing an optimal constant $c_2(f)$ appears via solution of an extremal problem

$$B_0 := \frac{1}{2} \inf_{f \in \mathbb{A}_0^{\uparrow}} \int_0^1 \dot{f}^2(t)\, \mathrm{d}t\ , \tag{2.4}$$

where $\mathbb{A}_0^{\uparrow}$ is the set of all non-decreasing absolutely continuous functions $f : [0,1] \to \mathbb{R}_+$ such that $f(0) = 0$ and $\int_0^1 f^{-2/p}(t)\,\mathrm{d}t \le 1$. Assuming that the infimum on the right hand side of (2.4) is attained at some f_*, we minimize $c_2(vf_*) = c_1 v^{-2/p} + B_0 v^2$ by taking $v = \left(\frac{c_1}{pB_0}\right)^{p/(p+1)}$. Hence, $f = vf_*$ gives the optimal value $c_2(f) = (p+1)\left(\frac{c_1^p B_0}{p^p}\right)^{1/(p+1)}$.
Since $p > 1$ by assumption, we have $1/(p+1) < 1/2$ and $p/(p+1) > 1/2$. Therefore, on the right hand side of (2.2), the first probability expression is a so-called "small ball probability" for $\mathbf{W}$ [i.e., probability that the Brownian motion stays in a narrow domain for a long time], whereas the second one is a "large deviation probability" for $\widetilde{W}$ [i.e., probability that the Brownian motion reaches a high level in a short time]. Estimation of $\gamma_{d,p}(T)$ requires thus a mixture of small ball and large deviation techniques.

The rest of the section is devoted to a rigorous proof of Theorem 1.1. To clarify the presentation, we admit a few technical results which will be proved in Section 3.

Proof of Theorem 1.1: upper bound. To get a rigorous upper bound for $\gamma_{d,p}(T)$, let $0 = t_0 < t_1 < t_2 < \cdots < t_N \le T$ and observe that

$$\gamma_{d,p}(T) \le \mathbb{P}\left\{ \sup_{t\in[t_{i-1},t_i]} \|\mathbf{W}(t)\| < \left(\sup_{t\in[0,t_i]} \widetilde{W}(t) + 1 \right)^{1/p}, \ \forall\, i \le N \right\}. \tag{2.5}$$

Let $0 < a_1 < \cdots < a_N$. By Anderson's inequality [see e.g. Lifshits (1995)], we have

$$\begin{aligned} &\mathbb{P}\left\{ \sup_{t\in[t_{i-1},t_i]} \|\mathbf{W}(t)\| < a_i, \ \forall\, i \le N \right\} \\ \le\ &\mathbb{P}\left\{ \sup_{t\in[t_{i-1},t_i]} \|\mathbf{W}(t)\| < a_i, \ \forall\, i \le N-1 \right\} \times \mathbb{P}\left\{ \sup_{t\in[0,t_N-t_{N-1}]} \|\mathbf{W}(t)\| < a_N \right\}, \end{aligned}$$

and by induction, this leads to:

$$\mathbb{P}\left\{ \sup_{t\in[t_{i-1},t_i]} \|\mathbf{W}(t)\| < a_i, \ \forall\, i \le N \right\} \le \prod_{i=1}^{N} \mathbb{P}\left\{ \sup_{t\in[0,t_i-t_{i-1}]} \|\mathbf{W}(t)\| < a_i \right\} .$$

At this stage, it is convenient to recall from Ciesielski and Taylor (1962) that

$$\mathbb{P}\left\{ \sup_{t\in[0,1]} \|\mathbf{W}(t)\| < x \right\} \sim c_3 \exp\left(-\frac{\kappa}{2x^2}\right), \qquad x \to 0+, \tag{2.6}$$

where $\kappa := j^2_{(d-2)/2}$ [$j_{(d-2)/2}$ denoting as before the smallest positive zero of the Bessel function $J_{(d-2)/2}$], and $c_3 = c_3(d)$ is a positive constant depending on d [whose value is explicitly known]. By scaling, for any $\varepsilon \in (0,1)$, there exists $c_4 = c_4(\varepsilon, d)$ such that for all $s, y > 0$,

$$\mathbb{P}\left\{ \sup_{t\in[0,s]} \|\mathbf{W}(t)\| < y \right\} \le c_4 \exp\left\{ -\frac{(1-\varepsilon)\kappa s}{2y^2} \right\}. \tag{2.7}$$

Accordingly,

$$\mathbb{P}\left\{ \sup_{t\in[t_{i-1},t_i]} \|\mathbf{W}(t)\| < a_i, \ \forall\, i \le N \right\} \le c_4^N \exp\left(-\frac{(1-\varepsilon)\kappa}{2} \sum_{i=1}^N \frac{t_i - t_{i-1}}{a_i^2} \right).$$

Going back to (2.5), and by conditioning upon the linear Brownian motion $\widetilde{W}$, we obtain:

$$\gamma_{d,p}(T) \le c_4^N \, \mathbb{E}\left\{ \exp\left(-\frac{(1-\varepsilon)\kappa}{2} \sum_{i=1}^N \frac{t_i - t_{i-1}}{[1 + \sup_{t\in[0,t_i]} \widetilde{W}(t)]^{2/p}} \right) \right\} .$$

For brevity, we write $\beta := 2/p$ and

$$S(t) := \sup_{u\in[0,t]} \widetilde{W}(u), \qquad t \ge 0.$$

Then we obtain

$$\begin{aligned} \gamma_{d,p}(T) \ &\le\ c_4^N \, \mathbb{E}\left\{ \exp\left(-\frac{(1-\varepsilon)\kappa}{2} \sum_{i=1}^N \frac{t_i - t_{i-1}}{[1+S(t_i)]^\beta} \right) \right\} \\ &=\ c_4^N \, \mathbb{E}\left\{ \exp\left(-\frac{(1-\varepsilon)\kappa}{2} \sum_{i=1}^N \frac{(\tau_i - \tau_{i-1}) T^{(p-1)/p}}{[T^{-1/2}+S(\tau_i)]^\beta} \right) \right\}, \end{aligned} \tag{2.8}$$

where $\tau_i := t_i/T$. We set now $\tau_i = (1-\varepsilon)^{N-i}$, $1 \le i \le N$. In view of the monotonicity of $t \mapsto S(t)$, we have

$$\begin{aligned} \sum_{i=1}^{N-1} \frac{\tau_i - \tau_{i-1}}{[T^{-1/2}+S(\tau_i)]^\beta} \ &\ge\ \sum_{i=1}^{N-1} \frac{(1-\varepsilon)(\tau_{i+1} - \tau_i)}{[T^{-1/2}+S(\tau_i)]^\beta} \\ &\ge\ (1-\varepsilon) \sum_{i=1}^N \int_{\tau_i}^{\tau_{i+1}} \frac{\mathrm{d}\tau}{[T^{-1/2}+S(\tau)]^\beta} \\ &=\ (1-\varepsilon) \int_{\tau_1}^1 \frac{\mathrm{d}\tau}{[T^{-1/2}+S(\tau)]^\beta} \ , \end{aligned}$$

so that

$$\gamma_{d,p}(T) \le c_4^N \, \mathbb{E}\left\{ \exp\left(-\frac{(1-\varepsilon)^2\kappa}{2} T^{(p-1)/p} \int_{\tau_1}^1 \frac{\mathrm{d}\tau}{[T^{-1/2}+S(\tau)]^\beta} \right) \right\}. \tag{2.9}$$

Observe that for any $b>0$,

$$\begin{aligned}
&\mathbb{E}\left\{ \exp\left(-b \int_{\tau_1}^1 \frac{\mathrm{d}\tau}{[T^{-1/2}+S(\tau)]^\beta} \right) \right\} \\
\le\ & \mathbb{E}\left\{ \exp\left(-b \int_{2\tau_1}^1 \frac{\mathrm{d}\tau}{[T^{-1/2}+S(\tau)]^\beta} \right) \mathbf{1}_{\{T^{-1/2} \le \varepsilon S(2\tau_1)\}} \right\} \\
&+\mathbb{E}\left\{ \exp\left(-b \int_{\tau_1}^{2\tau_1} \frac{\mathrm{d}\tau}{[T^{-1/2}+S(\tau)]^\beta} \right) \mathbf{1}_{\{T^{-1/2} > \varepsilon S(2\tau_1)\}} \right\} \\
\le\ & \mathbb{E}\left\{ \exp\left(-\frac{b}{(1+\varepsilon)^\beta} \int_{2\tau_1}^1 S(\tau)^{-\beta}\, \mathrm{d}\tau \right) \right\} + \exp\left(-\frac{b\tau_1 T^{1/p}}{(1+\varepsilon^{-1})^\beta} \right).
\end{aligned} \tag{2.10}$$

Taking $b=(1-\varepsilon)^2\kappa T^{(p-1)/p}/2$ and sending $T\to\infty$, we arrive at:

$$\begin{aligned}
&\limsup_{T\to\infty} T^{-(p-1)/(p+1)} \log \gamma_{d,p}(T) \\
\le\ & \inf_{\delta>0,\,\theta>0} \limsup_{T\to\infty} T^{-(p-1)/(p+1)} \log \mathbb{E}\left\{ \exp\left(-\frac{(1-\delta)\kappa T^{(p-1)/p}}{2} \int_\theta^1 S(\tau)^{-\beta}\, \mathrm{d}\tau \right) \right\}.
\end{aligned} \tag{2.11}$$

The question is now how to evaluate

$$\mathbb{E}\left\{ \exp\left(-\lambda \int_\theta^1 S^{-\beta}(t)\, \mathrm{d}t \right) \right\},$$

when $\lambda\to\infty$ and $\beta\in(0,2)$ is a fixed constant.

To obtain an upper bound for such an expression, we note that by Schilder's theorem [for justification, see (3.3) in Subsection 3.1],

$$\limsup_{x\to0+} x^{2/\beta} \log \mathbb{P}\left\{ \int_\theta^1 S^{-\beta}(t)\, \mathrm{d}t \le x \right\} \le -B_\theta, \tag{2.12}$$

where

$$B_\theta := \frac{1}{2} \inf_{f\in\mathbb{A}_\theta^\uparrow} \int_0^1 \dot{f}^2(t)\, \mathrm{d}t.$$

Here and in the sequel, $\dot{f}$ denotes the Radon–Nikodym derivative of f, and $\mathbb{A}_\theta^\uparrow$ is the set of all non-decreasing functions in the set $\mathbb{A}_\theta$ defined by

$$\mathbb{A}_\theta := \left\{ f:\, [0,1]\to\mathbb{R}_+,\ f(0)=0,\ f \text{ absolutely continuous},\ \int_\theta^1 f(t)^{-\beta}\, \mathrm{d}t \le 1 \right\}.$$

Clearly, $\inf_{f\in\mathbb{A}_\theta}\int_\theta^1 \dot f^2(t)\,\mathrm{d}t = \inf_{f\in\mathbb{A}_\theta^\uparrow}\int_\theta^1 \dot f^2(t)\,\mathrm{d}t$. [Indeed, for any $f\in\mathbb{A}_\theta$, we can take $g(t) := \sup_{s\in[0,t]} f(s)$, $t\in[0,1]$, to see that $g\in\mathbb{A}_\theta^\uparrow$ and $\int_\theta^1 \dot g^2(t)\,\mathrm{d}t \le \int_\theta^1 \dot f^2(t)\,\mathrm{d}t$.]

We now need the following elementary result.

Lemma 2.1. *Let $X\ge 0$ be a random variable. Let $\alpha>0$ and $B>0$. If*

$$\limsup_{x\to 0+} x^\alpha \log \mathbb{P}\{X\le x\} \le -B, \tag{2.13}$$

then

$$\limsup_{\lambda\to\infty} \lambda^{-\alpha/(\alpha+1)} \log \mathbb{E}\left[\mathrm{e}^{-\lambda X}\right] \le -\,\frac{(\alpha+1)B^{1/(\alpha+1)}}{\alpha^{\alpha/(\alpha+1)}}. \tag{2.14}$$

The proof of Lemma 2.1 is fairly standard: it suffices to write $\mathbb{E}\left[\mathrm{e}^{-\lambda X}\right] = \int_0^1 \mathbb{P}\{\mathrm{e}^{-\lambda X} > x\}\,\mathrm{d}x$, and estimate the integral using Laplace's method. We feel free to omit the details.

By combining (2.12) and (2.14) [with $\alpha = 2/\beta$] we obtain

$$\limsup_{\lambda\to\infty} \lambda^{-2/(2+\beta)} \log \mathbb{E}\left\{\exp\left(-\lambda\int_\theta^1 S^{-\beta}(t)\,\mathrm{d}t\right)\right\} \le -\,\frac{2+\beta}{\beta}\,\frac{B_\theta^{\frac{\beta}{2+\beta}}}{(2/\beta)^{\frac{2}{2+\beta}}}. \tag{2.15}$$

Next, plugging (2.15) into (2.11) with $\beta = 2/p$, and sending $\theta\to 0$ we arrive at:

$$\limsup_{T\to\infty} T^{-\frac{p-1}{p+1}} \log \gamma_{d,p}(T) \le -\,(p+1)\left(\frac{\kappa}{2p}\right)^{\frac{p}{p+1}} B_0^{\frac{1}{p+1}}, \tag{2.16}$$

where

$$B_0 := \frac{1}{2}\inf_{f\in\mathbb{A}_0}\int_0^1 \dot f^2(t)\,\mathrm{d}t = \lim_{\theta\to 0} B_\theta. \tag{2.17}$$

A few words about the limit relation in (2.17). First, we note that for $\beta\in(0,2)$, the infimum B_0 is finite since the set $\mathbb{A}_0$ contains appropriate power functions. The family of sets $\mathbb{A}_\theta$ being non-decreasing with respect to the parameter θ, the limit $\lim_{\theta\to 0} B_\theta$ exists and

$$\lim_{\theta\to 0} B_\theta \le B_0 < \infty.$$

To see why the second identity in (2.17) holds, we take $\theta_n = 1/n$, and for arbitrary $\varepsilon>0$, we take a sequence of functions $f_n\in\mathbb{A}_{\theta_n}$ such that

$$\frac{1}{2}\int_0^1 \dot f_n^2(t)\,\mathrm{d}t \le B_{\theta_n} + \varepsilon \le \lim_{\theta\to 0} B_\theta + \varepsilon.$$

The Strassen ball being compact in the space of continuous functions, (f_n) contains a subsequence uniformly converging to a limit function, say f. By Fatou's lemma, we have, for any m,

$$\int_{\theta_m}^1 f^{-\beta}(t)\,\mathrm{d}t \le \liminf_{n\to\infty} \int_{\theta_m}^1 f_n^{-\beta}(t)\,\mathrm{d}t \le 1,$$

from which it follows that $f \in \mathbb{A}_0$. Moreover,

$$B_0 \le \frac{1}{2}\int_0^1 \dot{f}^2(t)\,\mathrm{d}t \le \lim_{\theta\to 0} B_\theta + \varepsilon.$$

Sending $\varepsilon \to 0$, we obtain $B_0 \le \lim_{\theta\to 0} B_\theta$, and (2.17) is completely justified.

Now we only need to identify B_0. This variational problem leads to an ordinary differential equation. However, we choose a different way, representing B_0 as a solution of another, and well investigated, variational problem.

Let R denote a two-dimensional Bessel process [i.e., the Euclidean modulus of an $\mathbb{R}^2$-valued Brownian motion] starting from 0.

Lemma 2.2. *Let $\beta \in (0,2)$. Then*

$$\lim_{x\to 0+} x^{2/\beta} \log \mathbb{P}\left\{ \int_0^1 R^{-\beta}(t)\,\mathrm{d}t \le x \right\} = -\, \frac{2^{2/\beta-3}\pi}{(2-\beta)^{2/\beta-1}\beta}\, \frac{\Gamma^2(\frac{2-\beta}{2\beta})}{\Gamma^2(\frac{1}{\beta})}. \tag{2.18}$$

On the other hand, we also have

$$\lim_{x\to 0+} x^{2/\beta} \log \mathbb{P}\left\{ \int_0^1 R^{-\beta}(t)\,\mathrm{d}t \le x \right\} = -\frac{1}{2} \inf_{f\in\mathbb{A}_0} \int_0^1 \dot{f}^2(t)\,\mathrm{d}t = -B_0, \tag{2.19}$$

where B_0 is defined in (2.17).

By admitting Lemma 2.2 for the moment [its proof is postponed until Subsection 3.3], we are ready to complete the proof of the upper bound in Theorem 1.1. Taking $\beta := 2/p$ in Lemma 2.2, we get

$$B_0 = \frac{\pi\, p^p\, \Gamma^2(\frac{p-1}{2})}{8(p-1)^{p-1}\,\Gamma^2(\frac{p}{2})}\ .$$

Plugging this into (2.16) yields that

$$\limsup_{T\to\infty} T^{-\frac{p-1}{p+1}} \log \gamma_{d,p}(T) \le -\,(p+1)\left(\frac{\pi\,\kappa^p\,\Gamma^2(\frac{p-1}{2})}{2^{p+3}\,(p-1)^{p-1}\,\Gamma^2(\frac{p}{2})} \right)^{1/(p+1)},$$

which is the desired upper bound in Theorem 1.1. □

Proof of Theorem 1.1: lower bound. Take a function $h \in \mathbb{A}_0^{\uparrow}$ solving the variational problem in (2.12). Namely, let $h \in \mathbb{A}_0^{\uparrow}$ be such that

$$\frac{1}{2}\int_0^1 \dot{h}^2(t)\,\mathrm{d}t = B_0 \qquad \text{and} \qquad \int_0^1 h^{-\beta}(t)\,\mathrm{d}t = 1.$$

For any $\delta > 0$, consider a piecewise linear approximation h_δ of h such that

$$\frac{1}{2}\int_0^1 \dot{h}_\delta^2(t)\,\mathrm{d}t \le (1+\delta)B_0 \qquad \text{and} \qquad \int_0^1 h_\delta^{-\beta}(t)\,\mathrm{d}t = 1. \tag{2.20}$$

We bound our probability $\gamma_{d,p}$ by a split trick similar to that suggested in heuristic (2.2):

$$\begin{aligned}
\gamma_{d,p}(T) &\ge \mathbb{P}\left\{ \|\mathbf{W}(t)\|^p \le \varrho T^{p/(p+1)} h_\delta(t/T) \le 1 + \widetilde{W}(t),\ \forall\, t \in [0,T] \right\} \\
&= \mathbb{P}\left\{ \|\mathbf{W}(t)\| \le \varrho^{1/p} T^{1/(p+1)} h_\delta^{1/p}(t/T),\ \forall\, t \in [0,T] \right\} \times \\
&\qquad \mathbb{P}\left\{ \varrho T^{p/(p+1)} h_\delta(t/T) \le 1 + \widetilde{W}(t),\ \forall\, t \in [0,T] \right\},
\end{aligned}$$

where the additional parameter $\varrho > 0$ will be the subject of forthcoming optimization. For the first probability on the right hand side, the small ball estimate is well known [see Li (1999) and related works Berthet and Shi (2000), Lifshits and Linde (2002)].

$$\lim_{T\to\infty} T^{-(p-1)/(p+1)} \log \mathbb{P}\left\{ \|\mathbf{W}(t)\| \le \varrho^{1/p} T^{1/(p+1)} h_\delta^{1/p}(t/T),\ \forall\, t \in [0,T] \right\} = -\,\frac{\kappa}{2\varrho^\beta}, \tag{2.21}$$

where $\kappa = j^2_{(d-2)/2}$ as in (2.7). For the second probability, we have, by scaling,

$$\begin{aligned}
&\mathbb{P}\left\{ \varrho T^{p/(p+1)} h_\delta(t/T) \le 1 + \widetilde{W}(t),\ \forall\, t \in [0,T] \right\} \\
&\quad = \mathbb{P}\left\{ \varrho T^{(p-1)/2(p+1)} h_\delta(t) \le T^{-1/2} + \widetilde{W}(t),\ \forall\, t \in [0,1] \right\}.
\end{aligned}$$

This is essentially a large deviation probability but, unfortunately, the presence of a non-zero starting point on the right hand side prohibits a direct application of classical large deviation results. Instead, we offer the following palliative.

Lemma 2.3. *Let W be a standard Brownian motion and let $a, b, u > 0$. Then for every piecewise linear function $f(\cdot)$ with $f(0) = 0$,*

$$\liminf_{r\to\infty} r^{-2} \log \mathbb{P}\left\{ r f(t) \le a r^{-b} + W(t),\ \forall\, t \in [0,u] \right\} \ge -\,\frac{1}{2}\int_0^u \dot{f}^2(t)\,\mathrm{d}t.$$

By admitting Lemma 2.3 for the moment [its proof is postponed until Subsection 3.2] we are ready to complete the proof of the lower bound in Theorem 1.1. Taking in Lemma 2.3 $f = h_\delta$, $r = \varrho T^{(p-1)/2(p+1)}$, we obtain

$$\liminf_{T\to\infty} T^{-(p-1)/(p+1)} \log \mathbb{P}\left\{ \varrho T^{(p-1)/2(p+1)} h_\delta(t) \le T^{-1/2} + \widetilde{W}(t),\ \forall\, t \in [0,1] \right\} \ge -\frac{\varrho^2}{2} \int_0^1 \dot{h}_\delta^2(t)\,\mathrm{d}t \ge -(1+\delta)\varrho^2 B_0. \tag{2.22}$$

The estimates (2.21) and (2.22) together yield

$$\liminf_{T\to\infty} T^{-(p-1)/(p+1)} \log \gamma_{d,p}(T) \ge -\frac{\kappa}{2\varrho^\beta} \ - \ (1+\delta)\varrho^2 B_0.$$

By sending $\delta \to 0$ and maximizing the term on the right hand side via the choice $\varrho := (\frac{\kappa}{2pB_0})^{p/2(p+1)}$, we obtain:

$$\liminf_{T\to\infty} T^{-(p-1)/(p+1)} \log \gamma_{d,p}(T) \ge -(p+1) \left(\frac{\kappa}{2p}\right)^{\frac{p}{p+1}} B_0^{\frac{1}{p+1}},$$

which yields on the right hand side the same constant as in the upper bound (2.16). This implies the lower bound in Theorem 1.1. □

3 Technical details

This section contains the technical details which were left incomplete in Section 2 in the proof of Theorem 1.1. They are presented here in three distinct subsections. Subsection 3.1 summarizes the large deviation theory which we need, and provides a justification of (2.12). Subsections 3.2 and 3.3 are devoted to the proofs of Lemmas 2.3 and 2.2, respectively.

3.1 Large deviations

For the sake of completeness, we recall here a small part of large deviation theory which is used in the main proof. For further details, see e.g. Dembo and Zeitouni (1998), Lifshits (1995).

Let P be a centered Gaussian measure in a separable Banach space E. Let $|\cdot|$ denote the reproducing kernel Hilbert norm associated with P. Then for every measurable $A \subset E$, we have

$$-\frac{1}{2} \inf_{h\in A^\circ} |h|^2 \le \liminf_{r\to\infty} \frac{\log P(rA)}{r^2} \le \limsup_{r\to\infty} \frac{\log P(rA)}{r^2} \le -\frac{1}{2} \inf_{h\in \overline{A}} |h|^2. \tag{3.1}$$

The set A is called regular if $\inf_{h\in A^\circ}|h| = \inf_{h\in\overline{A}}|h|$ [where A° and $\overline{A}$ denote, respectively, the interior and the closure of A]. For regular sets, (3.1) yields

$$\lim_{r\to\infty}\frac{\log P(rA)}{r^2} = -\frac{1}{2}\inf_{h\in A}|h|^2. \tag{3.2}$$

Let $\beta > 0$ and $G : E \to \mathbb{R}^1$ be a β-homogeneous functional, i.e.,

$$G(\lambda y) = \lambda^\beta G(y), \qquad \lambda > 0, \;\; y \in E.$$

Let $A = \{y \in E : G(y) \geq 1\}$. If G is upper-semicontinuous, or, equivalently, if A is closed, we have from (3.1)

$$\limsup_{r\to\infty}\frac{\log P(y\in E : G(y)\geq r)}{r^{2/\beta}} = \limsup_{r\to\infty}\frac{\log P(r^{1/\beta}A)}{r^{2/\beta}} \leq -\frac{1}{2}\inf_{h\in A}|h|^2. \tag{3.3}$$

Moreover, if G is continuous, then, as we will see, A is regular, and we obtain from (3.2) that

$$\lim_{r\to\infty}\frac{\log P(y\in E :\; G(y)\geq r)}{r^{2/\beta}} = -\frac{1}{2}\inf_{h\in A}|h|^2. \tag{3.4}$$

It remains to check that A is regular. For any $\delta > 0$ and $h \in A$, we have $G((1+\delta)h) = (1+\delta)^\beta G(h) > 1$. Hence, by continuity of G, we have $(1+\delta)h \in A^\circ$ and

$$(1+\delta)|h| = |(1+\delta)h| \geq \inf_{\ell\in A^\circ}|\ell|.$$

Taking the minimum over all $h \in \overline{A}$, and making use of the fact $A = \overline{A}$, we get

$$\inf_{h\in\overline{A}}|h| = \inf_{h\in A}|h| \geq \inf_{\ell\in A^\circ}|\ell|.$$

Since the inverse inequality is trivial, the regularity of A is verified.

We need here only one particular case of all these inequalities — which is often referred to as the Schilder theorem [named after Schilder (1966)] — when P is the Wiener measure on $E = C([0,1],\mathbb{R}^d)$ and

$$|h|^2 = \begin{cases} \int_0^1 \dot{h}^2(t)\,\mathrm{d}t, & \text{if } h \text{ is absolutely continuous with } h(0)=0; \\ \infty, & \text{otherwise.} \end{cases}$$

3.2 Proof of Lemma 2.3

We prove Lemma 2.3 by induction over the number of linear pieces in the boundary function f. Consider first the linear boundary. Let $A > 0, H > 1$. Then

$$\begin{aligned}
&\mathbb{P}\left\{A + W(t) \geq Ht,\ \forall\, t \in [0,1] \text{ and } W(1) \geq H + \frac{1}{H}\right\} \\
\geq\ & \int_{H+\frac{1}{H}}^{H+\frac{2}{H}} \frac{\exp(-y^2/2)}{\sqrt{2\pi}}\, \mathbb{P}\left\{W(t) \geq Ht - A,\ \forall\, t \in [0,1] \mid W(1) = y\right\} \mathrm{d}y \\
\geq\ & \frac{\exp(-(H+\frac{2}{H})^2/2)}{\sqrt{2\pi}H}\, \mathbb{P}\left\{W(t) \geq Ht - A,\ \forall\, t \in [0,1] \mid W(1) = H\right\} \\
\geq\ & \frac{c_5 \exp(-H^2/2)}{H}\, \mathbb{P}\left\{\inf_{t\in[0,1]} W^0(t) \geq -A\right\} \\
\geq\ & c_5 H^{-1} \exp(-H^2/2) \min\{A^2;\ 1\},
\end{aligned}$$

where $W^0(t) := W(t) - tW(1)$, $t \in [0,1]$, is a standard Brownian bridge. Moreover, for induction argument, we need a scaled version of the proved inequality. Namely, for every $\Delta > 0$, we have

$$\begin{aligned}
&\mathbb{P}\left\{A + W(t) \geq Ht, \forall\, t \in [0,\Delta] \ \text{ and }\ W(\Delta) \geq \Delta H + \frac{1}{H}\right\} \\
&\quad \geq c_5\ (H\sqrt{\Delta})^{-1} \exp\left(-\frac{H^2\Delta}{2}\right) \min\left\{\frac{A^2}{\Delta};\ 1\right\},
\end{aligned}$$

which is valid under the assumptions $A > 0$ and $H\sqrt{\Delta} \geq 1$.

In particular, for $\Delta = u$, $A = ar^{-b}$ and $H = rK$, we obtain immediately

$$\liminf_{r\to\infty} r^{-2} \log \mathbb{P}\left\{rKt \leq ar^{-b} + W(t),\ \forall\, t \in [0,u]\right\} \geq -\frac{K^2 u}{2}. \tag{3.5}$$

Lemma 2.3 is now proved for linear boundaries $f(t) = Kt$.

Now we justify the induction. Let

$$f(t) = \begin{cases} Kt, & 0 \leq t \leq \Delta, \\ K\Delta + g(t-\Delta), & \Delta \leq t \leq u, \end{cases}$$

where g is a piecewise linear function having one less linear pieces than f. Since W has independent increments, we have,

$$\mathbb{P}\left\{A + W(t) \geq rf(t),\ \forall\, t \in [0,u]\right\} \geq p_1(r) \times p_2(r),$$

where

$$p_1(r) \;:=\; \mathbb{P}\left\{\frac{a}{r^b} + W(t) \geq rf(t),\, \forall t \in [0,\Delta] \text{ and } W(\Delta) \geq rf(\Delta) + \frac{1}{Kr}\right\};$$

$$p_2(r) \;:=\; \mathbb{P}\left\{\frac{1}{Kr} + W(t) - W(\Delta) \geq r(f(t) - f(\Delta),\ \forall\, t \in [\Delta, u]\right\}.$$

Since $f(t) = Kt$ on $[0,\Delta]$, it follows from (3.5) that

$$\liminf_{r\to\infty} r^{-2} \log p_1(r) \geq -\frac{K^2\Delta}{2}.$$

On the other hand, $p_2(r) = \{(Kr)^{-1} + W(s) \geq rg(s),\ \forall\, s \in [0, u-\Delta]\}$, so that by induction assumption,

$$\liminf_{r\to\infty} r^{-2} \log p_2(r) \geq -\frac{1}{2}\int_0^{u-\Delta} \dot{g}^2(s)\, \mathrm{d}s.$$

Assembling these pieces gives that

$$\begin{aligned}
&\liminf_{r\to\infty} r^{-2} \log \mathbb{P}\left\{\frac{a}{r^b} + W(t) \geq f(t),\ \forall\, t \in [0,u]\right\} \\
\geq\ & -\frac{K^2\Delta}{2} - \frac{1}{2}\int_0^{u-\Delta} \dot{g}^2(s)\, \mathrm{d}s \\
=\ & -\frac{1}{2}\int_0^u \dot{f}^2(t)\, \mathrm{d}t,
\end{aligned}$$

and we are done.

3.3 Proof of Lemma 2.2

Recall that R denotes a two-dimensional Bessel process starting from 0. Let $\beta \in (0,2)$ be a fixed constant, and write $\alpha := \frac{2\beta}{2-\beta}$.

Our starting point is the following theorem of Biane and Yor (1987), also stated as Corollary XI.1.12 in Revuz and Yor (1999).

Fact 3.1. *The random variables*

$$\int_0^1 R^{-\beta}(t)\, \mathrm{d}t \quad \text{and} \quad (\alpha/\beta)^\beta \left(\int_0^1 R^\alpha(t)\, \mathrm{d}t\right)^{-\beta/\alpha}$$

have the same distribution.

A great interest of this identity in our context is that instead of studying the *lower* tail behaviour of $\int_0^1 R^{-\beta}(t)\,\mathrm{d}t$, we only need to study the *upper* tail behaviour of $\int_0^1 R^{\alpha}(t)\,\mathrm{d}t$. For the latter problem, we use again the Schilder theorem [see (3.4)] which yields

$$\begin{aligned}
&\lim_{y\to\infty} y^{-2/\alpha}\log\mathbb{P}\left\{\int_0^1 R^{\alpha}(t)\,\mathrm{d}t > y\right\}\\
= \; & -\frac{1}{2}\,\inf\left\{\int_0^1 \dot{h}^2(t)\,\mathrm{d}t :\; h:[0,1]\to\mathbb{R}^2,\; h(0)=0,\right.\\
&\qquad\qquad \left.\int_0^1 |h(t)|^{\alpha}\,\mathrm{d}t \ge 1,\; h \text{ absolutely continuous}\right\}.
\end{aligned}$$

Next, it is easy to observe that the infimum is attained on the set of *real* positive increasing functions and hence it equals to $M^{-2/\alpha}$, where

$$M := \sup\left\{\int_0^1 |h(t)|^{\alpha}\,\mathrm{d}t :\; \int_0^1 \dot{h}^2(t)\,\mathrm{d}t \le 1,\; h\in\mathbb{B}\right\},$$

with $\mathbb{B} := \big\{h:[0,1]\to\mathbb{R}^1_+ \text{ absolutely continuous},\; h(0)=0\big\}$. We thus obtain

$$\lim_{y\to\infty} y^{-2/\alpha}\log\mathbb{P}\left\{\int_0^1 R^{\alpha}(t)\,\mathrm{d}t > y\right\} = -\frac{M^{-2/\alpha}}{2}. \tag{3.6}$$

Recall that the value of M is known: Strassen (1964) showed that $M = M_S$ where

$$M_S := \frac{2(2+\alpha)^{\alpha/2-1}}{\alpha^{\alpha/2}\left[\int_0^1 (1-t^{\alpha})^{-1/2}\,\mathrm{d}t\right]^{\alpha}} = \frac{2(2+\alpha)^{\alpha/2-1}\alpha^{\alpha/2}\,\Gamma^{\alpha}(\frac{1}{2}+\frac{1}{\alpha})}{\Gamma^{\alpha}(\frac{1}{\alpha})}. \tag{3.7}$$

Strassen proved this result for $\alpha \ge 1$ but it is easy to check that it is also valid for $\alpha\in(0,1)$. Indeed, in the latter case the functional

$$G_c(h) := \int_0^1 h^{\alpha}(t)\,\mathrm{d}t - c\int_0^1 \dot{h}^2(t)\,\mathrm{d}t \qquad (c>0)$$

is concave on $\mathbb{B}$. Strassen's calculation shows that for some $h_*\in\mathbb{B}$ one has

$$\int_0^1 h_*^{\alpha}(t)\,\mathrm{d}t = M_S, \qquad \int_0^1 \dot{h}^2(t)\,\mathrm{d}t = 1,$$

and for some $c>0$ the derivative of G_c vanishes at h_*. Hence, G_c attains its maximum at h_*, or, equivalently,

$$M_S = \sup\left\{\int_0^1 h^{\alpha}(t)\,\mathrm{d}t :\; \int_0^1 \dot{h}^2(t)\,\mathrm{d}t = 1,\; h\in\mathbb{B}\right\} = M.$$

Thus $M = M_S$ for every positive α.

Now, using sequentially Fact 3.1, (3.6) and (3.7), we obtain,

$$\begin{aligned}
&\lim_{x\to 0+} x^{2/\beta}\log \mathbb{P}\left\{\int_0^1 R^{-\beta}(t)\,\mathrm{d}t < x\right\} \\
=\ &\lim_{x\to 0+} x^{2/\beta}\log \mathbb{P}\left\{\int_0^1 R^{\alpha}(t)\,\mathrm{d}t > (\alpha/\beta)^\alpha x^{-\alpha/\beta}\right\} \\
=\ &-\frac{M^{-2/\alpha}\alpha^2}{2\beta^2} = -\frac{2}{M^{2/\alpha}(2-\beta)^2} = -\frac{2^{2/\beta-3}\pi}{(2-\beta)^{2/\beta-1}\beta}\ \frac{\Gamma^2(\frac{2-\beta}{2\beta})}{\Gamma^2(\frac{1}{\beta})}\ ,
\end{aligned}$$

as claimed in Lemma 2.2.

Now we prove the second part [identity (2.19)] of the lemma. The upper bound in (2.19) follows from the Schilder large deviation theorem (3.3), namely,

$$\limsup_{x\to 0+} x^{2/\beta}\log \mathbb{P}\left\{\int_0^1 R^{-\beta}(t)\,\mathrm{d}t \le x\right\} \le -\frac{1}{2}\inf_{f\in\mathbb{A}_0}\int_0^1 \dot f^2(t)\,\mathrm{d}t = -B_0.$$

The lower bound in (2.19) follows not from the general theory but from Lemma 2.3. Indeed, take small numbers $a > 0$, $\delta > 0$ and denote $\tau_a := \inf\{t : R(t) = a\}$ the first hitting time of the Bessel process. Then by the strong Markov property,

$$\begin{aligned}
&\mathbb{P}\left\{\int_0^1 R^{-\beta}(t)\,\mathrm{d}t \le (1+\delta)x\right\} \\
\ge\ &\mathbb{P}\left\{\int_0^{\tau_a} R^{-\beta}(t)\,\mathrm{d}t \le \delta x;\ \int_{\tau_a}^{\tau_a+1} R^{-\beta}(t)\,\mathrm{d}t \le x\right\} \\
=\ &\mathbb{P}\left\{\int_0^{\tau_a} R^{-\beta}(t)\,\mathrm{d}t \le \delta x\right\}\mathbb{P}\left\{\int_0^1 R^{-\beta}(t)\,\mathrm{d}t \le x \,\Big|\, R(0) = a\right\} \\
\ge\ &\mathbb{P}\left\{\int_0^{\tau_a} R^{-\beta}(t)\,\mathrm{d}t \le \delta x\right\}\mathbb{P}\left\{\int_0^1 |a + W(t)|^{-\beta}\,\mathrm{d}t \le x\right\}. \qquad (3.8)
\end{aligned}$$

By the scaling property of R, we have, for $x < \delta^{-1}a^{2-\beta}$,

$$\begin{aligned}
\mathbb{P}\left\{\int_0^{\tau_a} R^{-\beta}(t)\,\mathrm{d}t \le \delta x\right\} &= \mathbb{P}\left\{\int_0^{\tau_1} R^{-\beta}(t)\,\mathrm{d}t \le a^{\beta-2}\delta x\right\} \\
&\ge \mathbb{P}\left\{\int_0^1 R^{-\beta}(t)\,\mathrm{d}t \le a^{\beta-2}\delta x,\ \tau_1 \le 1\right\} \\
&= \mathbb{P}\left\{\int_0^1 R^{-\beta}(t)\,\mathrm{d}t \le a^{\beta-2}\delta x\right\},
\end{aligned}$$

the last equality following from the fact that $\{\int_0^1 R^{-\beta}(t)\,\mathrm{d}t \le a^{\beta-2}\delta x\} \subset \{\tau_1 \le 1\}$ for all $x < \delta^{-1}a^{2-\beta}$. Hence, by (2.18),

$$\liminf_{x\to 0+} x^{2/\beta}\log\mathbb{P}\left\{\int_0^{\tau_a} R^{-\beta}(t)\,\mathrm{d}t \le \delta x\right\} \ge -\frac{2^{2/\beta-3}\pi}{(2-\beta)^{2/\beta-1}\beta}\,\frac{\Gamma^2(\frac{2-\beta}{2\beta})}{\Gamma^2(\frac{1}{\beta})}\,\delta^{-2/\beta}a^{2(2-\beta)/\beta}. \tag{3.9}$$

We mention that it is possible, by means of stochastic calculus techniques, to show that $\int_0^{\tau_a} R^{-\beta}(t)\,\mathrm{d}t$ is distributed as $\{2/(2-\beta)\}^2 a^{2-\beta}/\sup_{0\le t\le 1} R^2(t)$, so that the "lim inf" expression on the left hand side of (3.9) is a true limit and its value actually is 0.

To estimate $\mathbb{P}\{\int_0^1 |a+W(t)|^{-\beta}\,\mathrm{d}t \le x\}$, we take a function h_δ from (2.20) and notice that for $\omega \in \{a+W(t) \ge x^{-1/\beta}h_\delta(t),\ \forall\, t\in[0,1]\}$, we have

$$\int_0^1 |a+W(t)|^{-\beta}\,\mathrm{d}t \le x\int_0^1 h_\delta^{-\beta}(t)\,\mathrm{d}t = x.$$

Hence, by Lemma 2.3,

$$\begin{aligned}
&\liminf_{x\to 0+} x^{2/\beta}\log\mathbb{P}\left\{\int_0^1 |a+W(t)|^{-\beta}\,\mathrm{d}t \le x\right\}\\
\ge\ &\liminf_{x\to 0+} x^{2/\beta}\log\mathbb{P}\left\{a+W(t)\ge x^{-1/\beta}h_\delta(t),\ \forall\, t\in[0,1]\right\}\\
\ge\ &-\frac{1}{2}\int_0^1 \dot h_\delta^2(t)\,\mathrm{d}t\\
\ge\ &-(1+\delta)B_0.
\end{aligned} \tag{3.10}$$

Combining (3.8)–(3.10) and choosing δ and a sufficiently small, we obtain:

$$\liminf_{x\to 0+} x^{2/\beta}\log\mathbb{P}\left\{\int_0^1 R^{-\beta}(t)\,\mathrm{d}t \le x\right\} \ge -B_0.$$

This yields the second part of Lemma 2.2.

4 Slow exit from more general domains

4.1 Domains with regular varying boundary

Let $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing continuous function starting from zero and p-regularly varying at infinity [with $p > 1$]: for any $a > 0$, $\Lambda(ar)/\Lambda(r) \to a^p$, $r\to\infty$. Let $\Lambda^{\leftarrow}$ denote the inverse function of Λ. Then $\Lambda^{\leftarrow}$ is $(1/p)$-regularly varying, so that

$$\nu(r) := r^{-1/p}\Lambda^{\leftarrow}(r)$$

is a slowly varying function. Let

$$D = D_{d,\Lambda} := \left\{(\mathbf{x}, y) := (x_1, \cdots, x_d, y) \in \mathbb{R}^{d+1} : \; y > \Lambda(\|\mathbf{x}\|)\right\}, \tag{4.1}$$

where $\|\mathbf{x}\|$ is the Euclidean norm of $\mathbf{x} := (x_1, \cdots, x_d) \in \mathbb{R}^d$. Rather sharp estimates for exit times from such domains may be found in Li (2001+).

We provide the following generalization of Theorem 1.1.

Theorem 4.1. *Let $d \geq 1$, Λ and ν be as above. Let $D = D_{d,\Lambda}$ be as in* (4.1)*. Assume that*

$$\lim_{T\to\infty} \frac{\nu\left(\nu^{-p/(p+1)}(T)\, T\right)}{\nu\,(T)} = 1. \tag{4.2}$$

Then

$$\lim_{T\to\infty} \frac{\log \mathbb{P}\{\tau_D > T\}}{f_p(T)} = -(p+1)\left(\frac{\pi\, j^{2p}_{(d-2)/2}}{2^{p+3}\,(p-1)^{p-1}}\; \frac{\Gamma^2(\frac{p-1}{2})}{\Gamma^2(\frac{p}{2})}\right)^{1/(p+1)},$$

where

$$f_p(T) := \frac{T^{(p-1)/(p+1)}}{\nu^{2p/(p+1)}\left(T^{p/(p+1)}\right)}. \tag{4.3}$$

Remark. The extra assumption (4.2) is verified e.g. by functions $\nu(T) = c\,(\log T)^{\alpha}$ [with $\alpha \in \mathbb{R}$] and by functions $\nu(T) = c\,\exp\{b(\log T)^{\alpha}\}$ [with $\alpha \in [0, 1/2)$].

Proof of Theorem 4.1. We only sketch the proof, starting with that of the lower bound. The splitting argument now reads:

$$\begin{aligned}
\mathbb{P}\{\tau_D > T\} &\geq \mathbb{P}\left\{\Lambda(\|W(t)\|) \leq \varrho T^{p/(p+1)} h_\delta(t/T) \leq 1 + \widetilde{W}(t),\ \forall\, t \in [0,T]\right\} \\
&= \mathbb{P}\left\{\Lambda(\|W(t)\|) \leq \varrho T^{p/(p+1)} h_\delta(t/T),\ \forall\, t \in [0,T]\right\} \times \\
&\qquad \mathbb{P}\left\{\varrho T^{p/(p+1)} h_\delta(t/T) \leq 1 + \widetilde{W}(t),\ \forall\, t \in [0,T]\right\},
\end{aligned}$$

and for the first factor we have

$$\begin{aligned}
&\log \mathbb{P}\left\{\Lambda(\|W(t)\|) \leq \varrho T^{p/(p+1)} h_\delta(t/T), \forall t \in [0,T]\right\} \\
=\ &\log \mathbb{P}\left\{\|W(t)\| \leq \Lambda^{\leftarrow}\left(\varrho T^{p/(p+1)} h_\delta(t/T)\right),\ \forall t \in [0,T]\right\} \\
=\ &\log \mathbb{P}\left\{\|W(t)\| \leq T^{-1/2}\Lambda^{\leftarrow}\left(\varrho T^{p/(p+1)} h_\delta(t)\right),\ \forall t \in [0,1]\right\} \\
\sim\ &-\frac{\kappa T}{2}\int_0^1 (\Lambda^{\leftarrow})^{-2}\left(\varrho T^{p/(p+1)} h_\delta(t)\right)\, \mathrm{d}t \\
=\ &-\frac{\kappa}{2\varrho^{2/p}}\, T^{(p-1)/(p+1)} \int_0^1 h_\delta^{-2/p}(t)\nu^{-2}\left(\varrho T^{p/(p+1)} h_\delta(t)\right)\, \mathrm{d}t \\
\sim\ &-\frac{\kappa}{2\varrho^{2/p}} T^{(p-1)/(p+1)}\nu^{-2}\left(\varrho T^{p/(p+1)}\right).
\end{aligned}$$

The rest of the proof strictly follows that of the lower bound in Theorem 1.1 after (2.21) with the replacement of κ by $\kappa\nu^{-2}\left(T^{p/(p+1)}\right)$. Note that the optimal choice of ϱ is $\varrho = \varrho(T) := (\frac{\kappa}{2pB_0})^{p/2(p+1)}\nu^{-p/(p+1)}(T^{p/(p+1)})$, and (4.2) provides the equivalence $\nu\left(\varrho T^{p/(p+1)}\right) \sim \nu\left(T^{p/(p+1)}\right)$ which considerably simplifies the calculation.

Now we turn to [the sketch of] the proof of the upper bound. We keep the notation of Section 2. In particular, we use a partition $\{t_0, \ldots, t_N\}$, the supremum $S(t)$, and the constant κ. Take a small constant $m > 0$ and a large constant $M > 0$. Introduce three events

$$\begin{aligned}
Q_- = Q_-(T) &:= \left\{S(t_1) \le \frac{m\ T^{p/(p+1)}}{\nu^{p/(p+1)}\left(T^{p/(p+1)}\right)}\right\}, \\
Q = Q(T) &:= \left\{\frac{m\ T^{p/(p+1)}}{\nu^{p/(p+1)}\left(T^{p/(p+1)}\right)} < S(t_1) \le S(T) < \frac{M\ T^{p/(p+1)}}{\nu^{p/(p+1)}\left(T^{p/(p+1)}\right)}\right\}, \\
Q_+ = Q_+(T) &:= \left\{S(T) \ge \frac{M\ T^{p/(p+1)}}{\nu^{p/(p+1)}\left(T^{p/(p+1)}\right)}\right\}.
\end{aligned}$$

There is no problem to bound the parts related to Q_- and Q_+. Indeed, by the classical large deviation estimate for $S(T)$ we have

$$\limsup_{T\to\infty} \frac{\log \mathbb{P}\{Q_+\}}{f_p(T)} \le -\frac{M^2}{2},$$

and we can choose M as large as possible. On the other hand, taking $t_1 = \tau_1 T$ for any fixed $\tau_1 \in (0,1)$, we have

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{f_p(T)} \log \mathbb{P}\{\tau_D \ge T;\ Q_-\} \\
\le\ &\limsup_{T\to\infty} \frac{1}{f_p(T)} \log \mathbb{P}\left\{\|W(t)\| \le \Lambda^{\leftarrow}\left(1 + \frac{m\ T^{p/(p+1)}}{\nu^{p/(p+1)}\left(T^{p/(p+1)}\right)}\right),\ \forall t \in [0, t_1]\right\} \\
=\ &-\frac{\kappa\tau_1}{2m^{2/p}},
\end{aligned}$$

the last inequality being a consequence of (2.6) and (4.2). So this part does not present any trouble either, as long as we choose $m > 0$ sufficiently small.

Finally, the estimate related to the main domain Q follows the scheme of Section 2. Namely, let

$$\gamma_Q(T) := \mathbb{P}\{\tau_D \ge T;\ Q\}.$$

Similarly to (2.5), we obtain

$$\gamma_Q(T) \le \mathbb{P}\left\{\sup_{t\in[t_{i-1},t_i]} \|\mathbf{W}(t)\| < \Lambda^{\leftarrow}\left(\sup_{t\in[0,t_i]} \widetilde{W}(t) + 1\right),\ \forall\, i \le N;\ Q\right\}.$$

Proceeding as in Section 2, we obtain the counterpart of (2.8), namely,

$$\gamma_Q(T) \le c_4^N\, \mathbb{E}\left\{ \mathbf{1}_Q \;\exp\left(-\frac{(1-\varepsilon)\kappa}{2} \sum_{i=1}^N \frac{t_i - t_{i-1}}{[1+S(t_i)]^{2/p} \nu^2 (1+S(t_i))} \right) \right\}.$$

Note that uniformly for all $i \le N$, we have, on the event Q,

$$\nu(1+S(t_i)) \sim \nu\left(\frac{T^{p/(p+1)}}{\nu^{p/(p+1)}\left(T^{p/(p+1)}\right)} \right) \sim \nu\left(T^{p/(p+1)}\right);$$

indeed, the first equivalence follows from the definition of Q and the Karamata representation for slowly varying functions, whereas the second is a consequence of condition (4.2). Accordingly, for all large T,

$$\gamma_Q(T) \le c_4^N\, \mathbb{E}\left\{ \exp\left(-\frac{(1-\varepsilon)^2\kappa}{2\nu^2\left(T^{p/(p+1)}\right)} \sum_{i=1}^N \frac{t_i - t_{i-1}}{[1+S(t_i)]^{2/p}} \right) \right\}.$$

Now we literally follow Section 2 with the sole replacement of κ by $\kappa\nu^{-2}\left(T^{p/(p+1)}\right)$. At the place of (2.9), we have

$$\gamma_Q(T) \le c_4^N\, \mathbb{E}\left\{ \exp\left(-\frac{(1-\varepsilon)^3\kappa\, T^{(p-1)/p}}{2\nu^2\left(T^{p/(p+1)}\right)} \int_{\tau_1}^1 \frac{\mathrm{d}\tau}{[T^{-1/2}+S(\tau)]^{2/p}} \right) \right\},$$

and taking $b = (1-\varepsilon)^3\kappa T^{(p-1)/p}/2\nu^2\left(T^{p/(p+1)}\right)$ in (2.10), we arrive at the following counterpart of (2.11):

$$\begin{aligned} &\limsup_{T\to\infty} \frac{\log \gamma_Q(T)}{f_p(T)} \\ \le\; &\inf_{\delta>0,\,\theta>0}\, \limsup_{T\to\infty} \frac{1}{f_p(T)} \log \mathbb{E}\left\{ \exp\left(-\frac{(1-\delta)\kappa T^{(p-1)/p}}{2\nu^2\left(T^{p/(p+1)}\right)} \int_\theta^1 S(\tau)^{-2/p}\,\mathrm{d}\tau \right) \right\}. \end{aligned}$$

We already know from (2.16), via the key estimate (2.15), that the expression on the right hand side equals $-(p+1)(\kappa/2p)^{\frac{p}{p+1}} B_0^{\frac{1}{p+1}}$, which yields the desired upper bound. □

4.2 Non-Euclidean norms

According to Theorem 7.2 from Port and Stone (1979) the following is valid. Let K be a non-empty connected open set in $\mathbb{R}^d$ that contains 0. Then

$$P\{\mathbf{W}(t) \in K,\ \forall t \in [0,T]\} \sim c\, \exp(-\lambda T), \qquad T \to \infty,$$

with some positive constant c and λ being the principal eigenvalue of laplacian $(-\frac{1}{2}\Delta)$ on K (with zero boundary condition).
We can transform this statement in

$$\mathbb{P}\left\{\mathbf{W}(s) \in xK, \forall s \in [0,1]\right\} \sim c \exp\left(-\frac{\lambda}{x^2}\right), \qquad x \to 0+,$$

by ordinary scaling arguments. Using the latter relation instead of our formula (2.6), one can easily obtain the results of this article for the exit times from a parabolic shape

$$D_{d,p,H} := \left\{(\mathbf{x}, y) \in \mathbb{R}^{d+1} : \ y > H(x)^p\right\},$$

for any norm $H(\cdot)$ in $\mathbb{R}^d$ equivalent to the Euclidean one. Obviously, the doubled principal eigenvalue of laplacian for the H-unit ball will everywhere replace $j^2_{(d-2)/2}$.

4.3 Open problem: quasi-conic domains

The natural question is to find out what happens if $p = 1$ and ν is an appropriate power of logarithmic function (using notation from Subsection 4.1). The answer should be something intermediate between the sub-exponential behaviour of the exit probability treated in the present paper and the polynomial behaviour which appears for purely conic domains.

The only known results are those of Li (2001+) but his upper and lower estimates are still of different orders of magnitude. It does not seem that our methods are appropriate for complete analysis of this case either.

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