Lower functions of empirical processes and Brownian sheets

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We give a complete characterization of the lower functions for the two-parameter Brownian sheet and for the uniform empirical process via integral criteria. Our result for the Brownian sheet can be viewed as a solution, in a very special case (the general case remains an open question), of the following problem: find the escape rate of infinite-dimensional Brownian motion. Our result for the empirical process disproves (and provides the correct form of) a conjecture in the book of Shorack and Wellner [18].

Key words and phrases: rate of escape, empirical process, infinite-dimensional Brownian motion, Brownian sheet, lower function.

1. Introduction and main results

1.1. Escape rate for Brownian motions

We dedicate this article to the centenary of the great mathematician A.N. Kolmogorov. His interest in the tests of asymptotic properties of random processes is well known – just recall the famous Kolmogorov-Petrovski-Erdős-Feller test for upper functions of the Wiener

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process and random walks, see Itô and McKean [10]. Our work provides a solution, in terms of integral tests, for some old problems related to escape rates of the Brownian sheet and the empirical process.

Consider $B_d := \{B_d(t),\ t \geq 0\}$, a standard Brownian motion taking values in $\mathbb{R}^d$. It is well-known that B_d is almost surely transient (i.e., $\|B_d(t)\| \to \infty$, a.s., for $t \to \infty$, where $\|\cdot\|$ denotes the Euclidean norm) if and only if $d \geq 3$. When this condition is fulfilled, the rate of escape of B_d was determined by Dvoretzky and Erdős [7]. Throughout the paper, unless stated otherwise, we write "i.o." to denote "infinitely often" when the relevant parameter goes to infinity.

Theorem A (Dvoretzky and Erdős [7]). *Let $d \geq 3$. Then for any non-decreasing function $f > 0$,*

$$\mathbb{P}\left(\|B_d(t)\| \leq \frac{t^{1/2}}{f(t)} \text{ i.o.} \right) = \begin{cases} 0 \\ 1 \end{cases} \iff \int^{\infty} \frac{\mathrm{d}t}{t\, f^{d-2}(t)} \begin{cases} < \infty \\ = \infty \end{cases}.$$

What happens in the spaces of infinite dimension? Let $X := \{X(t),\ t \geq 0\}$ be a Brownian motion in a real separable infinite-dimensional Banach space $(E, \|\cdot\|)$ with $X(0) = 0$. That is, X has independent stationary increments, continuous sample paths, and zero mean (i.e., for any continuous linear functional f and for any $t \geq 0$, $\mathbb{E}[f(X(t))] = 0$).

The following infinite-dimensional theorem is due to Erickson [9].

Theorem B (Erickson [9]). *Let X be a genuinely infinite-dimensional Brownian motion on E with $X(0) = 0$. If f is a slowly varying at infinity positive continuous function on $\mathbb{R}_+$, then for any $\delta > 0$,*

$$\int^{\infty} \frac{\mathbb{P}\{\|X(1)\| \leq (1-\delta)/f(t)\}}{t\, f^2(t)}\, \mathrm{d}t = \infty \implies \mathbb{P}\left(\|X(t)\| \leq \frac{t^{1/2}}{f(t)} \text{ i.o.} \right) = 1,$$
$$\int^{\infty} \frac{\mathbb{P}\{\|X(1)\| \leq (1+\delta)/f(t)\}}{t\, f^2(t)}\, \mathrm{d}t < \infty \implies \mathbb{P}\left(\|X(t)\| \leq \frac{t^{1/2}}{f(t)} \text{ i.o.} \right) = 0.$$

This theorem is useful in determining the correct function f together with the correct constant in the rate of escape of X, as long as we have sufficiently good information about the behaviour of $\mathbb{P}\{\|X(1)\| \le \varepsilon\}$ when $\varepsilon \to 0+$. (For precision upon the existence of such functions, see Cox [5].) We mention that the original theorem proved by Erickson is more general, and can be applied to any continuous semi-norm on E of rank at least 3 with respect to X.

It is a challenging problem to find a necessary and sufficient criterion like in Theorem A. In other words, what happens in the critical case ($\delta = 0$)? There has been so far no result available in this direction.

We analyze in the present paper a particular, but very important, example of infinite-dimensional Brownian motion. Consider the Brownian sheet $\mathbb{W} := \{\mathbb{W}(s,t),\ s \in [0,1],\ t \ge 0\}$. That is, $\mathbb{W}$ is a mean-zero Gaussian field with covariance

$$\mathbb{E}[\,\mathbb{W}(s,t)\mathbb{W}(s',t')\,] = \min(s,s')\ \min(t,t').$$

It is a common approach to treat multi-parameter processes as infinite-dimensional processes (see e.g. Khoshnevisan [11]). For each t, one can think of

$$X(t) := (s \mapsto \mathbb{W}(s,t),\ s \in [0,1]), \tag{1.1}$$

as an element of $C([0,1],\mathbb{R})$, the space of all continuous functions on $[0,1]$ endowed with the uniform norm

$$\|f\| := \sup_{s\in[0,1]} |f(s)|. \tag{1.2}$$

The importance of this particular Brownian motion is due to its close relationship (via the Kiefer process and the KMT approximation) with empirical processes. We will give more details on this topic later in Section 4.

Since $X(1)$ is a standard Brownian motion defined on $[0,1]$, a classical result of Chung [4] tells us that

$$\mathbb{P}\,(\,\|X(1)\| \le \varepsilon) \sim \frac{4}{\pi}\exp\left(-\frac{\pi^2}{8\varepsilon^2}\right), \qquad \varepsilon \to 0, \tag{1.3}$$

where $a(x) \sim b(x)$, $x \to x_0$, stands for $\lim_{x\to x_0} a(x)/b(x) = 1$. Applying Theorem B to $\mathbb{W}$ yields that for any slowly varying at infinity positive continuous function f, and for any

$\delta > 0$,

$$\int^{\infty} \frac{1}{t\, f^2(t)} \exp\Big(-\frac{\pi^2+\delta}{8} f^2(t)\Big)\, \mathrm{d}t = \infty \Longrightarrow \mathbb{P}\Big(\|X(t)\| \le \frac{t^{1/2}}{f(t)} \text{ i.o.}\Big) = 1,$$
$$\int^{\infty} \frac{1}{t\, f^2(t)} \exp\Big(-\frac{\pi^2-\delta}{8} f^2(t)\Big)\, \mathrm{d}t < \infty \Longrightarrow \mathbb{P}\Big(\|X(t)\| \le \frac{t^{1/2}}{f(t)} \text{ i.o.}\Big) = 0.$$

We establish the following integral test corresponding to the case $\delta = 0$.

Theorem 1.1. *Let X be as in* (1.1). *If $f > 0$ is non-decreasing, then*

$$\int^{\infty} \frac{f^6(t)}{t} \exp\Big(-\frac{\pi^2}{8} f^2(t)\Big)\, \mathrm{d}t \begin{cases} < \infty \\ = \infty \end{cases} \Longleftrightarrow \mathbb{P}\Big(\|X(t)\| \le \frac{t^{1/2}}{f(t)} \text{ i.o. for } t \to \infty \Big) = \begin{cases} 0 \\ 1 \end{cases},$$

where $\|\cdot\|$ is defined in (1.2).

The presence of the unusual power "6" in the test indicates that the Brownian sheet has exotic features of escape to infinity.

1.2. Escape rate for the empirical process

Let us mention the original motivation of this work. Our starting point was a conjecture of Shorack and Wellner [18] concerning the lower functions of empirical processes. Let $\{\alpha_n(t),\ t \in [0,1]\}$ be the uniform empirical process (for definition, see (4.1) in Section 4). We are interested in $\|\alpha_n\| := \sup_{s\in[0,1]} |\alpha_n(s)|$, the sup-norm of α_n. It is known (Mogulskii [17], Kuelbs [15]) that

$$\liminf_{n\to\infty} \sqrt{\log\log n}\, \|\alpha_n\| = \frac{\pi}{\sqrt{8}}, \qquad \text{a.s.} \tag{1.4}$$

What about the lower functions of $\|\alpha_n\|$? In the book of Shorack and Wellner [18] p. 530, it was conjectured that for any positive non-decreasing sequence (λ_n),

$$\mathbb{P}\Big(\|\alpha_n\| \le \frac{1}{\lambda_n}, \text{ i.o.}\Big) = \begin{cases} 0 \\ 1 \end{cases} \Longleftrightarrow \sum_n \frac{\lambda_n^3}{n} \exp\Big(-\frac{\pi^2}{8}\lambda_n^2\Big) \begin{cases} < \infty \\ = \infty \end{cases}. \tag{1.5}$$

As a by-product of the techniques developed for the proof of Theorem 1.1, we disprove the conjecture (1.5). In Section 4, we show that the correct answer is:

Theorem 1.2. *Let* (λ_n) *be non-decreasing, then*

$$\mathbb{P}\left(\|\alpha_n\| \le \frac{1}{\lambda_n}, \text{ i.o.}\right) = \begin{cases} 0 \\ 1 \end{cases} \iff \sum_n \frac{\lambda_n^7}{n} \exp\left(-\frac{\pi^2}{8}\lambda_n^2\right) \begin{cases} < \infty \\ = \infty \end{cases}.$$

Clearly, this theorem yields (1.4) as a special case.

1.3. Critical subsequences

A few words about some key technical points of the present work. In the terminology of Breiman [3], the integral test we are working on in Theorem 1.1 (or in Theorem 1.2) is called "delicate" in the sense that replacing f by a constant multiple of f may change the nature of the test, whereas the Dvoretzky–Erdős test stated in Theorem A is not that sensitive. It is commonly believed (see however Section 5 for some comments) that for a *delicate* integral test, one should examine the subsequence

$$t_i := \exp\left(\frac{i}{\log i}\right),$$

(or something similar such that $\log t_i$ is comparable to $i/\log i$ for large i), the so-called Erdős subsequence, referring to the pioneer work of Erdős [8] in his proof of the Kolmogorov test. Of course, one may sometimes use variants of the Erdős subsequence such as $\exp(\exp(i/\log i))$ if a process has a random clock which bears a logarithmic nature. For example, examination of Erdős subsequence led to the conjecture stated in (1.5).

We prove that the Erdős subsequence fails to work in our setting. Rather surprisingly, in the proof of Theorem 1.1, we have to use the somewhat unusual subsequence

$$t_i := \exp\left(\frac{i}{(\log i)^3}\right). \tag{1.6}$$

Although the thorough analysis of some aspects of planar Brownian motion shows that the subsequence defined in (1.6) is what we need, we do not have any good explanation of *why* a priori it should be so and why the Erdős subsequence fails to work in this case. The infinite-dimensional nature of our problem certainly plays a crucial role, and we are tempted to think that the Erdős subsequence would not work for any genuinely infinite-dimensional escape problem. A few more remarks about this can be find in Section 5.

The rest of the paper is organized as follows. We prove Theorem 1.1 in Section 2 by admitting a key probability estimate (Proposition 2.1) concerning the probability that a planar Brownian motion stays in a parallelogram. The proof of Proposition 2.1 is postponed to Section 3. Section 4 is devoted to the study of lower functions of empirical processes. In particular, we prove Theorem 1.2. Finally, some further remarks and questions are provided in Section 5.

Throughout the paper, the letter c with subscripts denotes unimportant constants which are finite and positive. We employ the usual notation $a_i \asymp b_i$ to denote that $0 < \liminf_i a_i/b_i \le \limsup_i a_i/b_i < \infty$.

2. Proof of Theorem 1.1

Throughout the section, $\mathbb{W} = \{\mathbb{W}(s,t),\ \ s \in \mathbb{R}_+,\, t \in \mathbb{R}_+\}$ is a Brownian sheet, and $f > 0$ is a non-decreasing function on $\mathbb{R}_+$. Let us recall the two parts of Theorem 1.1:

$$\text{(2.1)} \qquad \int^\infty \frac{f^6(t)}{t} \exp\left(-\frac{\pi^2}{8} f^2(t)\right) \mathrm{d}t < \infty \implies \mathbb{P}\left(\|\mathbb{W}(\bullet, t)\| \le \frac{t^{1/2}}{f(t)} \ \text{ i.o.}\right) = 0,$$

$$\text{(2.2)} \qquad \int^\infty \frac{f^6(t)}{t} \exp\left(-\frac{\pi^2}{8} f^2(t)\right) \mathrm{d}t = \infty \implies \mathbb{P}\left(\|\mathbb{W}(\bullet, t)\| \le \frac{t^{1/2}}{f(t)} \ \text{ i.o.}\right) = 1,$$

where $\|\cdot\|$ denotes as before the sup-norm.

The proof of (2.1), which is based on a Mogulskii-type minimal inequality, requires only standard techniques. The part (2.2), on the other hand, is the main contribution of the present paper. Dealing with infinite-dimensional Brownian motion, one can use some general Gaussian techniques such as Anderson's inequality, but even for particular case of the Brownian sheet, they do not yield (2.2). We will exploit some special properties of the Brownian sheet, and relate the latter to an $\mathbb{R}^2$-valued Brownian motion indexed by $\mathbb{R}_+$.

It is well known for the kind of integral tests in (2.1)–(2.2) that without loss of generality, one can consider only the "critical functions". Therefore, in what follows, we assume that

$$\text{(2.3)} \qquad \frac{\sqrt{\log\log t}}{2} \le f(t) \le \sqrt{\log\log t}\,, \qquad \text{for } t \ge t_0.$$

For rigorous justification, see e.g. Erdős [8], or Csáki [6].

We introduce the sequence $(t_i)_{i\geq 2}$ defined by

$$t_i = \exp\left(\frac{i}{(\log i)^3}\right). \tag{2.4}$$

(This is indeed the sequence we mentioned in (1.6).) It is easy to see that

$$\frac{t_{i+1}}{t_i} - 1 \asymp \frac{1}{(\log i)^3} \asymp \frac{1}{f^6(t_i)}, \tag{2.5}$$

and that

$$\int^{\infty} \frac{f^6(t)}{t} \exp\left(-\frac{\pi^2}{8} f^2(t)\right) \, \mathrm{d}t < \infty \iff \sum_i \exp\left(-\frac{\pi^2}{8} f^2(t_i)\right) < \infty. \tag{2.6}$$

Now we prove separately the two parts (2.1) and (2.2) of Theorem 1.1.

Proof of (2.1). Assume that f is such that

$$\int^{\infty} \frac{f^6(t)}{t} \exp\left(-\frac{\pi^2}{8} f^2(t)\right) \, \mathrm{d}t < \infty.$$

Let

$$\sigma_i := \inf\left\{t \geq t_i : \ \|\mathbb{W}(\bullet, t)\| \leq \frac{\sqrt{t_{i+1}}}{f(t_i)}\right\},$$

with the convention $\inf \emptyset := \infty$. Then

$$\begin{aligned} \{\sigma_i \leq t_{i+1}\} \cap \left\{\|\mathbb{W}(\bullet, t_{i+1}) - \mathbb{W}(\bullet, \sigma)\| \leq \sqrt{t_{i+1} - t_i}\right\} \\ \subset \left\{\|\mathbb{W}(\bullet, t_{i+1})\| \leq \frac{\sqrt{t_{i+1}}}{f(t_i)} + \sqrt{t_{i+1} - t_i}\right\}. \end{aligned}$$

We use the filtration generated by the second parameter: $\mathcal{F}_t := \sigma\{\mathbb{W}(u, v) : \ u \in [0, 1], \, v \in [0, t]\}$, $t \geq 0$. By the strong Markov property, on the event $\{\sigma_i \leq t_{i+1}\}$, we have

$$\mathbb{P}\left(\|\mathbb{W}(\bullet, t_{i+1}) - \mathbb{W}(\bullet, \sigma)\| \leq \sqrt{t_{i+1} - t_i} \mid \mathcal{F}_{\sigma_i}\right) \geq \mathbb{P}\left(\|\mathbb{W}(\bullet, 1)\| \leq 1\right) := c_1 > 0,$$

so that

$$\begin{aligned} \mathbb{P}\left(\sigma_i \leq t_{i+1}\right) &\leq \frac{1}{c_1} \mathbb{P}\left(\|\mathbb{W}(\bullet, t_{i+1})\| \leq \frac{\sqrt{t_{i+1}}}{f(t_i)} + \sqrt{t_{i+1} - t_i}\right) \\ &= \frac{1}{c_1} \mathbb{P}\left(\|\mathbb{W}(\bullet, 1)\| \leq \frac{1}{f(t_i)} + \frac{\sqrt{t_{i+1} - t_i}}{\sqrt{t_{i+1}}}\right) \\ &\leq \frac{1}{c_1} \mathbb{P}\left(\|\mathbb{W}(\bullet, 1)\| \leq \frac{1}{f(t_i)} + \frac{c_2}{f^3(t_i)}\right), \end{aligned}$$

the last inequality being a consequence of (2.5). Therefore, in light of (1.3), we get

$$\begin{aligned}\mathbb{P}(\sigma_i \le t_{i+1}) &\le c_3 \exp\left(-\frac{\pi^2}{8[1/f(t_i) + c_2/f^3(t_i)]^2}\right) \\ &\le c_4 \exp\left(-\frac{\pi^2}{8} f^2(t_i)\right).\end{aligned}$$

According to (2.6), this implies $\sum_i \mathbb{P}(\sigma_i \le t_{i+1}) < \infty$. Since

$$\{\sigma_i \le t_{i+1}\} = \left\{ \inf_{t \in [t_i, t_{i+1}]} \|\mathbb{W}(\bullet, t)\| \le \frac{\sqrt{t_{i+1}}}{f(t_i)} \right\},$$

it follows from the Borel–Cantelli lemma that almost surely for all large i,

$$\inf_{t \in [t_i, t_{i+1}]} \|\mathbb{W}(\bullet, t)\| > \frac{\sqrt{t_{i+1}}}{f(t_i)}.$$

Hence, for all $t \in [t_i, \ t_{i+1}]$ (recalling that f is non-decreasing)

$$\|\mathbb{W}(\bullet, t)\| > \frac{\sqrt{t_{i+1}}}{f(t_i)} \ge \frac{\sqrt{t}}{f(t)}.$$

This completes the proof of (2.1). □

Before tackling (2.2), we need the following notation and a probability estimate. For any pair of stochastic processes V_1 and V_2 with sample paths in $C[0,1]$, and for any $\varepsilon > 0$ and $\tau \in (0,1]$, we define

$$D(V_1, V_2, \varepsilon, \tau) := \left\{ \|V_1\| \le \varepsilon, \ \|\sqrt{1-\tau}\, V_1 + \sqrt{\tau}\, V_2\| \le \varepsilon \right\}.$$

Proposition 2.1. *Let W_1 and W_2 be independent one-dimensional Brownian motions. Then for $\varepsilon > 0$ and $\tau \in (0,1]$,*

$$\mathbb{P}(D(W_1, W_2, \varepsilon, \tau)) \le c_5 \exp\left(-\frac{\pi^2}{8\varepsilon^2} - \frac{c_6\, \tau^{1/3}}{\varepsilon^2}\right).$$

From the geometric point of wiew, this proposition concerns the probability that a planar Brownian motion stays in a parallelogram. It is worthwhile to point out the importance of the power (1/3) of τ on the right-hand side. Indeed, a direct application of Anderson's inequality would only yield the power 1, which would not suffice to prove (2.2).

By admitting Proposition 2.1 for the moment (its proof is postponed until Section 3), we are ready to proceed the calculations.

Proof of (2.2). Assume that

$$\int^{\infty} \frac{f^6(t)}{t} \exp\left(-\frac{\pi^2}{8} f^2(t)\right) \, dt = \infty,$$

and consider the sequence of events

$$A_i := \left\{ \|\mathbb{W}(\bullet, t_i)\| \le \frac{\sqrt{t_i}}{f(t_i)} \right\}, \qquad i \ge i_0,$$

with sufficiently large initial index i_0.

It follows from (1.3) that

$$\mathbb{P}(A_i) \asymp \exp\left(-\frac{\pi^2}{8} f^2(t_i)\right), \tag{2.7}$$

which implies, via (2.6), that $\sum_i \mathbb{P}(A_i) = \infty$.

In order to apply a version of the Borel–Cantelli lemma, we study $\sum\sum_{i<j\le N} \mathbb{P}(A_i A_j)$, and establish an upper bound of the double sum by a constant multiple of $[\, \sum_{i\le N} \mathbb{P}(A_j)\,]^2$. The desired bound for $\mathbb{P}(A_i A_j)$ will be obtained in three different ways, depending on the location of the pair (i, j). Denote

$$E_1 = E_1(N) := \left\{(i, j) : \; i_0 \le i < j \le N, \; j - i \le (\log i)^3 \right\},$$
$$E_2 = E_2(N) := \left\{(i, j) : \; i_0 \le i < j \le N, \; (\log i)^3 < j - i \le (\log i)^4 \right\},$$
$$E_3 = E_3(N) := \left\{(i, j) : \; i_0 \le i < j \le N, \; j - i > (\log i)^4 \right\}.$$

We start with easier cases where the classical Anderson inequality (see e.g. Lifshits [16], Chapter 11) provides a satisfactory bound: since $(t_j - t_i)^{-1/2}[\, \mathbb{W}(\bullet, t_j) - \mathbb{W}(\bullet, t_i)\,]$ and $t_i^{-1/2}\mathbb{W}(\bullet, t_i)$ are independent standard Brownian motions, we have

$$\mathbb{P}(A_i A_j) \le \mathbb{P}(A_i)\, \mathbb{P}\left(\|\mathbb{W}(\bullet, t_j - t_i)\| \le \frac{\sqrt{t_j}}{f(t_j)} \right)$$
$$= \mathbb{P}(A_i)\, \mathbb{P}\left(\|\mathbb{W}(\bullet, 1)\| \le \frac{1}{\sqrt{1 - (t_i/t_j)}\, f(t_j)} \right). \tag{2.8}$$

If $(i,j)\in E_3$, then $j-i>\frac{1}{2}(\log j)^4$, which, in view of (2.3), yields $t_i/t_j \le 1/f^2(t_j)$; hence, by means of (1.3),

$$\begin{aligned}
\mathbb{P}(A_iA_j) &\le \mathbb{P}(A_i)\,\mathbb{P}\left(\|\mathbb{W}(\bullet,1)\| \le \frac{1}{\sqrt{1-f^{-2}(t_j)}\,f(t_j)}\right)\\
&\le c_7\,\mathbb{P}(A_i)\,\exp\left(-\frac{\pi^2(1-f^{-2}(t_j))}{8}f^2(t_j)\right)\\
&\le c_8\,\mathbb{P}(A_i)\,\exp\left(-\frac{\pi^2}{8}f^2(t_j)\right)\\
&\le c_9\,\mathbb{P}(A_i)\,\mathbb{P}(A_j).
\end{aligned}$$

It follows that

$$\sum_{(i,j)\in E_3}\mathbb{P}(A_iA_j)\le c_{10}\left(\sum_{i\le N}\mathbb{P}(A_i)\right)^2. \tag{2.9}$$

Now let us consider the case $(i,j)\in E_2$. Inequality $j-i>(\log i)^3$ implies $\frac{t_i}{t_j}\le c_{11}<1$. In view of (2.8), (2.3) and (1.3), we have

$$\mathbb{P}(A_iA_j)\le \mathbb{P}(A_i)\,\mathbb{P}\left(\|\mathbb{W}(\bullet,1)\|\le \frac{1}{c_{12}\sqrt{\log j}}\right)\le c_{13}\,\mathbb{P}(A_i)\,j^{-c_{14}}.$$

Accordingly,

$$\sum_{(i,j)\in E_2}\mathbb{P}(A_iA_j)\le c_{13}\sum_{i\le N}\mathbb{P}(A_i)\sum_{j:\, i<j\le i+(\log i)^4} j^{-c_{14}}\le c_{15}\sum_{i\le N}\mathbb{P}(A_i). \tag{2.10}$$

Now we proceed to the upper bounds for $\mathbb{P}(A_iA_j)$ in the delicate case $(i,j)\in E_1$. Direct application of Anderson's inequality turns out to be too crude. This is where we need Proposition 2.1. Indeed, writing as before W_1 and W_2 for a pair of independent one-dimensional Brownian motions, we have

$$\begin{aligned}
\mathbb{P}(A_iA_j) &= \mathbb{P}\left(\|W_1\|\le\frac{1}{f(t_i)},\ \|\sqrt{1-\tau}\,W_1+\sqrt{\tau}\,W_2\|\le\frac{1}{f(t_j)}\right)\\
&\le \mathbb{P}\left(\|W_1\|\le\frac{1}{f(t_i)},\ \|\sqrt{1-\tau}\,W_1+\sqrt{\tau}\,W_2\|\le\frac{1}{f(t_i)}\right),
\end{aligned}$$

where $\tau=\tau_{ij}:=1-(t_i/t_j)$. If $(i,j)\in E_1$, then, again using (2.3), we derive

$$\tau\ge c_{16}\,\frac{j-i}{(\log i)^3}\ge\frac{c_{17}(j-i)}{f^6(t_i)}.$$

We apply Proposition 2.1 to $\varepsilon := 1/f(t_i)$ and obtain (using (2.7) at the last step)

$$\begin{aligned}\mathbb{P}(A_iA_j) &\le c_5 \exp\left(-\frac{\pi^2 f(t_i)}{8} - c_{18}\,(j-i)^{1/3}\right)\\ &\le c_{19}\,\mathbb{P}(A_i)\,\exp\left(-c_{18}\,(j-i)^{1/3}\right).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{(i,j)\in E_1} \mathbb{P}(A_iA_j) &\le c_{19} \sum_{i\le N} \mathbb{P}(A_i) \sum_{j>i} \exp\left(-c_{18}\,(j-i)^{1/3}\right)\\ &\le c_{20} \sum_{i\le N} \mathbb{P}(A_i).\end{aligned}$$

By combining this estimate with (2.9) and (2.10), we get:

$$\limsup_{N\to\infty} \frac{\sum\sum_{i_0\le i,j\le N} \mathbb{P}(A_iA_j)}{[\sum_{i_0\le i\le N} \mathbb{P}(A_i)\,]^2} \le c_{10} < \infty.$$

Next, we apply the following version of Borell-Cantelli lemma.

Lemma 2.2 (Kochen and Stone [12]). *Let (A_i) be a sequence of events such that $\sum_i \mathbb{P}(A_i) = \infty$. Then*

$$\mathbb{P}(A_i,\ \text{i.o.}) \ge \left[\liminf_{N\to\infty} \frac{\sum\sum_{i_0\le i,j\le N} \mathbb{P}(A_iA_j)}{[\sum_{i_0\le i\le N} \mathbb{P}(A_i)\,]^2}\right]^{-1}.$$

According to Lemma 2.2, we have

$$\mathbb{P}\left(\|\mathbb{W}(\bullet,t_i)\| \le \frac{\sqrt{t_i}}{f(t_i)},\ \text{i.o.}\right) > 0. \tag{2.11}$$

Instead of the subsequence $t_i = \exp(i/(\log i)^3)$ defined in (2.4), we can also consider the subsequence $\widehat{t}_i = \lfloor \exp(i/(\log i)^3)\rfloor$. Since $\widehat{t}_i \le t_i \le \widehat{t}_i + 1$, the argument leading to (2.11) also applies to $(\widehat{t}_i)$ in place of (t_i), possibly with different constants. Accordingly, we have

$$\mathbb{P}\left(\|\mathbb{W}(\bullet,\widehat{t}_i)\| \le \frac{\sqrt{\widehat{t}_i}}{f(\widehat{t}_i)},\ \text{i.o.}\right) > 0,$$

which, in turn, yields that

$$\mathbb{P}\left(\|\mathbb{W}(\bullet,n)\| \le \frac{\sqrt{n}}{f(n)},\ \text{i.o.}\right) > 0.$$

Note that $\mathbb{W}(\bullet, n) = \sum_{k=1}^{n} [\,\mathbb{W}(\bullet, k) - \mathbb{W}(\bullet, k-1)\,]$, and that $\{\mathbb{W}(\bullet, k) - \mathbb{W}(\bullet, k-1),\ k \geq 1\}$ is a sequence of iid random variables taking values in $(C([0,1], \mathbb{R}),\ \|\cdot\|)$. Since any finite permutation of $\{\mathbb{W}(\bullet, k) - \mathbb{W}(\bullet, k-1),\ k \geq 1\}$ does not affect the event $\{\|\mathbb{W}(\bullet, n)\| \leq \sqrt{n}/f(n),\ \text{i.o.}\}$, the Hewitt–Savage 0–1 law confirms that

$$\mathbb{P}\left(\|\mathbb{W}(\bullet, n)\| \leq \frac{\sqrt{n}}{f(n)},\ \text{i.o.} \right) = 1.$$

This completes the proof of (2.2). □

3. Proof of Proposition 2.1

The sole goal of this section is to prove Proposition 2.1. Let us first recall the statement of the proposition: let W_1 and W_2 be independent Wiener processes, then there exist finite and positive constants c_5 and c_6 such that for $\varepsilon > 0$ and $\tau \in (0, 1]$,

$$\begin{aligned} &\mathbb{P}\left(|W_1(t)| \leq \varepsilon,\ |\sqrt{1-\tau}\, W_1(t) + \sqrt{\tau}\, W_2(t)| \leq \varepsilon,\ \text{for all } t \in [0,1]\right) \\ &\quad \leq c_5 \exp\left(-\frac{\pi^2}{8\varepsilon^2} - \frac{c_6\, \tau^{1/3}}{\varepsilon^2} \right). \end{aligned}$$

For every $h > 0$, we define a sequence of stopping times $\theta_k = \theta_k(h)$ depending only on the process W_2. Let $\theta_0 = 0$ and then

$$\begin{aligned} \theta_{2k+1} &:= \inf\{t > \theta_{2k} : |W_2(t)| = 2h\}, \\ \theta_{2k+2} &:= \inf\{t > \theta_{2k+1} : |W_2(t)| = h\}, \qquad k \geq 0. \end{aligned}$$

Denote also $\theta'_k := \min\{\theta_k, 1\}$. Furthermore, define the duration of upward crossing

$$Y_k(h) := \theta_{2k+1} - \theta_{2k} \quad (k \geq 0)$$

and the duration of downward crossing

$$Z_k(h) := \theta_{2k} - \theta_{2k-1} \quad (k \geq 1).$$

Similarly, $Y'_k(h)$ and $Z'_k(h)$ are defined via θ'_k.

Notice some scaling and moment properties of $Y_k(h)$ and $Z_k(h)$. For upward crossing times, we have $Y_k(h) \stackrel{d}{=} h^2 Y_k(1)$. The variables $\{Y_k(1), k \geq 0\}$ are independent and identically distributed for $k \geq 1$. It is well known that some exponential moments, e.g. $\mathbb{E}\exp\{Y_k(1)/4\}$ are finite. Hence, by the exponential Chebyshev inequality, for every $b > \mathbb{E}Y_0(1)$, there exists a positive constant c_b such that for all h and all integer N,

$$\mathbb{P}\left(\sum_{k=0}^{N-1} Y_k(h) \geq bh^2 N\right) \leq \exp\{-c_b N\}. \tag{3.1}$$

On the other hand, for downward crossing duration, we have not only the scaling property $Z_k(h) \stackrel{d}{=} h^2 Z_k(1)$ for i.i.d. variables $Z_k(h)$, but also the stability of order (1/2). Namely,

$$\sum_{k=1}^{N} Z_k(1) \stackrel{d}{=} Z_1(N) \stackrel{d}{=} N^2 Z_1(1).$$

In particular, we have

$$\begin{aligned}\mathbb{P}\left(\sum_{k=1}^{N} Z_k(h) \leq 1\right) &= \mathbb{P}\left(N^2 h^2 Z_1(1) \leq 1\right) \\ &= \mathbb{P}\left(\sup_{0\leq t\leq N^{-2}h^{-2}} W_2(t) \geq 1\right) \\ &\leq 2\exp\left\{-\frac{N^2h^2}{2}\right\}. \qquad (3.2)\end{aligned}$$

Given N and h, consider the following events:

$$\begin{aligned} D_1 &:= \{\, \|W_1\| \leq \varepsilon\}\,; \\ D_{2,1} &:= \left\{\sum_{k=0}^{N-1} Y_k(h) \geq bh^2 N\right\}; \\ D_{2,2} &:= \left\{\sum_{k=1}^{N} Z_k(h) \leq 1\right\}; \\ D_{2,3} &:= (D_{2,1} \cup D_{2,2})^c. \end{aligned}$$

Set, for brevity, $D := D(W_1, W_2, \varepsilon, \tau)$. Since $D \subset D_1$, we have

$$\begin{aligned} D &= (D\, D_{2,1}) \cup (D\, D_{2,2}) \cup (D\, D_{2,3}) \\ &\subset (D_1 D_{2,1}) \cup (D_1 D_{2,2}) \cup (D\, D_{2,3}). \end{aligned}$$

Using the estimates (3.1), (3.2) and the independence of W_1 and W_2, we obtain

$$\mathbb{P}\left((D_1 D_{2,1}) \cup (D_1 D_{2,2})\right) \leq \mathbb{P}(D_1)\left(\exp\{-c_b N\} + 2\exp\{\frac{-N^2h^2}{2}\}\right). \tag{3.3}$$

Now consider the sample paths from $D\,D_{2,3}$. Notice that in this case

$$\sum_{k=0}^{N-1} Y_k(h) \le bh^2 N, \tag{3.4}$$

and

$$\sum_{k=1}^{N} Z_k(h) > 1.$$

Hence, $\theta_{2N} > 1$, and we have

$$\sum_{k=0}^{N-1} Y_k'(h) + \sum_{k=1}^{N} Z_k'(h) = 1. \tag{3.5}$$

The key observation reads as follows: every sample path from $D\,D_{2,3}$ satisfies

$$W_1(t) \in \begin{cases} [-\varepsilon, \varepsilon], & \text{if } t \in \cup_{k=0}^{N-1}[\theta_{2k}', \theta_{2k+1}']; \\ [-\varepsilon, -\varepsilon + 2\varepsilon_h]\,\mathrm{sign}\,\{W_2(\theta_{2k-1})\}, & \text{if } t \in \cup_{k=1}^{N}[\theta_{2k-1}', \theta_{2k}']; \end{cases}$$

where ε_h is the half-width of the interval in which the process W_1 has to stay when W_2 reaches the level h. Moreover, if $h > 2\sqrt{\tau}\varepsilon$, this width obeys the bounds

$$\varepsilon_h := \left(\varepsilon - \frac{\sqrt{\tau}}{2\sqrt{1-\tau}}\left(h - \frac{\sqrt{\tau}\varepsilon}{1+\sqrt{1-\tau}}\right)\right)_+ \le \varepsilon - \frac{\sqrt{\tau}h}{4},$$

thus

$$\varepsilon_h^{-2} \ge \varepsilon^{-2}\left(1 - \frac{\sqrt{\tau}h}{4\varepsilon}\right)^{-2} \ge \varepsilon^{-2} + \frac{\sqrt{\tau}h}{2\varepsilon^3}. \tag{3.6}$$

Hence, by Anderson's inequality and small ball bound (1.3) for W_1 we have

$$\mathbb{P}\left(D\,D_{2,3}|W_2\right) \le \left(\frac{4}{\pi}\right)^{2N} \prod_{k=0}^{N-1} \exp\left\{-\frac{\pi^2}{8\varepsilon^2} Y_k'(h)\right\} \prod_{k=1}^{N} \exp\left\{-\frac{\pi^2}{8\varepsilon_h^2} Z_k'(h)\right\},$$

while by (3.5) and (3.6)

$$\begin{aligned} &\varepsilon^{-2}\sum_{k=0}^{N-1} Y_k'(h) + \varepsilon_h^{-2}\sum_{k=1}^{N} Z_k'(h) \\ &\quad \ge \varepsilon^{-2}\left(\sum_{k=0}^{N-1} Y_k'(h) + \sum_{k=1}^{N} Z_k'(h)\right) + \frac{\sqrt{\tau}h}{2\varepsilon^3}\sum_{k=1}^{N} Z_k'(h) \\ &\quad = \varepsilon^{-2} + \frac{\sqrt{\tau}h}{2\varepsilon^3}\left(1 - \sum_{k=0}^{N-1} Y_k'(h)\right). \end{aligned}$$

Replace $\sum_{k=0}^{N-1} Y'_k(h)$ by the larger sum $\sum_{k=0}^{N-1} Y_k(h)$ and apply (3.4). In this way, we obtain the uniform bound for conditional probability

$$\mathbb{P}\left(D\, D_{2,3}|W_2\right) \le \mathbf{1}_{D_{2,3}} \left(\frac{4}{\pi}\right)^{2N} \exp\left\{-\frac{\pi^2}{8\varepsilon^2} - \frac{\pi^2}{16\varepsilon^3}\sqrt{\tau}h(1-bh^2N)\right\}.$$

and we get an estimate

$$\mathbb{P}\left(D\, D_{2,3}\right) \le \left(\frac{4}{\pi}\right)^{2N} \exp\left\{-\frac{\pi^2}{8\varepsilon^2} - \frac{\pi^2}{16\varepsilon^3}\sqrt{\tau}h(1-bh^2N)\right\}. \tag{3.7}$$

Now we specify the parameters of our construction. Let

$$\begin{aligned} h &:= 2\varepsilon\tau^{-1/6}; \\ b &:= \max\left\{\frac{4\ln(4/\pi)}{\pi^2}, \mathbb{E}Y_0(1)\right\} + 1; \\ N &:= \left\lfloor \frac{1}{2bh^2} \right\rfloor. \end{aligned}$$

Consider two cases. If $2bh^2 \ge 1$, then

$$\varepsilon^{-2}\tau^{1/3} = 4h^{-2} \le 8b,$$

and the statement of Proposition 2.1 is trivial, since

$$\mathbb{P}(D) \le \mathbb{P}(D_1) \le \frac{4}{\pi}\exp\left\{-\frac{\pi^2}{8\varepsilon^2}\right\} \le \frac{4\mathrm{e}}{\pi}\exp\left\{-\frac{\pi^2}{8\varepsilon^2} - \frac{\tau^{1/3}}{8b\varepsilon^2}\right\}.$$

On the other hand, if $2bh^2 \le 1$, we derive from (3.7) the estimate

$$\mathbb{P}\left(D\, D_{2,3}\right) \le \exp\left\{-\frac{\pi^2}{8\varepsilon^2} - \left(\frac{\pi^2}{16} - \frac{\log(4/\pi)}{4b}\right)\frac{\tau^{1/3}}{\varepsilon^2}\right\}.$$

The proof of Proposition 2.1 is, therefore, complete by application of the latter bound jointly with (3.3). □

4. Application to empirical processes

Let U_1, U_2, $\cdots$ be a sequence of independent random variables having the common uniform distribution in $(0,1)$. Let

$$\alpha_n(t) := n^{1/2}\left(\sum_{i=1}^{n} \mathbf{1}_{\{U_i \le t\}} - t\right), \qquad t \in [0,1], \tag{4.1}$$

which is the uniform empirical process based on the first n observations. As before, we work under the sup-norm $\|f\| := \sup_{t\in[0,1]} |f(t)|$.

Our goal is to prove Theorem 1.2 characterizing the lower functions for empirical process. In order to do it, we first obtain its Gaussian companion – a similar result for the Kiefer process $\{K_n(t);\, n \geq 0,\, t \in [0,1]\}$. By definition, the latter is a centered Gaussian random field with covariance

$$\mathbb{E}[\, K_n(s) K_m(t)\,] := \min\{n, m\}\ (\min\{s, t\} - st)\,.$$

Here is the version of Theorem 1.2 for the Kiefer process.

Theorem 4.1. *Let $K_n(t)$ be a Kiefer process. Then for every positive and increasing sequence (λ_n),*

$$\mathbb{P}\left(\|K_n\| \leq \frac{\sqrt{n}}{\lambda_n},\ \text{i.o.} \right) = \begin{cases} 0 \\ 1 \end{cases} \iff \sum_n \frac{\lambda_n^7}{n} \exp\left(-\frac{\pi^2}{8}\lambda_n^2\right) \begin{cases} < \infty \\ = \infty \end{cases}.$$

In the proof of Theorem 1.1, we only had to limit ourselves to the study of "critical functions" in the sense of (2.3). Exactly for the same reason, in order to prove Theorems 1.2 and 4.1, we can assume without loss of generality that

$$\frac{\sqrt{\log\log n}}{2} \leq \lambda_n \leq \sqrt{\log\log n}\,, \qquad \text{for } n \geq n_0.$$

Under this assumption, the equivalence between Theorems 1.2 and 4.1 is immediate. Indeed, a strong approximation theorem of Komlós, Major and Tusnády [14] provides a joint construction of α_n and K_n such that

$$\left\| \alpha_n - \frac{K_n}{\sqrt{n}} \right\| = o\left(\frac{\log^2 n}{n^{1/2}}\right) = o\left(\lambda_n^{-3}\right) = o\left(\left|\frac{1}{\lambda_n} - \frac{1}{\lambda_n \pm 1/\lambda_n}\right|\right)$$

almost surely while the test series converges (resp. diverges) simultaneously for the sequences λ_n and $\lambda_n \pm 1/\lambda_n$.

The proof of Theorem 4.1 is in the same spirit as that of Theorem 1.1, with two exceptions. Instead of using (1.3) for the sup-norm of Brownian motion, we use the corresponding result for the Brownian bridge (noting that K_1 is a standard Brownian bridge):

$$\mathbb{P}\left(\|K_1\| \leq \varepsilon \right) \sim \frac{\sqrt{2\pi}}{\varepsilon} \exp\left(-\frac{\pi^2}{8\varepsilon^2}\right), \qquad \varepsilon \to 0, \tag{4.2}$$

(The computation of the distribution of $\|K_1\|$ goes back at least to Kolmogorov [13].) Also, the role of Proposition 2.1 is played by the following result. Recall our notation

$$D(V_1, V_2, \varepsilon, \tau) := \left\{ \|V_1\| \le \varepsilon,\ \|\sqrt{1-\tau}\, V_1 + \sqrt{\tau}\, V_2\| \le \varepsilon \right\},$$

for any stochastic processes V_1 and V_2.

Proposition 4.2. (i) *If $\overset{\circ}{W}_1$ is a standard Brownian bridge independent of the Wiener process W_2, then for any $\varepsilon \in (0,1]$ and $\tau \in (0,1]$,*

$$\mathbb{P}\left(D(\overset{\circ}{W}_1, W_2, \varepsilon, \tau) \right) \le \frac{c_{21}}{\varepsilon} \exp\left(-\frac{\pi^2}{8\varepsilon^2} - \frac{c_{22}\, \tau^{1/3}}{\varepsilon^2} \right). \tag{4.3}$$

(ii) *Let $\overset{\circ}{W}_1$ and $\overset{\circ}{W}_2$ be independent Brownian bridges. Then for any $\varepsilon \in (0,1]$ and $\tau \in (0,1]$,*

$$\mathbb{P}\left(D(\overset{\circ}{W}_1, \overset{\circ}{W}_2, \varepsilon, \tau) \right) \le c_{23} \max\left\{ \frac{1}{\varepsilon}, \frac{\sqrt{\tau}}{\varepsilon^4} \right\} \exp\left(-\frac{\pi^2}{8\varepsilon^2} - \frac{c_{24}\, \tau^{1/3}}{\varepsilon^2} \right) . \tag{4.4}$$

Although Part (i) of the proposition does not directly contribute to the proof of Theorem 4.1, it is used to prove Part (ii).

Once Proposition 4.2 is established, the proof of Theorem 4.1 goes exactly along the same lines as of Theorem 1.1, and we feel free to omit the details. The rest of this section is devoted to the proof of Proposition 4.2.

Proof of Proposition 4.2. We start with the proof of (4.3). Let $q := 1 - \varepsilon^2$, and write for brevity $\|v(\cdot)\|_{[0,q]} := \sup_{0 \le t \le q} |v(t)|$ and

$$D_q(V_1, V_2, \varepsilon, \tau) := \left\{ \|V_1\|_{[0,q]} \le \varepsilon,\ \|\sqrt{1-\tau}\, V_1 + \sqrt{\tau}\, V_2\|_{[0,q]} \le \varepsilon \right\}.$$

By the full probability formula, we have

$$\begin{aligned}
&\mathbb{P}\left(D_q(W_1, W_2, \varepsilon, \tau) \right) \\
&\quad = \int_{-\infty}^{\infty} \mathbb{P}\left(D_q(W_1, W_2, \varepsilon, \tau) \mid W_1(1) = r \right) \frac{\exp\{-r^2/2\}}{\sqrt{2\pi}}\, \mathrm{d}r \\
&\quad \ge c_{25}\, \varepsilon \inf_{|r| \le \varepsilon/2} \mathbb{P}\left(D_q(W_1, W_2, \varepsilon, \tau) \mid W_1(1) = r \right) \\
&\quad = c_{25}\, \varepsilon \inf_{|r| \le \varepsilon/2} \mathbb{P}\left(D_q(\overset{\circ}{W}_1 + r\mathrm{Id}, W_2, \varepsilon, \tau) \right) \\
&\quad = c_{25}\, \varepsilon \inf_{|r| \le \varepsilon/2} \mathbb{P}\left(D_q(\overset{\circ}{W}_1 + r h_q, W_2, \varepsilon, \tau) \right) ,
\end{aligned}$$

where

$$h_q(t) := \begin{cases} t & \text{for } 0 \le t \le q; \\ q(1-t)/\varepsilon^2 & \text{for } q \le t \le 1; \end{cases}$$

is an admissible shift of Brownian bridge and the Cameron–Martin density of the corresponding shifted measure is

$$p_{rh_q}(v) = \exp\left\{ -\frac{r^2 q}{2\varepsilon^2} + \frac{r}{\varepsilon^2} v(q) \right\}.$$

It is worthwhile to notice that under assumptions $|r| \le \varepsilon/2$ and $\|\mathring{W}_1\|_{[0,q]} \le \varepsilon$, we have

$$p_{rh_q}(\mathring{W}_1) \ge \exp\left\{ -\frac{1}{8} - \frac{|\mathring{W}_1(q)|}{4\varepsilon} \right\} \ge \mathrm{e}^{-3/8}.$$

We obtain via the Cameron–Martin formula,

$$\begin{aligned} \mathbb{P}\left(D_q(\mathring{W}_1 + rh_q, W_2, \varepsilon, \tau) \right) &= \mathbb{E}\left[\mathbf{1}_{D_q(\mathring{W}_1, W_2, \varepsilon, \tau)} p_{rh_q}(\mathring{W}_1) \right] \\ &\ge \mathrm{e}^{-3/8}\, \mathbb{P}\left(D_q(\mathring{W}_1, W_2, \varepsilon, \tau) \right) \\ &\ge \mathrm{e}^{-3/8}\, \mathbb{P}\left(D(\mathring{W}_1, W_2, \varepsilon, \tau) \right). \end{aligned}$$

We infer from this calculation that

$$\mathbb{P}\left(D_q(W_1, W_2, \varepsilon, \tau) \right) \ge c_{26}\, \varepsilon\, \mathbb{P}\left(D(\mathring{W}_1, W_2, \varepsilon, \tau) \right). \tag{4.5}$$

On the other hand, by scaling and Proposition 2.1, we have

$$\begin{aligned} \mathbb{P}\left(D_q(W_1, W_2, \varepsilon, \tau) \right) &= \mathbb{P}\left(D(W_1, W_2, \frac{\varepsilon}{\sqrt{q}}, \tau) \right) \\ &\le c_5 \exp\left(-\frac{\pi^2 q^2}{8\varepsilon^2} - \frac{c_6\, \tau^{1/3} q^2}{\varepsilon^2} \right) \\ &\le c_{27} \exp\left(-\frac{\pi^2}{8\varepsilon^2} - \frac{c_6\, \tau^{1/3}}{\varepsilon^2} \right). \end{aligned}$$

In view of (4.5), this yields (4.3) and thus the first part of the proposition.

It remains to check the second part. If $\tau^{1/3} \le \varepsilon^2$, then there is nothing to prove, since by (4.2), we trivially have

$$\begin{aligned}
\mathbb{P}\left(\|\overset{\circ}{W}_1\| \le \varepsilon,\ \|\sqrt{1-\tau}\,\overset{\circ}{W}_1 + \sqrt{\tau}\,\overset{\circ}{W}_2\| \le \varepsilon \right) &\le \mathbb{P}\left(\|\overset{\circ}{W}_1\| \le \varepsilon \right) \\
&\le \frac{c_{28}}{\varepsilon} \exp\left\{ -\frac{\pi^2}{8\varepsilon^2} \right\} \\
&\le \frac{c_{29}}{\varepsilon} \exp\left\{ -\frac{\pi^2}{8\varepsilon^2} - \frac{\tau^{1/3}}{\varepsilon^2} \right\}.
\end{aligned}$$

We assume now $\tau^{1/3} > \varepsilon^2$. Without loss of generality, we consider for $i = 1$ and 2,

$$\overset{\circ}{W}_i(t) := W_i(t) - tW_i(1), \qquad t \in [0,1],$$

and we write $\overset{\circ}{W}_i = W_i - W_i(1)\,\mathrm{Id})$, where W_1 and W_2 are independent Wiener processes. Note that the random variable $W_2(1)$ is independent of the processes $\overset{\circ}{W}_1$ and $\overset{\circ}{W}_2$. Observe that $\mathbb{P}(|W_2(1)| \le x) \ge x/2$ for $x \in [0,1]$. Hence, writing

$$p_\varepsilon := \mathbb{P}(\,\|\overset{\circ}{W}_1\| \le \varepsilon,\ \|\sqrt{1-\tau}\,\overset{\circ}{W}_1 + \sqrt{\tau}\,\overset{\circ}{W}_2\| \le \varepsilon),$$

we have

$$\begin{aligned}
p_\varepsilon &= \frac{\mathbb{P}(\,\|\overset{\circ}{W}_1\| \le \varepsilon,\ \|\sqrt{1-\tau}\,\overset{\circ}{W}_1 + \sqrt{\tau}\,W_2 - \sqrt{\tau}\,W_2(1)\,\mathrm{Id}\| \le \varepsilon,\ |W_2(1)| \le \varepsilon^3/\sqrt{\tau}\,)}{\mathbb{P}\,(\,|W_2(1)| \le \varepsilon^3/\sqrt{\tau}\,)} \\
&\le \frac{\mathbb{P}(\,\|\overset{\circ}{W}_1\| \le \varepsilon,\ \|\sqrt{1-\tau}\,\overset{\circ}{W}_1 + \sqrt{\tau}\,W_2\| \le \varepsilon + \varepsilon^3)}{\varepsilon^3/2\sqrt{\tau}}.
\end{aligned}$$

In light of (4.3) (which we have just proved), this yields

$$\begin{aligned}
p_\varepsilon &\le 2c_{21}\,\frac{\sqrt{\tau}}{\varepsilon^4} \exp\left(-\frac{\pi^2}{8(\varepsilon+\varepsilon^3)^2} - \frac{c_{22}\,\tau^{1/3}}{(\varepsilon+\varepsilon^3)^2} \right) \\
&\le c_{30}\,\frac{\sqrt{\tau}}{\varepsilon^4} \exp\left(-\frac{\pi^2}{8\varepsilon^2} - \frac{c_{22}\,\tau^{1/3}}{\varepsilon^2} \right).
\end{aligned}$$

This implies (4.4) and completes the proof of Proposition 4.2. □

5. Remarks and questions

This final section contains a few remarks and questions related to the lower functions of infinite-dimensional Brownian motion and empirical processes.

(1) A standard time inversion argument immediately yields the following companion of Theorem 1.1 for small times: Let X be as in (1.1). If $f > 0$ is non-decreasing, then

$$\int_{0+} \frac{f^6(t)}{t} \exp\left(-\frac{\pi^2}{8} f^2(t)\right) \mathrm{d}t \begin{cases} < \infty \\ = \infty \end{cases}$$
$$\iff \mathbb{P}\left(\|X(t)\| \leq \frac{t^{1/2}}{f(1/t)} \text{ i.o. for } t \to 0 \right) = \begin{cases} 0 \\ 1 \end{cases},$$

where $\|\cdot\|$ is defined in (1.2).

(2) The present paper focuses on the rate of escape problem for a particular infinite-dimensional Brownian motion in a particular space (the Brownian sheet under the sup-norm). It would be very interesting to characterize the escape rate of more general infinite-dimensional Brownian motions.

Looking at the two parts (2.1) and (2.2) of Theorem 1.1, part (2.1) can be extended with no extra efforts to any infinite-dimensional Brownian motion X taking values in a separable Banach space $(E, \|\cdot\|)$, as soon as one has sufficiently good information about the asymptotic behaviour of $\mathbb{P}(\|X(1)\| \leq \varepsilon)$. Part (2.2), on the other hand, seems far more delicate, and this is why we had to use special properties of our example.

A question we ask ourselves is whether the behaviour of $\mathbb{P}(\|X(1)\| \leq \varepsilon)$ suffices to determine the escape rate of X.

(3) If we are not so ambitious, then we might like to study the same Brownian sheet $\mathbb{W}$ under the L^2-norm $\|g\|_2 := [\int_0^1 g^2(t)\, \mathrm{d}t\,]^{1/2}$. Is it possible to characterize $\mathbb{P}\{\|\mathbb{W}(\bullet, t)\|_2 \leq \sqrt{t}/f(t),\ \text{i.o.}\}$ via an integral criterion? One can find a partial answer in Albin's work [2]. By using his Theorem 4, we get, for any $T > 1$,

$$\mathbb{P}\left(\min_{1 \leq t \leq T} \frac{\|\mathbb{W}(\bullet, t)\|_2}{\sqrt{t}} \leq \varepsilon \right) \sim \frac{c \, \log T}{\varepsilon^3} \exp\left\{ -\frac{1}{8\varepsilon^2} \right\}, \qquad \varepsilon \to 0.$$

An application of the Borel-Cantelli lemma yields

$$\int^{\infty} \frac{f^3(t)}{t} \exp\left(-\frac{f^2(t)}{8}\right) \mathrm{d}t < \infty \implies \mathbb{P}\left(\|\mathbb{W}(\bullet, t)\|_2 \leq \frac{t^{1/2}}{f(t)} \text{ i.o. for } t \to \infty \right) = 0.$$

Similarly, for the Kiefer process we have

$$\int^{\infty} \frac{f^4(t)}{t} \exp\left(-\frac{f^2(t)}{8}\right) \mathrm{d}t < \infty \implies \mathbb{P}\left(\|K_n\|_2 \leq \frac{n^{1/2}}{f(n)} \text{ i.o. for } n \to \infty \right) = 0.$$

Both integrals on the left-hand sides suggest that $t_i := \exp\left(\frac{i}{(\log i)^2}\right)$ could serve here as a critical subsequence. Unfortunately, the problem of necessary and sufficient test remains open, to the best of our knowledge.

(4) It is not the first time that the Erdős subsequence fails to work in a delicate integral test. Indeed, in an attempt to characterize the upper functions of the square integral of (one-dimensional) Brownian motion, Albin [1] showed that the Erdős subsequence is not suitable.

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