

A Ray-Knight theorem for symmetric Markov processes

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Abstract

Let X be a strongly symmetric recurrent Markov process with state space S and let L_t^x denote the local time of X at $x \in S$. For a fixed element 0 in the state space S , let

$$\tau(t) \stackrel{def}{=} \inf\{s : L_s^0 > t\}.$$

The 0 -potential density, $u_{\{0\}}(x, y)$, of the process X killed at $T_0 = \inf\{s : X_s = 0\}$, is symmetric and positive definite. Let $\eta = \{\eta_x; x \in S\}$ be a mean-zero Gaussian process with covariance

$$E_\eta(\eta_x \eta_y) = u_{\{0\}}(x, y).$$

The main result of this paper is the following generalization of the classical second Ray-Knight Theorem: For any $b \in \mathbb{R}$ and $t > 0$

$$\{L_{\tau(t)}^x + \frac{1}{2}(\eta_x + b)^2; x \in S\} \stackrel{law}{=} \left\{ \frac{1}{2} \left(\eta_x + \sqrt{2t + b^2} \right)^2; x \in S \right\}.$$

A version of this theorem is also given when X is transient.

1 Introduction

The goal of this paper is to generalize the second Ray-Knight Theorem to all strongly symmetric Markov processes with finite 1-potential densities. In

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Chapter XI, Section 2, [13] the authors use the terminology “second Ray–Knight Theorem” to describe the law of the Brownian local time process $\{L_{\tau(t)}^x; x \geq 0\}$, where $\tau(t) \stackrel{def}{=} \inf\{s : L_s^0 > t\}$ is the right continuous inverse of the mapping $t \mapsto L_t^0$. In its standard formulation, this theorem says that for any fixed $t > 0$, the process $\{L_{\tau(t)}^x; x \geq 0\}$, under the measure P^0 , is a Markov process in $x \geq 0$, and identifies this process as a squared Bessel process of dimension 0 starting at t . For a general Markov process X , the methods of [6] can be used to show that the analogous local time process will not be a Markov process in the state variable x unless X has continuous paths.

There is an alternate formulation of the second Ray–Knight Theorem. It states that for any $t > 0$, under the measure $P^0 \times \tilde{P}^0$,

$$(1.1) \quad \{L_{\tau(t)}^x + \frac{1}{2}\tilde{B}_x^2; x \geq 0\} \stackrel{law}{=} \{\frac{1}{2}(\tilde{B}_x + \sqrt{2t})^2; x \geq 0\}$$

where $\{\tilde{B}_x/\sqrt{2}; x \geq 0\}$ is a real-valued Brownian motion, with measure \tilde{P}^0 , independent of the original Brownian motion. The equivalence of this with the standard formulation mentioned above is a consequence of the additivity property of squared Bessel processes, (see e.g. Chapter XI, Theorem 1.2, [13]). (1.1) is the version of the second Ray–Knight Theorem which we generalize to strongly symmetric Markov processes, by replacing \tilde{B}_x by a different mean zero Gaussian process.

Let $X = (\Omega, \mathcal{F}_t, X_t, P^x)$, $t \in R^+$, denote a strongly symmetric standard Markov process with state space S , which is a locally compact separable metric space, and with reference measure m . The full definition of these terms is given in [9]. Strong symmetry means that for each $\alpha > 0$ and $x \in S$, the potential measures

$$(1.2) \quad U^\alpha(x, A) = \int_0^\infty e^{-\alpha t} E^x(1_A(X_t)) dt$$

are absolutely continuous with respect to the reference measure m , and we can find densities $u^\alpha(x, y)$, for the potential measures, which are symmetric in x, y . Throughout this paper we assume that the densities $u^\alpha(x, y)$ are finite for some, hence all $\alpha > 0$. This is the necessary and sufficient condition for the existence of local times, see e.g. Theorem 3.2 of [9].

Let L_t^x denote the local time of X at $x \in S$. Heuristically, L_t^x is the amount of time that the process spends at x , up to time t . We can define

$$(1.3) \quad L_t^x = \lim_{\epsilon \rightarrow 0} \int_0^t f_{\epsilon, x}(X_s) ds$$

where $f_{\epsilon,x}$ is an approximate delta-function at x . Specifically, we take the support of $f_{\epsilon,x}$ to be $B(x, \epsilon)$, the ball of radius ϵ centered at x and $\int f_{\epsilon,x}(y) dm(y) = 1$. Convergence here is locally uniform in t , almost surely. Hence L_t^x inherits from $\int_0^t f_{\epsilon,x}(X_s) ds$ both continuity in t and the additivity property: $L_{t+s}^x = L_t^x + L_s^x \circ \theta_t$. Let 0 denote a fixed element in the state space S and let

$$(1.4) \quad \tau(t) \stackrel{def}{=} \inf\{s : L_s^0 > t\}.$$

$\tau(t)$ is the right continuous inverse of the mapping $t \mapsto L_t^0$.

Let Y denote the process X killed when it first hits 0. Y is a strongly symmetric Markov process. We use $u_{\{0\}}^\alpha(x, y)$ to denote its α -potential densities. These densities, and in particular the 0-potential density $u_{\{0\}}(x, y)$, are finite and positive definite. See Lemma 5.1. Let $\eta = \{\eta_x; x \in S\}$ be a mean-zero Gaussian process with covariance

$$(1.5) \quad E_\eta(\eta_x \eta_y) = u_{\{0\}}(x, y).$$

(We define P_η to be the measure for η and denote the corresponding expectation operator by E_η).

The main result of this paper is the following generalization of the classical second Ray-Knight Theorem (1.1).

Theorem 1.1 *Let $X = (\Omega, \mathcal{F}_t, X_t, P^x)$ be a strongly symmetric recurrent Markov process and let $\{L_{\tau(t)}^x; x \in S\}$ be the local time process for X as defined above.*

For any $b \in R$ and $t > 0$, under the measure $P^0 \times P_\eta$,

$$(1.6) \quad \{L_{\tau(t)}^x + \frac{1}{2}(\eta_x + b)^2; x \in S\} \stackrel{law}{=} \{\frac{1}{2}(\eta_x + \sqrt{2t + b^2})^2; x \in S\}.$$

When X is a real-valued symmetric Lévy process the Gaussian process in Theorem 1.1 has stationary increments, with $\eta_0 = 0$. In particular when X is a real-valued symmetric stable process of index $\beta \in (1, 2]$, the Gaussian process is a fractional Brownian motion of index $\beta - 1$. This is shown in Section 6.

Theorem 1.1 is reminiscent of Dynkin's isomorphism theorem [3], which is used in [9], [10], [4], and [5] to obtain many interesting facts about the local time process $\{L_t^x; x \in S\}$, by exploiting properties of associated Gaussian processes. However, Dynkin's theorem refers to the total local time $\{L_\lambda^x; x \in S\}$, for the process killed at an independent exponential time λ , or

to $\{L_\infty^x; x \in S\}$ for transient processes. Because of this, Dynkin's theorem lends itself primarily to the study of properties of $\{L_t^x; x \in S\}$ which are uniform in t . Theorem 1.1 allows one to obtain results about $\{L_t^x; x \in S\}$ which may vary with t .

Theorem 1.1 is used in [1] to show that the most visited site up to time t , of a symmetric stable process, is transient. This result is extended to a larger class of Lévy processes in [8]. In [11], Theorem 1.1 is used to give a relatively simple proof of the necessary and sufficient condition for the joint continuity of the local time of symmetric Lévy processes. (Actually, the proofs in [11] are also valid for all strongly symmetric Markov processes.)

In Theorem 1.1 the Markov process X is taken to be recurrent. If it is transient then with probability one, $L_\infty^0 < \infty$. Consequently, (1.6) can not hold for transient processes. To see this note that it follows immediately from (1.6) that $L_{\tau(t)}^0 = t$ for all t , which is impossible if $t > L_\infty^0$.

We now present a generalization of (1.6) so that it also holds when X is transient. When X is transient, its 0-potential, $u(0, 0)$ is finite and, under P^0 , L_∞^0 is an exponential random variable with mean $u(0, 0)$. When X is recurrent, its 0-potential, $u(0, 0)$ is infinite. In the next theorem we define ρ to be an exponential random variable with mean $u(0, 0)$, when $u(0, 0) < \infty$ and set ρ to be identically equal to ∞ when $u(0, 0) = \infty$. Let $\tau^-(t) = \inf\{s : L_s^0 \geq t\}$, the left continuous inverse of $t \mapsto L_t^0$. Let η be as defined in (1.5) and take η and ρ to be independent.

Theorem 1.2 *Let $X = (\Omega, \mathcal{F}_t, X_t, P^x)$ be a strongly symmetric Markov process and $h_x \stackrel{\text{def}}{=} P^x(T_0 < \infty)$. For any $b \in R^1$ and $t \in (0, \infty]$, under the measure $P^0 \times P_\eta$, we have*

$$(1.7) \quad \{L_{\tau^-(t \wedge L_\infty^0)}^x + \frac{1}{2}(\eta_x + h_x b)^2; x \in S\} \\ \stackrel{\text{law}}{=} \left\{ \frac{1}{2} \left(\eta_x + h_x \sqrt{2(t \wedge \rho) + b^2} \right)^2; x \in S \right\}.$$

In particular, when X is transient

$$(1.8) \quad \{L_{\tau^-(L_\infty^0)}^x + \frac{1}{2}(\eta_x + h_x b)^2; x \in S\} \stackrel{\text{law}}{=} \left\{ \frac{1}{2} \left(\eta_x + h_x \sqrt{2\rho + b^2} \right)^2; x \in S \right\}.$$

An important tool used to prove these results is a new interpretation of a familiar formula due to M. Kac for the moment generating function of the

total accumulated local time process. This is given in Section 3. In Section 2 we present some preliminaries concerning Gaussian processes. In Section 4 we prove an isomorphism theorem closely related to Theorems 1.1 and 1.2, which facilitates their proofs. Section 5 is devoted to the proofs of Theorems 1.1 and 1.2. In Section 7 we use the Kac type formula to give an easy proof of the first Ray–Knight theorem.

To apply Theorems 1.1 and 1.2 one must obtain the 0–potential density $u_{\{0\}}(x, y)$, of the process X killed at $T_0 = \inf\{s : X_s = 0\}$, in order to identify the Gaussian process η , defined in (1.5). In Section 6 we do this when X is a real–valued symmetric Lévy process. It turns out that when X is recurrent, η has stationary increments. In Remark 6.1, we show that when X is a symmetric stable process with index $\beta \in (1, 2]$, η is a fractional Brownian motion with index $\beta - 1$.

2 Preliminaries on Gaussian processes

In Corollary 2.1 and Lemma 2.2 we establish some equivalence relationships between the sums of squares of two independent Gaussian processes which play an important role in the proof of Theorems 1.1 and 1.2. They are consequences of the following routine calculation which we provide for the convenience of the reader.

Lemma 2.1 *Let $\zeta = (\zeta_1, \dots, \zeta_n)$ be a mean zero, n -dimensional Gaussian random variable with covariance matrix Σ . Assume that Σ is invertible. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be an n -dimensional vector and Λ an $n \times n$ diagonal matrix with λ_j as its j -th diagonal entry. Let $u = (u_1, \dots, u_n)$ be an n -dimensional vector. We can choose $\lambda_i > 0, i = 1 \dots, n$ sufficiently small so that $(\Sigma^{-1} - \Lambda)$ is invertible and*

$$(2.1) \quad \begin{aligned} E \exp \left(\sum_{i=1}^n \lambda_i (\zeta_i + u_i)^2 / 2 \right) \\ = \frac{1}{(\det(I - \Sigma \Lambda))^{1/2}} \exp \left(\frac{(\gamma \cdot u)}{2} + \frac{(\gamma \tilde{\Sigma} \gamma^t)}{2} \right) \end{aligned}$$

where $\tilde{\Sigma} \stackrel{\text{def}}{=} (\Sigma^{-1} - \Lambda)^{-1}$ and $\gamma = (\lambda_1 u_1, \dots, \lambda_n u_n)$.

Proof We write

$$(2.2) \quad E \exp \left(\sum_{i=1}^n \lambda_i (\zeta_i + u_i)^2 / 2 \right)$$

$$= \exp\left(\frac{(\gamma \cdot u)}{2}\right) E \exp\left(\sum_{i=1}^n \lambda_i (\zeta_i^2/2 + u_i \zeta_i)\right)$$

and

$$(2.3) \quad \begin{aligned} & E \exp\left(\sum_{i=1}^n \lambda_i (\zeta_i^2/2 + u_i \zeta_i)\right) \\ &= \frac{1}{(\det \Sigma)^{1/2}} \int \exp\left((\gamma \cdot \zeta) - \frac{\zeta (\Sigma^{-1} - \Lambda) \zeta^t}{2}\right) d\zeta \\ &= \frac{(\det \tilde{\Sigma})^{1/2}}{(\det \Sigma)^{1/2}} \tilde{E} e^{(\gamma \cdot \xi)} \end{aligned}$$

where ξ is an n -dimensional Gaussian random variable with mean zero and covariance matrix $\tilde{\Sigma}$ and \tilde{E} is expectation with respect to the probability measure of ξ . Clearly

$$(2.4) \quad \tilde{E} e^{(\gamma \cdot \xi)} = \exp\left(\frac{(\gamma \tilde{\Sigma} \gamma^t)}{2}\right).$$

Putting these together gives us (2.1).

We have the following immediate Corollary of Lemma 2.1.

Corollary 2.1 *Let $\eta = \{\eta_x; x \in S\}$ be a mean zero Gaussian process and f_x a real valued function on S . It follows from Lemma 2.1 that for $a^2 + b^2 = c^2 + d^2$*

$$(2.5) \quad \{(\eta_x + f_x a)^2 + (\tilde{\eta}_x + f_x b)^2; x \in S\} \stackrel{law}{=} \{(\eta_x + f_x c)^2 + (\tilde{\eta}_x + f_x d)^2; x \in S\}.$$

where $\tilde{\eta}$ is an independent copy of η .

Let $G = \{G_x; x \in S\}$ be a mean zero Gaussian process with covariance $u(x, y)$. We use the standard decomposition to write $G_x = \rho_x + h_x G_0$, where $h_x = u(x, 0)/u(0, 0)$ and $\rho_x = G_x - h_x G_0$. Thus ρ_x and $h_x G_0$ are independent Gaussian processes.

Lemma 2.2 *Let \tilde{G} be an independent copy of G . Then*

$$(2.6) \quad \left\{ \frac{G_x^2}{2} + \frac{\tilde{G}_x^2}{2}; x \in S \right\} \stackrel{law}{=} \left\{ \frac{\rho_x^2}{2} + \frac{(\tilde{\rho}_x + h_x \sqrt{2T})^2}{2}; x \in S \right\}$$

where T is an exponential random variable with mean EG_0^2 , independent of G and \tilde{G} .

Proof Let E_ρ denote expectation with respect to $\{\rho_x; x \in S\}$ and E_{G_0} denote expectation with respect to G_0 . Clearly, $E_G = E_\rho E_{G_0}$. Let $\gamma = (\gamma_1, \dots, \gamma_m)$ and $\tau = (\tau_1, \dots, \tau_m)$ be two vectors. We write $\gamma \cdot \tau^2 = \sum_{i=1}^m \gamma_i \tau_i^2$. Let $\lambda = (\lambda_0, \dots, \lambda_n)$ and $\bar{\lambda} = (\lambda_1, \dots, \lambda_n)$. Let $G = (G_{x_0}, \dots, G_{x_n})$ and let $h = (h_{x_0}, \dots, h_{x_n})$, where $x_0 = 0$. Note that for all $\lambda_0, \dots, \lambda_n$ sufficiently small we have

$$(2.7) \quad E_G e^{\lambda \cdot G^2/2} E_G e^{\bar{\lambda} \cdot \tilde{G}^2/2} = E_{G_0} \left(e^{\lambda_0 h_0^2 (G_0^2 + \tilde{G}_0^2)/2} \left(E_\rho e^{\bar{\lambda} \cdot (\rho + h G_0)^2/2} \right)^2 \right)$$

where $\rho = (\rho_{x_1}, \dots, \rho_{x_n})$. Also, by Lemma 2.1

$$(2.8) \quad \left(E_\rho e^{\bar{\lambda} \cdot (\rho + h G_0)^2/2} \right)^2 = E_\rho e^{\bar{\lambda} \cdot \rho^2/2} E_\rho e^{\bar{\lambda} \cdot (\rho + h \sqrt{G_0^2 + \tilde{G}_0^2})^2/2}.$$

Combining these we see that the left hand side of (2.7)

$$(2.9) \quad = E_\rho e^{\bar{\lambda} \cdot \rho^2/2} E_{G_0} E_\rho e^{\lambda \cdot (\rho + h \sqrt{G_0^2 + \tilde{G}_0^2})^2/2}.$$

Lastly, set $E G_0^2 = 1/q$, and note that

$$(2.10) \quad \begin{aligned} E_{G_0} E_\rho e^{\lambda \cdot (\rho + h \sqrt{G_0^2 + \tilde{G}_0^2})^2/2} \\ &= \frac{q}{\pi} \int_0^\infty \int_0^\infty E_\rho e^{\lambda \cdot (\rho + h \sqrt{2(s+t)})^2/2} e^{-(s+t)q} \frac{1}{\sqrt{st}} ds dt \\ &= E e^{\lambda \cdot (\rho + h \sqrt{2T})^2/2}. \end{aligned}$$

Substituting this in (2.9) completes the proof of this lemma.

3 A Kac type formula

We give a version of Kac's formula for the moment generating function of the total accumulated local time process which enables us to easily obtain the classical first and second Ray–Knight theorems and generalize the second Ray–Knight theorem. In what follows we use the notation $A^{(1)}$ for the matrix obtained by replacing each of the n entries in the first column of the $n \times n$ matrix A by the number one.

Lemma 3.1 *Let X be a strongly symmetric Markov process with 0-potential density $u(x, y)$. Let L_∞ denote the total accumulated local time of X . Let Σ*

be the matrix with elements $\Sigma_{i,j} = u(x_i, x_j)$, $i, j = 1, \dots, n$ and assume that Σ is invertible. Let $y = x_1$. Let Λ be the matrix with elements $\{\Lambda\}_{i,j} = \lambda_i \delta_{i,j}$. For all $\lambda_1, \dots, \lambda_n$ sufficiently small we have

$$(3.1) \quad E^y \exp \left(\sum_{i=1}^n \lambda_i L_{\infty}^{x_i} \right) = \frac{\det \left((I - \Sigma \Lambda)^{(1)} \right)}{\det (I - \Sigma \Lambda)}.$$

Proof By Kac's moment formula

$$(3.2) \quad \begin{aligned} & E^y \left(\left(\sum_{j=1}^n \lambda_j L_{\infty}^{x_j} \right)^k \right) \\ &= k! \sum_{j_1, \dots, j_k=1}^n u(y, x_{j_1}) \lambda_{j_1} u(x_{j_1}, x_{j_2}) \lambda_{j_2} u(x_{j_2}, x_{j_3}) \cdots \\ & \quad u(x_{j_{k-2}}, x_{j_{k-1}}) \lambda_{j_{k-1}} u(x_{j_{k-1}}, x_{j_k}) \lambda_{j_k} \end{aligned}$$

for all k . The proof of this is standard, see e.g. [9], (4.7) or [7].

Let $\beta = (u(y, x_1) \lambda_1, \dots, u(y, x_n) \lambda_n)$ and $\bar{1}$ be the transpose of an n -dimensional vector with all of its elements equal to one. Observe that $\sum_{j_k=1}^n u(x_{j_{k-1}}, x_{j_k}) \lambda_{j_k}$ is an $n \times 1$ matrix with entries $\{\Sigma \Lambda \bar{1}\}_{j_{k-1}}$, $j_{k-1} = 1, \dots, n$.

Note also that $(\Sigma \Lambda)^2 \bar{1}$ is an $n \times 1$ matrix and

$$(3.3) \quad \sum_{j_{k-1}=1}^n u(x_{j_{k-2}}, x_{j_{k-1}}) \lambda_{j_{k-1}} \{\Sigma \Lambda \bar{1}\}_{j_{k-1}} = \{(\Sigma \Lambda)^2 \bar{1}\}_{j_{k-2}}.$$

Iterating this relationship we get

$$(3.4) \quad E^y \left(\left(\sum_{j=1}^n \lambda_j L_{\infty}^{x_j} \right)^k \right) = k! \beta (\Sigma \Lambda)^{k-1} \bar{1}$$

$$(3.5) \quad = k! \sum_{i=1}^n \{(\Sigma \Lambda)^k\}_{1,i}$$

where we use the facts that $x_1 = y$ and $\beta (\Sigma \Lambda)^{k-1}$ is an n -dimensional vector which is the same as the first row of $(\Sigma \Lambda)^k$. It follows from this that

$$(3.6) \quad E^y \exp \left(\sum_{i=1}^n \lambda_i L_{\infty}^{x_i} \right) = \sum_{i=1}^n \left\{ \sum_{k=0}^{\infty} (\Sigma \Lambda)^k \right\}_{1,i}$$

$$\begin{aligned}
&= \sum_{i=1}^n \{(I - \Sigma\Lambda)^{-1}\}_{1,i} \\
&= \frac{1}{\det(I - \Sigma\Lambda)} \sum_{i=1}^n \{\text{Adj}(I - \Sigma\Lambda)\}_{1,i}
\end{aligned}$$

Recall that for an $n \times n$ matrix $A = \{a_{i,j}\}$,

$$\det A = \sum_{j=1}^n (-1)^{1+j} a_{j,1} \det A_{\{j\},\{1\}},$$

where $A_{\{j\},\{1\}}$ is the matrix obtained by deleting the j -th row and 1-st column of A . Also, $\{\text{Adj} A\}_{1,j} = (-1)^{1+j} \det A_{\{j\},\{1\}}$. Consequently

$$(3.7) \quad \sum_{j=1}^n \{\text{Adj} A\}_{1,j} = \sum_{j=1}^n (-1)^{1+j} \det A_{\{j\},\{1\}} = \det(A^{(1)}).$$

(3.1) now follows from (3.6) and (3.7). This completes the proof of this lemma.

Remark 3.1 The 0-potential $u(x, y)$ in Lemma 3.1 is positive definite, see e.g. [9], Theorem 3.3. Let $G = \{G(x); x \in S\}$ be a mean zero Gaussian process with covariance $u(x, y)$. Note that $(\det(I - \Sigma\Lambda))^{-1}$ is the moment generating function of $G_n^2/2 + \tilde{G}_n^2/2$, where $G_n = (G_{x_1}, \dots, G_{x_n})$, and similarly for \tilde{G}_n . (See Lemma 2.1). Suppose that we can find a stochastic process $\{H_x; x \in S\}$ such that $(\det((I - \Sigma\Lambda)^{(1)}))^{-1}$ is the moment generating function of H_{x_1}, \dots, H_{x_n} , and suppose this can be done for all finite joint distributions. Then we would have that under the measure $P_y \times P_H \times P_G \times P_{\tilde{G}}$

$$(3.8) \quad \{L_\infty^x + H_x; x \in S\} \stackrel{\text{law}}{=} \{G_x^2/2 + \tilde{G}_x^2/2; x \in S\}.$$

4 A related isomorphism theorem

In this section we prove an isomorphism theorem closely related to Theorems 1.1 and 1.2 which facilitates their proof. Essentially, for the proofs of Theorems 1.1 and 1.2, we will only have to verify that the conditions of this section apply.

Let Z is a strongly symmetric Markov process with state space S and 0-potential density $v(x, y)$ which can be written in the following form:

$$(4.1) \quad v(x, y) = v_*(x, y) + 1/q$$

where, $q > 0$ and $v_*(x, y)$ is positive definite and satisfies $v_*(0, y) \equiv 0$ for all $y \in S$. Let $\{\eta_x; x \in S\}$ be the mean zero Gaussian process with covariance $v_*(x, y)$. We have the following isomorphism theorem.

Theorem 4.1 *Let Z be a strongly symmetric Markov process with 0-potential density $v(x, y)$ satisfying (4.1), and let $\{L_\infty^x; x \in S\}$ denote the total accumulated local time of Z . Then under the measure $P_0 \times P_\eta$*

$$(4.2) \quad \{L_\infty^x + (\eta_x + b)^2/2; x \in S\} \stackrel{law}{=} \{(\eta_x + \sqrt{2T + b^2})^2/2; x \in S\}.$$

for all $b \in R$, where T is an exponential random variable with mean $1/q$.

Proof Set

$$(4.3) \quad \phi_x = \eta_x + G_0$$

where $G_0 \sim N(0, 1/q)$ so that $\{\phi_x; x \in E\}$ is the mean zero Gaussian process with covariance function $v(x, y)$.

Let Σ be the covariance matrix of $\phi_{x_1}, \dots, \phi_{x_n}$ where $x_1 = 0$ and x_2, \dots, x_n are arbitrary. Note that $\{\Sigma\}_{i,j} = v(x_i, x_j)$ and in particular $\{\Sigma\}_{j,1} = v(x_j, 0) = 1/q$. Consider $\det((I - \Sigma\Lambda)^{(1)})$. Subtract the first row of the matrix in this determinant from each of the other rows. Using this row operation it is clear that

$$(4.4) \quad \det((I - \Sigma\Lambda)^{(1)}) = \det(I - \tilde{\Sigma}\tilde{\Lambda})$$

where $\tilde{\Sigma}$ is the covariance matrix of $\eta_{x_2}, \dots, \eta_{x_n}$ and $\tilde{\Lambda}$ is the natural restriction of Λ , i.e. $\{\tilde{\Lambda}\}_{i,j} = \lambda_{i+1}\delta_{i,j}$, $1 \leq i, j \leq n-1$. That is, the reciprocal of (4.4) is the moment generating function of $(1/2)(\eta_{x_1}^2, \dots, \eta_{x_n}^2) + (1/2)(\tilde{\eta}_{x_1}^2, \dots, \tilde{\eta}_{x_n}^2)$, where $\tilde{\eta}$ is an independent copy of η . (Note that we can include $\eta_{x_1}^2$ since it is equal to 0). Taking into account the comments made in Remark 3.1 we see that we have obtained

$$(4.5) \quad \{L_\infty^x + \eta_x^2/2 + \tilde{\eta}_x^2/2; x \in S\} \stackrel{law}{=} \{\phi_x^2/2 + \tilde{\phi}_x^2/2; x \in S\}$$

under the measure $P_0 \times P_\phi \times P_{\tilde{\phi}}$.

To obtain (4.2) we note that it follows from Lemma 2.2 that

$$(4.6) \quad \{\phi_x^2/2 + \tilde{\phi}_x^2/2; x \in S\} \stackrel{law}{=} \{\tilde{\eta}_x^2/2 + (\eta_x + \sqrt{2T})^2/2; x \in S\}$$

since by assumption $h_x = v(x, 0)/v(0, 0) \equiv 1$. Substituting this in (4.5) we get

$$(4.7) \quad \{L_\infty^x + \eta_x^2/2 + \tilde{\eta}_x^2/2; x \in S\} \stackrel{law}{\equiv} \{\tilde{\eta}_x^2/2 + (\eta_x + \sqrt{2T})^2/2; x \in S\}.$$

which implies that

$$(4.8) \quad \{L_\infty^x + \eta_x^2/2; x \in S\} \stackrel{law}{\equiv} \{(\eta_x + \sqrt{2T})^2/2; x \in S\}.$$

Since T is independent of the Gaussian processes it follows from Corollary 2.1 that

$$\{(\eta_x + \sqrt{2T})^2/2 + (\tilde{\eta}_x + b)^2/2; x \in S\} \stackrel{law}{\equiv} \{\eta_x^2/2 + (\tilde{\eta}_x + \sqrt{2T + b^2})^2/2; x \in S\}.$$

Thus we see that adding $(\tilde{\eta}_x + b)^2/2$ to each side of (4.8) gives us (4.2).

5 Proofs of Theorems 1.1–1.2

Let X be a strongly symmetric Markov process. Two Markov processes determined by X play an important role in these proofs. The first is the process $Y = \{Y_t; t \in R^+\}$, which is X killed when it first hits 0. This process is introduced prior to the statement of Theorem 1.1. To define the second, let $T = T(q)$ be an exponential random variable with mean $1/q$. We define $Z = \{Z_t; t \in R^+\}$ by

$$(5.1) \quad Z_t = \begin{cases} X_t & \text{if } t < \tau(T) \\ \Delta & \text{otherwise} \end{cases}$$

where $\tau(t)$ is defined in (1.4).

Recall that $u_{\{0\}}(x, y)$ denotes the 0-potential density of Y ; consequently $u_{\{0\}}(0, y) \equiv 0$, for all $y \in S$.

Lemma 5.1 *Both Y and Z are strongly symmetric Markov processes. If X is recurrent and $\tilde{u}(x, y)$ denotes the 0-potential density of Z , then*

$$(5.2) \quad \tilde{u}(x, y) = u_{\{0\}}(x, y) + 1/q.$$

Furthermore, both $u_{\{0\}}(x, y)$ and $\tilde{u}(x, y)$ are finite, symmetric and positive-definite.

Proof of Lemma 5.1: The fact that Y is strongly symmetric follows easily from Hunt's switching identity, see e.g. [9], Section 3, where the reader will also find a proof that $u_{\{0\}}(x, y)$ is positive-definite. (Although the results of that reference are stated for the process killed on exiting a compact set, the proofs hold just as well for $u_{\{0\}}(x, y)$). The finiteness of $u_{\{0\}}(x, y)$ follows from the finiteness of $u^1(x, y)$ and the well-known fact that $L_{T_0}^x$ is exponential under P^x , hence $L_{T_0}^x$ would be infinite a.s. if $u_{\{0\}}(x, x) = E^x(L_{T_0}^x)$ were infinite. The strong symmetry of Z follows from more general results in [12] or [14]. Finally, to prove (5.2) we use the fact that $\tau(T) = T_0 + \tau(T) \circ \theta_{T_0}$ and the strong Markov property at T_0 to see that

$$\begin{aligned}
(5.3) \quad \tilde{u}(x, y) &= E^x(L_{\tau(T)}^y) = E^x(L_{T_0}^y) + E^0(L_{\tau(T)}^y) \\
&= E^x(L_{T_0}^y) + E^y(L_{\tau(T)}^0) = E^x(L_{T_0}^y) + E^y(T) \\
&= u_{\{0\}}(x, y) + 1/q
\end{aligned}$$

where the third equality uses the symmetry of $\tilde{u}(x, y)$ in the form

$$E^0(L_{\tau(T)}^y) = \tilde{u}(0, y) = \tilde{u}(y, 0) = E^y(L_{\tau(T)}^0).$$

This completes the proof of Lemma 5.1.

Proof of Theorem 1.1: Given X , with local time process $\{L_t^x; x \in S\}$, we consider the associated Markov process Z defined in (5.1) with 0-potential $\tilde{u}(x, y)$. By Lemma 5.1, Z satisfies the conditions of Theorem 4.1. Also, the total accumulated local time process for Z is precisely $\{L_{\tau(T)}^x; x \in S\}$. Hence by Theorem 4.1, under the measure $P_0 \times P_\eta$

$$(5.4) \quad \{L_{\tau(T)}^x + (\eta_x + b)^2/2; x \in S\} \stackrel{\text{law}}{=} \{(\eta_x + \sqrt{2T + b^2})^2/2; x \in S\}$$

for all $b \in R$. (1.6) follows from this by taking the Laplace transform.

Proof of Theorem 1.2: Suppose now that the Markov process X is transient. Let $u(x, y)$ denote its 0-potential, which is finite. Under P^0 , the total accumulated local time of X , L_∞^0 has an exponential distribution with mean $u(0, 0)$. Let ρ to be an independent exponential random variable with mean $u(0, 0)$. Consider the Markov processes Y and Z defined in (5.1), for the transient process X .

Lemma 5.2 *The potential density of the Markov process Z with respect to m is given by*

$$(5.5) \quad \tilde{u}(x, y) = u_{\{0\}}(x, y) + h_x h_y \frac{1}{q'}$$

where $h_x = P^x\{T_0 < \infty\} = \frac{u(x,0)}{u(0,0)}$ and $\frac{1}{q'} = E(T \wedge \rho)$. Both $\tilde{u}(x, y)$ and $u_{\{0\}}(x, y)$ are symmetric and positive definite and $u_{\{0\}}(0, y) \equiv 0$, for all $y \in S$.

Proof of Lemma 5.2: The fact that the potential of Z is absolutely continuous with respect to m , is symmetric and positive definite is the content of Lemma 5.1. Let $\tilde{u}(x, y)$ be its density with respect to m . Then

$$\begin{aligned}\tilde{u}(x, y) &= u_{\{0\}}(x, y) + P^x\{T_0 < \infty\}\tilde{u}(0, y) \\ &= u_{\{0\}}(x, y) + P^x\{T_0 < \infty\}\tilde{u}(y, 0) \\ &= u_{\{0\}}(x, y) + P^x\{T_0 < \infty\}P^y\{T_0 < \infty\}\tilde{u}(0, 0),\end{aligned}$$

where the second equality follows from the symmetry of \tilde{u} . We now recall that $\tilde{u}(0, 0) = E^0(L_{\tau(T)}^0)$, and that $L_{\tau(T)}^0$ is equal to the minimum of the two exponential random variables, T and L_∞^0 . This completes the proof of Lemma 5.2.

Note that Lemma 5.2 is actually a generalization of Lemma 5.1 since when X is recurrent $h_x = P^x\{T_0 < \infty\} \equiv 1$.

Proof of Theorem 1.2 continued: Let A_t be a continuous additive functional of X and let $R_A(\omega) \stackrel{def}{=} \inf\{t \mid A_t(\omega) > 0\}$. We call A_t a local time of X at $x \in S$ if $P^x(R_A = 0) = 1$, and $P^y(R_A = 0) = 0$ for all $y \neq x$. If A_t and B_t are two local times for X at $x \in S$, then $A_t = cB_t$ for some constant $c > 0$. It is easy to check that L_t^x defined in (1.3) is a local time of X at x with potential

$$(5.6) \quad E^y \left(\int_0^\infty e^{-\alpha t} dL_t^x \right) = u^\alpha(y, x), \quad \forall \alpha > 0.$$

We call L_t^x the normalized local time of X at x . However, note that for any real valued function $g(x) > 0$, $x \in S$, $g(x)L_t^x$ is also a local time for X at x , which we call a non-normalized local time of X at x , (unless $g(x) \equiv 1$).

Dividing (5.5) by $h_x h_y$ we get

$$(5.7) \quad \frac{\tilde{u}(x, y)}{h_x h_y} = \frac{u_{\{0\}}(x, y)}{h_x h_y} + \frac{1}{q'}.$$

It is easy to see that the function h_x is excessive for X, Y and Z . Let X^h be an h -path transform of X taken with respect to h_x and let $P^{x/h}$ denote the measure corresponding to X^h . Denote by Y^h and Z^h the corresponding

h -path transforms of Y and Z . Note that $\frac{u(x,y)}{h_x h_y}$, $\frac{u_{\{0\}}(x,y)}{h_x h_y}$ and $\frac{\tilde{u}(x,y)}{h_x h_y}$ are the potential densities of X^h , Y^h and Z^h respectively, all with respect to $h_x^2 m(dx)$. Clearly they are all symmetric. Let \hat{L}_t^y be the normalized local time at y of X^h with respect to $h_x^2 m(dx)$, so that

$$(5.8) \quad E^{x/h} \int_0^\infty e^{-\alpha t} d\hat{L}_t^y = \frac{u^\alpha(x,y)}{h_x h_y}.$$

Consider $\{\eta_x/h_x; x \in S\}$. This is a mean zero Gaussian process with covariance $\frac{u_{\{0\}}(x,y)}{h_x h_y}$. By (5.7), Z^h satisfies the conditions of Theorem 4.1. Also, the total accumulated local time process for Z^h is precisely $\{\hat{L}_{\tau(T)}^x; x \in S\}$. Hence by Theorem 4.1, under $P^{0/h} \times P_\eta$

$$(5.9) \quad \left\{ \hat{L}_{\tau(T)}^x + \frac{(\eta_x/h_x + b)^2}{2}; x \in S \right\} \stackrel{law}{=} \left\{ \frac{(\eta_x/h_x + \sqrt{2V + b^2})^2}{2}; x \in S \right\}$$

where V is an exponential random variable with parameter q' and T is an exponential random variable with parameter q . Equivalently, under $P^{0/h} \times P_\eta$

$$(5.10) \quad \left\{ \tilde{L}_{\tau(T)}^x + \frac{(\eta_x + h_x b)^2}{2}; x \in S \right\} \stackrel{law}{=} \left\{ \frac{(\eta_x + h_x \sqrt{2V + b^2})^2}{2}; x \in S \right\}$$

where $\tilde{L}_t^x = h_x^2 \hat{L}_t^x$. $\tilde{L} \stackrel{def}{=} \{\tilde{L}_t^x; x \in S\}$ is a non-normalized local time for X^h . Its potential with respect to the process X^h is

$$(5.11) \quad E^{x/h} \int_0^\infty e^{-\alpha t} h^2(y) d\hat{L}_t^y = \frac{u^\alpha(x,y) h_y^2}{h_x h_y} = \frac{u^\alpha(x,y) h_y}{h_x}$$

which is the α -potential of X^h with respect to m . This is precisely the potential of L with respect to the process X^h , where $L \stackrel{def}{=} \{L_t^x; x \in S\}$ is the normalized local time for X . Since X^h is X killed at the last exit from 0, conditioned to hit 0 after time 0, we see that $\tilde{L}_t^x = L_t^x$ until the last exit of X from 0. Let $\tau_t^- \stackrel{def}{=} \inf\{s : L_s^0 \geq t\}$. It follows that $\tilde{L}_{\tau(T)}^x = L_{\tau^-(T \wedge L_\infty^0)}^x$. Therefore (5.10) is equivalent to the statement that under $P^0 \times P_\eta$

$$\left\{ L_{\tau^-(T \wedge L_\infty^0)}^x + \frac{(\eta_x + h_x b)^2}{2}; x \in S \right\} \stackrel{law}{=} \left\{ \frac{(\eta_x + h_x \sqrt{2(T \wedge \rho) + b^2})^2}{2}; x \in S \right\}$$

where ρ is an exponentially distributed random variable with mean $u(0,0)$ independent of T . Since this is the Laplace transform of (1.7), the proof of Theorem 1.2 is completed.

6 $u_{\{0\}}(x, y)$ for symmetric Lévy processes

To apply Theorems 1.1 and 1.2 one must obtain the 0-potential density $u_{\{0\}}(x, y)$, of the process X killed at $T_0 = \inf\{s : X_s = 0\}$, in order to identify the Gaussian process η , defined in (1.5). In this section we do this when X is a real-valued symmetric Lévy processes.

Let X be a real-valued symmetric Lévy process defined by

$$(6.1) \quad Ee^{i\lambda X_t} = e^{-t\psi(|\lambda|)}.$$

We assume that X has a local time, which is equivalent to the condition that $1/(1 + \psi(|\lambda|)) \in L^1$. It follows simply from (6.1) that the α -potential density of X

$$(6.2) \quad u^\alpha(x, y) = \frac{1}{\pi} \int_0^\infty \frac{\cos(\lambda(x - y))}{\alpha + \psi(\lambda)} d\lambda.$$

Since, in this case, $u^\alpha(x, y)$ depends only on $|x - y|$, we also write it as $u^\alpha(x - y)$.

In our presentation we distinguish between whether X is transient or recurrent. Consider the integral

$$(6.3) \quad \int_0^1 \frac{1}{\psi(\lambda)} d\lambda.$$

The Lévy process X is transient when this integral is finite and is recurrent when this integral is infinite.

Theorem 6.1 *When X is recurrent*

$$(6.4) \quad u_{\{0\}}(x, y) = \phi(x) + \phi(y) - \phi(x - y)$$

where

$$(6.5) \quad \phi(x) \stackrel{\text{def}}{=} \frac{1}{\pi} \int_0^\infty \frac{1 - \cos \lambda x}{\psi(\lambda)} d\lambda \quad x \in R.$$

Consequently the Gaussian process η defined in (1.5) is such that $\eta(0) = 0$ and

$$(6.6) \quad E(\eta(x) - \eta(y))^2 = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\lambda(x - y))}{\psi(\lambda)} d\lambda.$$

When X is transient we can define the stationary Gaussian process $\{G_x, x \in R\}$ as the process with covariance $u(x, y) \stackrel{\text{def}}{=} u^0(x, y)$, given in (6.2). Let ρ_x

be the projection of G_x in the subspace of L^2 orthogonal to G_0 . That is, we write $G_x = \rho_x + h_x G_0$, where $h_x = P^x\{T_0 < \infty\} = u(x, 0)/u(0, 0)$ and $\rho_x = G_x - h_x G_0$. Thus ρ_x and $h_x G_0$ are independent Gaussian processes. In this case

$$(6.7) \quad u_{\{0\}}(x, y) = E(\rho_x \rho_y)$$

$$(6.8) \quad = u(x, y) - \frac{u(x, 0)u(y, 0)}{u(0, 0)}.$$

Proof It follows from (6.2) that

$$(6.9) \quad u^\alpha(0) - u^\alpha(x) = \frac{1}{\pi} \int_0^\infty \frac{1 - \cos(\lambda x)}{\alpha + \psi(\lambda)} d\lambda \\ \stackrel{def}{=} \phi_\alpha(x)$$

Consequently

$$(6.10) \quad \lim_{\alpha \rightarrow 0} (u^\alpha(0) - u^\alpha(x)) = \frac{1}{\pi} \int_0^\infty \frac{1 - \cos(\lambda x)}{\psi(\lambda)} d\lambda \\ = \phi(x).$$

Using the Markov property at time T_0 shows that

$$(6.11) \quad u^\alpha(x, y) = E^x \left(\int_0^\infty e^{-\alpha t} dL_t^y \right) \\ = E^x \left(\int_0^{T_0} e^{-\alpha t} dL_t^y \right) + E^x \left(\int_{T_0}^\infty e^{-\alpha t} dL_t^y \right) \\ = u_{\{0\}}^\alpha(x, y) + E^x(e^{-\alpha T_0}) u^\alpha(0, y).$$

It follows from (6.11) that

$$(6.12) \quad u_{\{0\}}^\alpha(x, y) - u_{\{0\}}^\alpha(x, 0) \\ = u^\alpha(x - y) - u^\alpha(x) - \{u^\alpha(y) - u^\alpha(0)\} E^x(e^{-\alpha T_0}) \\ = \phi_\alpha(x) + \phi_\alpha(y) E^x(e^{-\alpha T_0}) - \phi_\alpha(x - y).$$

The assumption that X is recurrent means that $P^x(T_0 < \infty) = 1$. Consequently, $\lim_{\alpha \rightarrow 0} E^x(e^{-\alpha T_0}) = 1$. Also, clearly, $u_{\{0\}}^\alpha(x, y)$ tends to $u_{\{0\}}(x, y)$ as α decreases to zero. Therefore we can take the limits in (6.12) and use the fact that $u_{\{0\}}^\alpha(x, 0) = 0$ to obtain (6.4). Since

$$(6.13) \quad E(\eta(x) - \eta(y))^2 = u_{\{0\}}(x, x) + u_{\{0\}}(y, y) - 2u_{\{0\}}(x, y)$$

we get (6.6).

Now suppose that X is transient. Since the 0-potential of X exists in this case we can simply take the limit $\alpha \rightarrow 0$ in (6.11) to get

$$(6.14) \quad u_{\{0\}}(x, y) = u(x, y) - P^x\{T_0 < \infty\} u^\alpha(0, y).$$

Using $P^x\{T_0 < \infty\} = u(x, 0)/u(0, 0)$ we get (6.7).

Remark 6.1 *When X is a symmetric stable process with index $\beta \in (1, 2]$, $\psi(|\lambda|) = |\lambda|^\beta$. Using this in (6.5) and making the change of variables $\lambda|x - y| = s$, we see that*

$$(6.15) \quad E(\eta(x) - \eta(y))^2 = c_\beta |x - y|^{\beta-1}$$

where

$$(6.16) \quad c_\beta = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos s}{s^\beta} ds.$$

7 First Ray–Knight Theorem

We show that the first Ray–Knight theorem is an immediate consequence of our version of Kac’s moment formula.

Theorem 7.1 *(First Ray–Knight Theorem) Let B be Brownian motion on R^+ , starting from $y > 0$ and let $\{L_t^x; (x, t) \in R^+ \times R^+\}$ denote its local time process. Let T_0 denote the first hitting time of 0. Then, under the measure $P^y \times \tilde{P}_1^0 \times \tilde{P}_2^0$*

$$(7.1) \quad \{L_{T_0}^x; 0 \leq x \leq y\} \stackrel{\text{law}}{=} \{\tilde{B}_{1,x}^2 + \tilde{B}_{2,x}^2; 0 \leq x \leq y\}$$

where $\{\tilde{B}_{i,x}; x \geq 0\}$, $i = 1, 2$ are independent real valued Brownian motions with measures \tilde{P}_i^0 , independent of the original Brownian motion B .

Proof We apply Lemma 3.1 to the process Y which is Brownian motion, starting at $y > 0$ and killed the first time it hits zero. In this case L_{T_0} is the total accumulated local time of Y . The 0-potential of Y is $u_{\{0\}}(x, y) = 2(|x| \wedge |y|)$ when $xy \geq 0$ and $u_{\{0\}}(x, y) = 0$ when $xy < 0$. This is well known and is also included in Theorem 6.1. We take $y = x_1 > \cdots > x_n > 0$ and $\Sigma_{i,j} = u_{\{0\}}(x_i, x_j)$ in Lemma 3.1. In the next paragraph we show that $\det((I - \Sigma\Lambda)^{(1)}) = 1$, which implies that H_x in Remark 3.1 is identically zero.

Consequently (7.1) follows from (3.8) since the Gaussian process G in (3.8), with covariance $2(x \wedge y)$ for $x, y > 0$, is equal in law to $\sqrt{2}B$.

Let D be the matrix obtained by subtracting the first row of $(I - \Sigma\Lambda)^{(1)}$ from each of the other rows. Clearly, $D_{j,j} = 1$, $j = 1, \dots, n$; $D_{j,1} = 0$, $j = 2, \dots, n$; and $D_{j,k} = 0$, $k = j + 1, \dots, n$. This shows that the matrix $\widetilde{D} \stackrel{\text{def}}{=} \{D_{i,j}\}_{i,j=2}^n$ is a lower triangular matrix, i.e., all its entries above the diagonal are equal to zero. Thus $\det((I - \Sigma\Lambda)^{(1)}) = \det(D) = D_{1,1} \det(\widetilde{D}) = 1$. This completes the proof of this theorem.

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