Long excursions of a random walk

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Summary. In [1] and [9] it was proved that the length of the longest excursion among the first n excursions of a plane random walk is nearly equal to the total sum of the lengths of these excursions. In this paper several results are proved in the same spirit, for plane random walks and for random walks in higher dimensions.

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1. Introduction

Consider a simple symmetric random walk on the lattice $\mathbb{Z}^2$. This means that if the moving particle is in $x \in \mathbb{Z}^2$ at the moment n then at the moment $n+1$ the particle moves with equal probabilities to any of the 4 neighbours of x independently of how the particle achieved x. Let S_n be the location of the particle after n steps and assume that $S_0 = 0$. Introduce the following notations:

(i) $\xi(x,n) := \#\{k:\ 0 < k \le n,\ S_k = x\},\ (x \in \mathbb{Z}^2,\ n = 1,2,\ldots)$

(ii) $\xi(x,t)$ $(x \in \mathbb{Z}^2,\ t \ge 0)$ is the continuous process obtained by linear interpolation from $\xi(x,n)$ with $\xi(x,0) = 0$,

(iii) $\rho_0 := 0$,
$\rho_k := \min\{n:\ n > \rho_{k-1},\ S_n = 0\}, \qquad k \ge 1,$

(iv) $r(k) := \rho_k - \rho_{k-1}$,

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(v) $M_n(1) \geq M_n(2) \geq \ldots \geq M_n(\xi(0,n)+1)$ are the order statistics of the sequence

$$r(1), r(2), \ldots, r(\xi(0,n)), n - \rho_{\xi(0,n)},$$

(vi) $\mathcal{H}$ is the set of functions

$$h(m_1, m_2, \tau, x) = \begin{cases} m_1 & \text{if} \quad 0 \leq x < \tau, \\ m_2 & \text{if} \quad \tau \leq x \leq 1, \end{cases}$$

where $0 \leq m_1 \leq m_2 \leq 1,\ 0 \leq \tau \leq 1$.

In words, $\xi(x,n)$ counts the number of visits to the site x during the first n steps and is called the local time of the random walk; ρ_n, which denotes the time of the n-th return to zero, is the "inverse" of $\xi(0,k)$ in some sense. The sections between two consecutive returns are called excursions and so $\{r(k),\ k = 1, 2, \ldots\}$ denotes the sequence of the lengths of the excursions. Clearly, the sequence $\{M_n(1), \ldots\}$ denotes the ordered lengths of the excursions. In this paper we study some properties of the "long" excursions, which in spite of the recurrence of the random walk behaves quite differently from the case of dimension 1.

In [1] we proved the following Strassen type

THEOREM A. *With probability one, the set of limit points of the sequence*

$$\left\{ \frac{\pi \xi(0, nt)}{(\log n) \log_3 n},\ 0 \leq t \leq 1 \right\}$$

is $\mathcal{H}$.

Here and in what follows, $\log n := \log \max(n, e)$, and $\log_p n$ is the p-th iteration of log.

The proof of Theorem A is based on the following

LEMMA A.

$$\lim_{n \to \infty} \frac{M_n(1) + M_n(2)}{n} = 1 \quad \text{a.s.}$$

We note that by the proof of Theorem A (cf. [1]) we easily get the following

THEOREM 1. *With probability one, the set of limit points of the sequence*

$$\left\{ \frac{\xi(0, nt)}{\xi(0, n)},\ 0 \leq t \leq 1 \right\}$$

is $\mathcal{H}$.

Lemma A easily implies

PROPOSITION A. $\rho_{n+1} - \rho_n$ *is either much smaller or much bigger than* ρ_n.

In order to give a more accurate form of Proposition A, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of positive numbers with

$$\sum_{n=1}^{\infty} \frac{1}{n\alpha_n} < \infty, \qquad \alpha_n \geq \log_2 n, \tag{1.1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n\beta_n} = \infty, \qquad \inf_{n\geq 1} \beta_n > 0. \tag{1.2}$$

Further let

$$a_n = \exp\left(\frac{n}{\alpha_n}\right), \quad b_n = \exp\left(\frac{n}{\beta_n}\right), \quad c_n = \exp\left(\frac{n}{(1+\varepsilon)\beta_n}\right) \ (\varepsilon > 0). \tag{1.3}$$

Finally let

$$T_n = \frac{\rho_{n+1} - \rho_n}{\rho_n}.$$

Then we have

THEOREM 2.

(i) $$T_n \notin (a_n^{-1}, a_n) \qquad \text{a.s.}$$

for all but finitely many n,

(ii) $$T_n \in (b_n^{-1}, c_n^{-1}) \quad \text{i.o. a.s.}$$

(iii) $$T_n \in (c_n, b_n) \quad \text{i.o. a.s.}$$

EXAMPLE 1.

$$\alpha_n = (\log n)^{1+\delta} \quad (\delta > 0), \quad \beta_n = \log n$$

i.e.,

$$a_n = \exp\left(\frac{n}{(\log n)^{1+\delta}}\right), \quad b_n = \exp\left(\frac{n}{\log n}\right), \quad c_n = \exp\left(\frac{n}{(1+\varepsilon)\log n}\right)$$

satisfy the above conditions.

Theorem 2 clearly implies:

COROLLARY 1. *Let*

$$T_n^* = \frac{\rho_{n+1} - \rho_n}{\rho_{n+1}} \qquad (n = 0, 1, 2, \ldots).$$

Then

$$T_n^* \notin (a_n^{-1}, 1 - a_n^{-1}) \qquad \text{a.s.,}$$

for all but finitely many n,

$$T_n^* \in (0, c_n^{-1}), \qquad \text{i.o. a.s.}$$
$$T_n^* \in (1 - c_n^{-1}, 1) \qquad \text{i.o. a.s.}$$

Lemma A also implies that $\rho_{n+k} - \rho_n$ for any $k = 1, 2, \ldots$, is either much bigger or much smaller than ρ_n. A refined form of this statement is given in the next corollary.

COROLLARY 2. *Let $\{a_n\}$ be defined as above and assume that $\{a_n\}$ is a non-decreasing sequence. Further let*

$$\begin{aligned} a_n^* &= \exp\left(\sum_{j=n}^{\infty} a_j^{-1}\right) - 1, \\ T_n^{(k)} &= \frac{\rho_{n+k} - \rho_n}{\rho_n}, \quad (k = 1, 2, \ldots) \\ B_n &= \bigcup_{k=1}^{\infty} \{T_n^{(k)} \in (a_n^*, a_n)\}. \end{aligned}$$

Then among the events B_n only finitely many might occur with probability 1.

Note that if

$$a_n = \exp\left(\frac{n}{(\log n)^{1+\delta}}\right) \qquad (\delta > 0)$$

then

$$a_n^* \le \exp\left(-\frac{n}{(\log n)^{1+2\delta}}\right).$$

The above results also imply that the waiting time

$$\nu(n) = \min\{k : \ k \ge n, \ \rho_{k+1} - \rho_k \ge \rho_n\}$$

is finite with probability 1 for any finite n. In fact we prove

THEOREM 3. *Let $\{f_n\}$ be a positive sequence such that $n \mapsto f_n/n$ is non-decreasing, then*

$$\limsup_{n\to\infty} \frac{\nu(n)}{f_n} = \begin{cases} 0 \\ \infty \end{cases}, \text{ a.s.} \iff \sum_n \frac{1}{f_n} \begin{cases} < \infty \\ = \infty \end{cases}.$$

Corollary 1 claims that T_n^* is either nearly 0 or nearly 1. At the same time Theorem 3 suggests that between n and $2n$ there exists about 1 integer k for which T_k^* is nearly 1. This idea is formulated in our following

THEOREM 4.

(i) $$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=0}^{n} T_k^* = 1 \qquad \text{a.s.},$$

(ii) $$\lim_{n\to\infty} \mathbf{P}\left\{\frac{\sum_{k=0}^n T_k^* - \log n}{\sqrt{\log n}} \le x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du,$$

(iii) $$\limsup_{n\to\infty} \frac{\sum_{k=0}^n T_k^* - \log n}{\sqrt{2(\log n)(\log_3 n)}} = 1 \qquad \text{a.s.}$$

Now consider a simple symmetric random walk on the lattice $\mathbb{Z}^d$, which means that at each step, the moving particle visits each of its $(2d)$ neighbours (on $\mathbb{Z}^d$) with equal probability $1/(2d)$. Let S_n be the location of the particle after n steps and assume that $S_0 = 0$. Since the random walk in $\mathbb{Z}^d$, $d \geq 3$, is transient, the above results cannot be true. However if we consider the longest excursion away from some $x \in \mathbb{Z}^d$ completed by the time n, then it can be long. For $i \geq 0$, define the random variable $\kappa(i)$ by

$$S_{i+j} \neq S_i, \ j = 1, 2, \ldots, \kappa(i) - 1, \quad S_{i+\kappa(i)} = S_i.$$

(If such $\kappa(i)$ does not exist, we set $\kappa(i) := \infty$). Let

$$R(n) := \max\{\kappa(i) : \ i + \kappa(i) \leq n\},$$

which in words denotes the length of the longest completed excursion (away from any point) at time n.

THEOREM 5. *Let $d \geq 3$. With probability one,*

$$\lim_{n\to\infty} \frac{\log R(n)}{\log n} = \begin{cases} 1 & \text{if} \quad d = 3, 4, \\ \frac{2}{d-2} & \text{if} \quad d \geq 5. \end{cases}$$

2. Preliminary results

THEOREM B. ([2]) *For any $\delta > 0$ we have*

$$\exp\left((1-\delta)\frac{\pi n}{\log_2 n}\right) \leq \rho_n \leq \exp(n(\log n)^{1+\delta}) \quad \text{a.s.},$$

for all but finitely many n. When n goes to infinity,

$$\mathbf{P}\{\rho_1 > n\} = \mathbf{P}\{\xi(0, n) = 0\} = \pi(\log n)^{-1} + O((\log n)^{-2}),$$

and

$$\mathbf{P}\{\xi(0, n) \geq x \log n\} = e^{-\pi x}(1 + O((\log n)^{-1/4}))$$

uniformly for $x \in (0, (\log n)^{3/4})$.

Since $\{\rho_n \leq k\} = \{\xi(0, k) \geq n\}$, Theorem B implies

PROPOSITION B. *For any $2 \leq m < n$ we have*

$$\left|\mathbf{P}\{m < \rho_1 < n\} - \frac{\pi \log(n/m)}{(\log n)\log m}\right| \leq C\left(\frac{1}{(\log m)^2} + \frac{1}{(\log n)^2}\right)$$

where $C > 0$ is an absolute constant. Furthermore,

$$\mathbf{P}\{\rho_n \le e^{n/z}\} = e^{-\pi z}(1 + O(n^{-1/7}))$$

uniformly for $z \in (0, n^{3/7})$ and

$$\mathbf{P}\{\rho_n > e^{n/z}\} \le C(1 - e^{-\pi z}) \le C\pi z, \quad z \in [n^{-1/7}, 1].$$

Similarly for any $0 \le z_1 < z_2 < n^{3/7}$ and $z_2 - z_1 \le 1$ we have

$$\left| \mathbf{P}\left\{ \exp\left(\frac{n}{z_2}\right) \le \rho_n < \exp\left(\frac{n}{z_1}\right) \right\} - \pi e^{-\pi z_1}(z_2 - z_1) \right| \le \frac{C}{n^{1/7}}.$$

In the proof of Theorem 4 we use results of Rényi [6], [7] concerning extreme elements. Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. random variables having continuous distribution. ξ_k is called an extreme element if $\xi_k > \max_{1 \le j \le k-1} \xi_j$. By convention, ξ_1 is counted as an extreme element. Let μ_n be the number of extreme elements among $\xi_1, \ldots, \xi_n$.

THEOREM C. ([6], [7])

(i) $$\lim_{n\to\infty} \frac{\mu_n}{\log n} = 1 \quad \text{a.s.},$$

(ii) $$\lim_{n\to\infty} \mathbf{P}\left\{ \frac{\mu_n - \log n}{\sqrt{\log n}} \le x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du,$$

(iii) $$\limsup_{n\to\infty} \frac{\mu_n - \log n}{\sqrt{2(\log n)(\log_3 n)}} = 1 \quad \text{a.s.}$$

In the proof of Theorem 5, we need the following two results concerning the simple symmetric random walk on $\mathbb{Z}^d$ $(d \ge 3)$.

THEOREM D. ([4]) *Let $d \ge 3$. Then as $n \to \infty$*

$$\mathbf{P}\{\rho_1 = 2n\} \sim (1 - \gamma(d))^2 \mathbf{P}\{S_{2n} = 0\},$$

where $\gamma(d) = \mathbf{P}\{\rho_1 < \infty\}$.

THEOREM E. ([3]) *Let $f(n) \uparrow \infty$ be a positive integer valued function and let E_n be the event that the paths*

$$\{S_0, S_1, \ldots, S_n\} \quad \text{and} \quad \{S_{n+f(n)+1}, S_{n+f(n)+2}, \ldots\}$$

have points in common. Then

(i) *for $d = 3$, if $f(n) = n(\varphi(n))^2$ and $\varphi(n)$ is non-decreasing, then*

$$\mathbf{P}\{E_n \text{ i.o.}\} = 0 \quad \text{or} \quad 1 \tag{2.1}$$

depending on whether $\sum_{k=1}^{\infty}(\varphi(2^k))^{-1}$ *converges or diverges,*

(ii) *for* $d=4$, *if* $f(n)=n\psi(n)$ *and* $\psi(n)$ *is non-decreasing, then we have* (2.1) *depending on whether* $\sum_{k=1}^{\infty}(k\psi(2^k))^{-1}$ *converges or diverges,*

(iii) *for* $d\geq 5$, *if*

$$\sup_{m\geq n}\frac{f(m)}{m}\geq C\frac{f(n)}{n}$$

(*for some* $C>0$) *then we have* (2.1) *depending on whether* $\sum_{k=1}^{\infty}(f(k))^{(2-d)/d}$ *converges or diverges.*

3. Proof of Theorem 2

Let $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy (1.1) and (1.2), respectively, and let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be as in (1.3). Fix a constant $\delta\in(0,1/3)$. Introduce the following notations:

$$\begin{aligned}
d_n &= \exp\left((1-\delta)\frac{\pi n}{\log_2 n}\right),\\
e_n &= \exp(n(\log n)^{1+\delta}),\\
A_n^{(1)} &= \{T_n\in(a_n^{-1},a_n)\},\\
A_n^{(2)} &= \{T_n\in(b_n^{-1},c_n^{-1})\},\\
A_n^{(3)} &= \{T_n\in(c_n,b_n)\}.
\end{aligned}$$

Clearly we have

$$A_n^{(1)}=\{A_n^{(1)}\cap\mathcal{B}_n\}\cup\{A_n^{(1)}\cap\mathcal{B}_n^c\},$$

where $\mathcal{B}_n:=\{\rho_n\in(d_n,e_n)\}$ and $\mathcal{B}_n^c$ denotes its complementary. By Theorem B among the events

$$A_n^{(i)}\cap\mathcal{B}_n^c\,,$$

only finitely many might occur with probability 1.

Another application of Theorem B yields that

$$\begin{aligned}
\mathbf{P}\left\{A_n^{(1)}\cap\mathcal{B}_n\right\} &= \mathbf{E}\left[\mathbf{1}_{\mathcal{B}_n}\mathbf{P}\left\{A_n^{(1)}\mid\rho_n\right\}\right]=\\
&= \mathbf{E}\left[\mathbf{1}_{\mathcal{B}_n}\left(\frac{2\pi\log a_n}{\log(a_n^{-1}\rho_n)\cdot\log(a_n\rho_n)}+O\left(\frac{1}{\log^2(a_n^{-1}\rho_n)}\right)\right)\right].
\end{aligned}$$

On $\mathcal{B}_n$, we have (recalling that $\alpha_n\geq\log_2 n$)

$$\log(a_n^{-1}\rho_n)=\log\rho_n-\frac{n}{\alpha_n}\geq(1-\delta)\frac{\pi n}{\log_2 n}-\frac{n}{\log_2 n}\geq\frac{n}{\log_2 n},\tag{3.1}$$

so that

$$\mathbf{E}\left[\mathbf{1}_{\mathcal{B}_n}\cdot O\left(\frac{1}{\log^2(a_n^{-1}\rho_n)}\right)\right] = O\left(\frac{(\log_2 n)^2}{n^2}\right).$$

On the other hand, since (3.1) also tells that $\log(a_n^{-1}\rho_n) \geq (\log \rho_n)/2$ on $\mathcal{B}_n$, we have

$$\mathbf{E}\left[\mathbf{1}_{\mathcal{B}_n}\cdot \frac{\log a_n}{\log(a_n^{-1}\rho_n)\cdot\log(a_n\rho_n)}\right] \leq 2(\log a_n)\,\mathbf{E}\left[\frac{\mathbf{1}_{\mathcal{B}_n}}{(\log\rho_n)^2}\right],$$

which, by writing $N := \lfloor(\log_2 n)/((1-\delta)\pi)\rfloor$ and using Proposition B, is

$$\begin{aligned}
&\leq 2(\log a_n)\sum_{i=0}^{N}\mathbf{E}\left[\frac{\mathbf{1}_{\{\exp(n/(i+1))\leq\rho_n<\exp(n/i)\}}}{(n/(i+1))^2}\right] = \\
&= 2(\log a_n)\sum_{i=0}^{N}\frac{\pi\exp(-\pi i)+O(n^{-1/7})}{(n/(i+1))^2} = \\
&= O\left(\frac{\log a_n}{n^2}\right) = \\
&= O\left(\frac{1}{n\alpha_n}\right).
\end{aligned}$$

Therefore,

$$\mathbf{P}\left\{A_n^{(1)}\cap\mathcal{B}_n\right\} = O\left(\frac{(\log_2 n)^2}{n^2}\right) + O\left(\frac{1}{n\alpha_n}\right),$$

which is summable for n. We obtain (i) of Theorem 2 by an application of the Borel–Cantelli lemma.

Now, we turn to the proof of (ii) and (iii). We first assume that for some n_0,

$$\beta_n \leq (\log n)^2, \qquad n\geq n_0.$$

We fix a constant $\theta > 1/(\inf_{n\geq1}\beta_n)$. Introduce the following notations:

$$\begin{aligned}
\mathcal{E}_n &= \{\rho_n\in(e^{\theta n}, e^{2\theta n})\},\\
p_n^{(2)}(x) &= \mathbf{P}\left\{\frac{\rho_1}{x}\in(b_n^{-1}, c_n^{-1})\right\},\\
p_n^{(3)}(x) &= \mathbf{P}\left\{\frac{\rho_1}{x}\in(c_n, b_n)\right\}.
\end{aligned}$$

We have, for $i = 2$ or 3,

$$\mathbf{P}\left\{\mathcal{E}_n\cap A_n^{(i)}\right\} = \mathbf{E}\left[\mathbf{1}_{\mathcal{E}_n}\mathbf{P}\left\{A_n^{(i)}\mid\rho_n\right\}\right] = \mathbf{E}\left[\mathbf{1}_{\mathcal{E}_n}\cdot p_n^{(i)}(\rho_n)\right].$$

By Proposition B, there exist two positive constants C_1 and C_2 such that for all n,

$$\frac{C_1}{n\beta_n} \leq p_n^{(i)}(e^{nz}) \leq \frac{C_2}{n\beta_n}, \qquad i = 2 \text{ or } 3, \tag{3.2}$$

uniformly for $z \in (\theta, 2\theta)$. Therefore,

$$\frac{C_1}{n\beta_n}\mathbf{P}\{\mathcal{E}_n\} \le \mathbf{P}\{\mathcal{E}_n \cap A_n^{(i)}\} \le \frac{C_2}{n\beta_n}\mathbf{P}\{\mathcal{E}_n\}.$$

Since $C_3 \le \mathbf{P}\{\mathcal{E}_n\} \le C_4$ for some positive constants C_3 and C_4, we obtain:

$$\frac{C_5}{n\beta_n} \le \mathbf{P}\{\mathcal{E}_n \cap A_n^{(i)}\} \le \frac{C_6}{n\beta_n}. \tag{3.3}$$

As a consequence, $\sum_n \mathbf{P}\{\mathcal{E}_n \cap A_n^{(i)}\} = \infty$, for $i = 2$ or 3.

In order to complete the proof of (ii) and (iii) of Theorem 2 (in case $\beta_n \le (\log n)^2$), we prove that the events $A_n^{(i)}$ are nearly pairwise independent. Let $0 < n < N$. Then we have

$$\mathbf{P}\{A_n^{(i)}\mathcal{E}_n A_N^{(i)}\mathcal{E}_N\} = \mathbf{E}\left[\mathbf{1}_{A_n^{(i)}\mathcal{E}_n\mathcal{E}_N} \cdot p_N^{(i)}(\rho_N)\right],$$

which, according to (3.2), is

$$\le C_2\,\mathbf{E}\left[\frac{\mathbf{1}_{A_n^{(i)}\mathcal{E}_n\mathcal{E}_N}}{N\beta_N}\right] \le \frac{C_2}{N\beta_N}\mathbf{P}\{A_n^{(i)}\mathcal{E}_n\}.$$

This, jointly considered with (3.3) and the fact that $\sum_n \mathbf{P}\{\mathcal{E}_n \cap A_n^{(i)}\} = \infty$, yields

$$\liminf_{m\to\infty} \frac{\displaystyle\sum_{n=1}^{m}\sum_{N=1}^{m}\mathbf{P}\{A_n^{(i)}\mathcal{E}_n A_N^{(i)}\mathcal{E}_N\}}{\left(\displaystyle\sum_{n=1}^{m}\mathbf{P}\{A_n^{(i)}\mathcal{E}_n\}\right)^2} < \infty.$$

According to the Kochen and Stone's version ([5]) of the Borel–Cantelli lemma, we have, for $i = 2$ or 3, $\mathbf{P}\{A_n^{(i)}\mathcal{E}_n \text{ i.o.}\} > 0$. Consequently,

$$\mathbf{P}\{A_n^{(i)} \text{ i.o.}\} > 0.$$

Since the event $\{A_n^{(i)} \text{ i.o.}\}$ is invariant under any finite permutation between the i.i.d.r.v.'s $\{\rho_{n+1} - \rho_n\}_{n=0}^{\infty}$, an immediate application of the Hewitt–Savage zero-one law yields that $\mathbf{P}\{A_n^{(i)} \text{ i.o.}\} = 1$ for $i = 2$ or 3. This completes the proof of Theorem 2 (ii) and (iii) in case $\beta_n \le (\log n)^2$.

For general $\{\beta_n\}$, we define

$$\widetilde{\beta}_n := \min\left(\beta_n,\ (\log n)^2\right),$$

which satisfies (1.2), such that $\widetilde{\beta}_n \le (\log n)^2$. According to what we have just proved, with probability one,

$$T_n \in \left(\exp\left(\frac{n}{(1+\varepsilon)\widetilde{\beta}_n}\right),\ \exp\left(\frac{n}{\widetilde{\beta}_n}\right)\right), \tag{3.4}$$

along a (random) subsequence $\{n(i)\}_{i\geq 1}$. If there existed a sub-subsequence of $\{n(i)\}_{i\geq 1}$, say $\{n(i(k))\}_{k\geq 1}$, such that $\beta_{n(i(k))} > \log^2(n(i(k)))$ for $k \geq 1$, then $\widetilde{\beta}_{n(i(k))} = \log^2(n(i(k)))$, and according to (i) of Theorem 2,

$$T_{n(i(k))} \notin \left(\exp\left(-\frac{n(i(k))}{\widetilde{\beta}_{n(i(k))}}\right), \exp\left(\frac{n(i(k))}{\widetilde{\beta}_{n(i(k))}}\right)\right),$$

for all large k. This would contradict (3.4). As a consequence, $\beta_{n(i)} \leq \log^2(n(i))$ for all large i, say $i \geq i_0$. Thus $\widetilde{\beta}_{n(i)} = \beta_{n(i)}$ for $i \geq i_0$, which in view of (3.4) implies

$$T_{n(i)} \in \left(\exp\left(\frac{n(i)}{(1+\varepsilon)\widetilde{\beta}_{n(i)}}\right), \exp\left(\frac{n(i)}{\widetilde{\beta}_{n(i)}}\right)\right), \qquad i \geq i_0.$$

This proves Theorem 2 (iii). A similar argument yields Theorem 2 (ii) for general $\{\beta_n\}$. □

Corollary 2 follows easily from Theorem 2 and the next lemma.

LEMMA 1. *Let $\{g_n\}$ be a positive non-decreasing sequence such that $\sum_n (1/g_n) < \infty$. If $\{\rho_n\}_{n=1}^{\infty}$ is an increasing sequence with $\rho_0 := 0$, satisfying*

$$\frac{\rho_{n+1} - \rho_n}{\rho_n} \notin (g_n^{-1}, g_n), \qquad n \geq n_0,$$

then

$$\frac{\rho_{n+k} - \rho_n}{\rho_n} \notin (g_n^*, g_n), \qquad n \geq n_0, \ \ k \geq 1,$$

where $g_n^* := \exp\left(\sum_{j=n}^{\infty} g_j^{-1}\right) - 1$.

4. Proof of Theorem 3

At first we prove two lemmas.

LEMMA 2. *For all integers $n \geq 1$ and $a \geq 2n$,*

$$\mathbf{P}\{\nu(n) > a\} \leq \frac{Cn}{a} + \frac{C}{n^{1/7}},$$

where C is an absolute, positive constant.

Proof. Clearly we have

$$\mathbf{P}\{\nu(n) > a\} = \mathbf{P}\{r(n+1) < \rho_n, \ \ldots, \ r(a) < \rho_n\}.$$

Since $r(n+1),\ldots,r(a)$, ρ_n are independent, we get

$$\mathbf{P}\{\nu(n)>a\}=\mathbf{E}(F^{a-n}(\rho_n)),$$

where $F(x)=\mathbf{P}\{\rho_1<x\}$ is the distribution function of ρ_1. By Theorem B, $F(x)\le 1-C_7/\log x$ for all $x\ge 2$ if $C_7>0$ is small enough. Hence

$$\begin{aligned}\mathbf{P}\{\nu(n)>a\} &\le \mathbf{E}\left(1-\frac{C_7}{\log\rho_n}\right)^{a-n}\le \mathbf{E}\exp\left(-\frac{C_7(a-n)}{\log\rho_n}\right)\le\\ &\le \exp\left(-\frac{C_7(a-n)}{n}\right)+\mathbf{P}\{\log\rho_n>n^{8/7}\}+\\ &\quad +\mathbf{E}\left(\exp\left(-\frac{C_7(a-n)}{\log\rho_n}\right)\mathbf{1}_{\{n\le\log\rho_n\le n^{8/7}\}}\right).\end{aligned}$$

Proposition B with $z=n^{-1/7}$ gives

$$\mathbf{P}\{\log\rho_n>n^{8/7}\}\le\frac{C_8}{n^{1/7}}.$$

Integration by parts yields

$$\begin{aligned}&\mathbf{E}\left(\exp\left(-\frac{C_7(a-n)}{\log\rho_n}\right)\mathbf{1}_{\{n\le\log\rho_n\le n^{8/7}\}}\right)=\\ &= -\int_{[n,n^{8/7}]}\exp\left(-\frac{C_7(a-n)}{x}\right)d_x\mathbf{P}\{\log\rho_n>x\}=\\ &= \exp\left(-\frac{C_7(a-n)}{n}\right)\mathbf{P}\{\log\rho_n>n\}-\\ &\quad -\exp\left(-\frac{C_7(a-n)}{n^{8/7}}\right)\mathbf{P}\{\log\rho_n>n^{8/7}\}+\\ &\quad +\int_{[n,n^{8/7}]}\frac{C_7(a-n)}{x^2}\exp\left(-\frac{C_7(a-n)}{x}\right)\mathbf{P}\{\log\rho_n>x\}\,dx.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\mathbf{P}\{\nu(n)>a\} &\le 2\exp\left(-\frac{C_7(a-n)}{n}\right)+\frac{C_8}{n^{1/7}}+\\ &\quad +\frac{C_7(a-n)}{n}\int_{n^{-1/7}}^{1}\exp\left(-\frac{C_7(a-n)z}{n}\right)\mathbf{P}\{\rho_n>e^{n/z}\}\,dz,\end{aligned}$$

which, by Proposition B, yields

$$\mathbf{P}\{\nu(n)>a\}\le 2\exp\left(-\frac{C_7(a-n)}{n}\right)+\frac{C_8}{n^{1/7}}+$$

$$+\frac{C_9(a-n)}{n}\int_0^\infty \exp\left(-\frac{C_7(a-n)z}{n}\right)\left(1-e^{-\pi z}\right)dz =$$
$$= \quad 2\exp\left(-\frac{C_7(a-n)}{n}\right)+\frac{C_8}{n^{1/7}}+\frac{(C_9/C_7)\,\pi n}{C_7(a-n)+\pi n}.$$

Since $a \geq 2n$ and $\sup_{x>0}(xe^{-x}) < \infty$, this yields the lemma. □

LEMMA 3. *If $n \geq n_0$ is even, and if a is an integer with $n < a \leq n^{8/7}$, then*

$$\mathbf{P}\{e^a < \rho_n - \rho_{n/2} < e^{2a},\ \rho_{n/2} < e^n,\ r(i) < e^a \text{ for } n+1 \leq i \leq a\} \geq \frac{C_{10}\,n}{a},$$

for some absolute constant $C_{10} > 0$.

Proof. By the strong Markov property,

$$\begin{aligned} &\mathbf{P}\{e^a < \rho_n - \rho_{n/2} < e^{2a},\ \rho_{n/2} < e^n,\ r(i) < e^a \text{ for } n+1 \leq i \leq a\} = \\ = \quad &\mathbf{P}\{e^a < \rho_{n/2} < e^{2a}\}\mathbf{P}\{\rho_{n/2} < e^n\}(\mathbf{P}\{\rho_1 < e^a\})^{a-n}. \end{aligned}$$

By Proposition B and Theorem B,

$$\begin{aligned} \mathbf{P}\{e^a < \rho_{n/2} < e^{2a}\} \quad &\geq \quad \frac{C_{11}\,n}{a}, \\ \mathbf{P}\{\rho_{n/2} < e^n\} \quad &\geq \quad C_{12}, \\ (\mathbf{P}\{\rho_1 < e^a\})^{a-n} \quad &\geq \quad (\mathbf{P}\{\rho_1 < e^a\})^a \geq \left(1-\frac{C_{13}}{a}\right)^a \geq C_{14}. \end{aligned}$$

The last inequality holds in the case $a > C_{13}$ but is trivially true if $a \leq C_{13}$ (possibly with a smaller value of C_{14}) since $\min_{1 \leq a \leq C_{13}}(\mathbf{P}\{\rho_1 < e^a\})^a > 0$. Assembling these pieces completes the proof of Lemma 3. □

Now, we turn to the proof of Theorem 3.

Let $\{f_n\}$ be a positive sequence such that $n \mapsto f_n/n$ is non-decreasing.

First, assume $\sum_n(1/f_n) < \infty$. Thus $(f_n/n) \to \infty$. Let $n_k = 2^k$ for all $k \geq 1$, which implies $\sum_k(n_k/f_{n_k}) < \infty$. By Lemma 2, for all large k,

$$\mathbf{P}\{\nu(n_{k+1}) > f_{n_k}\} \leq \frac{C_{15}\,n_{k+1}}{f_{n_k}} + \frac{C_{15}}{n_{k+1}^{1/7}}.$$

Therefore $\sum_k \mathbf{P}\{\nu(n_{k+1}) > f_{n_k}\} < \infty$. By the Borel–Cantelli lemma and the monotonicity of $n \mapsto \nu(n)$, we have

$$\limsup_{n\to\infty}\frac{\nu(n)}{f_n} \leq 1 \qquad \text{a.s.}$$

Since replacing f_n by a constant multiple of f_n does not change the nature of the test, we conclude that the "lim sup" expression vanishes almost surely. This yields the first part of Theorem 3.

To prove the other part, we assume $\sum_n (1/f_n) = \infty$. Let again $n_k = 2^k$, which implies $\sum_k (n_k/f_{n_k}) = \infty$. It is well-known for the proof of this kind of integral test that we can assume without loss of generality that

$$n < f_n \le n^{8/7},$$

when n is sufficiently large. (For a rigorous justification, we may use a similar argument as the one at the end of Section 3). Define for large k (say $k \ge k_0$) the events

$$\begin{aligned} A_k &= \{e^{f_{n_k}} < \rho_{n_k} - \rho_{n_k/2} < e^{2f_{n_k}},\ \rho_{n_k/2} < e^{n_k}, \\ &\qquad r(i) < e^{f_{n_k}} \text{ for all } n_k + 1 \le i \le f_{n_k}\}. \end{aligned}$$

By Lemma 3, for all $k \ge k_0$,

$$\mathbf{P}\{A_k\} \ge \frac{C_{10}\, n_k}{f_{n_k}},$$

which implies $\sum_{k \ge k_0} \mathbf{P}\{A_k\} = \infty$.

We now estimate $\mathbf{P}\{A_k \cap A_\ell\}$ for $\ell \ge k+2$ (and $k \ge k_0$). Observe that on A_k, we have

$$\rho_{f_{n_k}} < e^{2f_{n_k}} + e^{n_k} + (f_{n_k} - n_k)e^{f_{n_k}} < e^{f_{n_\ell}},$$

using the fact that $4f_{n_k} \le f_{n_\ell}$ (which is a consequence of the monotonicity of $n \mapsto f_n/n$). On the other hand, $\rho_{n_\ell} > e^{f_{n_\ell}}$ on A_ℓ. This means that $f_{n_k} < n_\ell$ if $A_k \cap A_\ell \ne \emptyset$. Write

$$\ell_0 = \ell_0(k) := \inf\{\ell \ge k :\ n_\ell > f_{n_k}\},$$

and we have just proved that $A_k \cap A_\ell = \emptyset$ for $\ell < \ell_0$.

When $\ell \ge \ell_0 + 1$, we have $n_\ell/2 > f_{n_k}$, so that the event A_k is independent of the variable $\rho_{n_\ell} - \rho_{n_\ell/2}$. Accordingly, for $\ell \ge \ell_0 + 1$,

$$\begin{aligned} \mathbf{P}\{A_k \cap A_\ell\} &\le \mathbf{P}\{A_k\}\,\mathbf{P}\{\rho_{n_\ell} - \rho_{n_\ell/2} > e^{f_{n_\ell}}\} \le \\ &\le \mathbf{P}\{\rho_{n_k} - \rho_{n_k/2} > e^{f_{n_k}}\}\,\mathbf{P}\{\rho_{n_\ell} - \rho_{n_\ell/2} > e^{f_{n_\ell}}\} = \\ &= \mathbf{P}\{\rho_{n_k/2} > e^{f_{n_k}}\}\,\mathbf{P}\{\rho_{n_\ell/2} > e^{f_{n_\ell}}\}, \end{aligned}$$

which yields that for $\ell \ge \ell_0 + 1$,

$$\mathbf{P}\{A_k \cap A_\ell\} \le C_{16}\, \frac{n_k}{f_{n_k}}\, \frac{n_\ell}{f_{n_\ell}} \le C_{17}\, \mathbf{P}\{A_k\}\mathbf{P}\{A_\ell\}.$$

Therefore,

$$\sum_{k_0\le k\le \ell\le n} \mathbf{P}\{A_k\cap A_\ell\} = \sum_{k=k_0}^{n} \left(\mathbf{P}\{A_k\} + \mathbf{P}\{A_k\cap A_{k+1}\} + \mathbf{P}\{A_k\cap A_{\ell_0}\}\right) + \\ + \sum_{k=k_0}^{n}\sum_{\ell=\ell_0+1}^{n} \mathbf{P}\{A_k\cap A_\ell\} \le \\ \le 3\sum_{k=k_0}^{n} \mathbf{P}\{A_k\} + C_{17}\sum_{k=k_0}^{n}\sum_{\ell=k}^{n}\mathbf{P}\{A_k\}\mathbf{P}\{A_\ell\}.$$

Since $\sum_{k\ge k_0} \mathbf{P}\{A_k\} = \infty$, this yields

$$\liminf_{n\to\infty} \frac{\sum_{k=k_0}^{n}\sum_{\ell=k_0}^{n}\mathbf{P}\{A_k\cap A_\ell\}}{\left(\sum_{k=k_0}^{n}\mathbf{P}\{A_k\}\right)^2} < \infty.$$

According to Kochen and Stone's version of the Borel–Cantelli lemma ([5]), $\mathbf{P}\{A_k \text{ i.o.}\} > 0$. Since $A_k \subset \{\nu(n_k) > f_{n_k}\}$, we obtain:

$$\mathbf{P}\left\{\limsup_{n\to\infty} \frac{\nu(n)}{f_n} \ge 1\right\} > 0.$$

Clearly, $\{\limsup_{n\to\infty}(\nu(n)/f_n) \ge 1\}$ is invariant under any finite permutation between the i.i.d.r.v.'s $\{\rho_{n+1}-\rho_n\}_{n=0}^{\infty}$, the Hewitt–Savage zero-one law yields that

$$\limsup_{n\to\infty} \frac{\nu(n)}{f_n} \ge 1 \qquad \text{a.s.}$$

Replacing f_n by an arbitrary multiple of f_n implies that the "lim sup" expression is infinity almost surely. This completes the proof of Theorem 3. □

5. Proof of Theorem 4

First we define continuous random variables which are close to the discrete variables $r_k = \rho_k - \rho_{k-1}$. Let $\{\zeta_k\}_{k=1}^{\infty}$ be an i.i.d. sequence of random variables, uniformly distributed on $(0,1)$ and also independent of the sequence $\{r_k\}_{k=1}^{\infty}$. Define

$$\widehat{r}_k = r_k + \zeta_k, \quad \widehat{\rho}_k = \sum_{i=1}^{k}\widehat{r}_i, \quad \widehat{T}_k = \frac{\widehat{r}_{k+1}}{\widehat{\rho}_{k+1}}, \quad k = 1,2,\ldots$$

Then $\{\widehat{r}_k\}_{k=1}^{\infty}$ is a sequence of i.i.d. random variables with continuous distribution. Now let $\widehat{Z}_1 = 1$ and for $k>1$ put $\widehat{Z}_k = 1$ if $\widehat{r}_{k+1}$ is an extreme element, i.e. $\widehat{r}_{k+1} > \max_{1\le i\le k}\widehat{r}_i$ and let $\widehat{Z}_k = 0$ otherwise .

LEMMA 4. *As $k \to \infty$, we have*

$$|\widehat{Z}_k - T_k^*| = O\left(\exp\left(-\frac{k}{\log^2 k}\right)\right) \qquad \text{a.s.}$$

Proof. We have

$$|\widehat{Z}_k - T_k^*| \le |\widehat{Z}_k - \widehat{T}_k| + |\widehat{T}_k - T_k^*|.$$

Simple algebra shows, using $0 \le \zeta_k \le 1$, that

$$|\widehat{T}_k - T_k^*| = \left|\frac{\widehat{r}_{k+1}}{\widehat{\rho}_{k+1}} - \frac{r_{k+1}}{\rho_{k+1}}\right| \le \frac{k+1}{\rho_{k+1}} = O\left(\exp\left(-\frac{k}{\log^2 k}\right)\right) \qquad \text{a.s.,}$$

the last identity following from Theorem B. This combined with Corollary 1 yields

$$\widehat{T}_k \notin \left(\exp\left(-\frac{k}{\log^2 k}\right), 1 - \exp\left(-\frac{k}{\log^2 k}\right)\right) \qquad \text{a.s.,}$$

for all but finitely many k. This means that for k large enough, $\widehat{Z}_k = 1$ implies $\widehat{T}_k \in \left(1 - \exp\left(-\frac{k}{\log^2 k}\right), 1\right)$, while $\widehat{Z}_k = 0$ implies $\widehat{T}_k \in \left(0, \exp\left(-\frac{k}{\log^2 k}\right)\right)$, so we have also

$$|\widehat{Z}_k - \widehat{T}_k| = O\left(\exp\left(-\frac{k}{\log^2 k}\right)\right),$$

proving Lemma 4. □

As a consequence, we have

$$\left|\sum_{k=1}^n \widehat{Z}_k - \sum_{k=1}^n T_k^*\right| = O(1) \qquad \text{a.s.,}$$

as $k \to \infty$. Since $\sum_{k=1}^n \widehat{Z}_k$ is the number of extreme elements up to n from a continuous distribution, Theorem 4 follows from Theorem C. □

6. Proof of Theorem 5

In this section, $\{S_n\}_{n=0}^\infty$ denotes a simple symmeric random walk on $\mathbb{Z}^d$ $(d \ge 3)$, starting from 0. Let E_n be the event as in Theorem E. Then $\mathbf{P}\{E_n \text{ i.o.}\} = 0$ if

$$f(n) \ge f^*(n) := \begin{cases} n(\log n)^{1+\varepsilon} & \text{if} \quad d = 3 \\ n(\log_2 n)^{1+\varepsilon} & \text{if} \quad d = 4 \\ n^{2/(d-2)}(\log n)^{2(1+\varepsilon)/(d-2)} & \text{if} \quad d \ge 5. \end{cases} \tag{6.1}$$

Observe that if there exists an excursion of length $g(n)$ away from S_n, i.e. if

$$S_{n+j} \ne S_n, \;\; j = 1, 2, \ldots, g(n), \;\; S_{n+g(n)+1} = S_n,$$

then E_n does not hold with $f(n) = g(n)$, i.e. $g(n)$ must be less than $f^*(n)$ of (6.1). Let $N := n + f^*(n) = h(n)$, i.e. $n = h^{-1}(N)$. Hence

$$R(N) \le f^*(n) = f^*(h^{-1}(N)),$$

which, in turn, easily implies that

$$\limsup_{n\to\infty} \frac{\log R(n)}{\log n} \le \begin{cases} 1 & \text{if} \quad d = 3, 4, \\ \frac{2}{d-2} & \text{if} \quad d \ge 5, \end{cases}$$

almost surely.

Now we turn to the proof of the lower bound in Theorem 5. Fix $0 < a < \min(\frac{2}{d-2}, 1)$. Note that $a < 1$. It suffices to show that

$$\liminf_{n\to\infty} \frac{\log R(n)}{\log n} \ge a \qquad \text{a.s.} \tag{6.2}$$

To this end, recall (see for example [8] p. 183) that $\mathbf{P}\{S_{2n} = 0\} \sim 2(d/4\pi)^{d/2} n^{-d/2}$ $(n \to \infty)$. Therefore, an application of Theorem D reveals that for all $n \ge 1$,

$$\frac{C_{18}}{n^{(d-2)/2}} \le \mathbf{P}\{n \le \rho_1 < \infty\} \le \frac{C_{19}}{n^{(d-2)/2}}, \tag{6.3}$$

where C_{18} and C_{19} are two positive constants depending only on d. Define for $n > j \ge 0$,

$$\begin{aligned} \tau_j &:= \inf\{k > j : S_k = S_j\}, \\ A_j(n) &:= \{j + n^a < \tau_j \le j + \Theta n^a\}, \end{aligned}$$

where the constant $\Theta = \Theta(d) > 1$ is chosen to satisfy $\Theta > (C_{19}/C_{18})^{2/(d-2)}$.

For all $j \le n - \Theta n^a$, we have

$$\begin{aligned} \mathbf{P}\{A_j(n)\} &= \mathbf{P}\{n^a < \rho_1 \le \Theta n^a\} = \\ &= \mathbf{P}\{n^a < \rho_1 < \infty\} - \mathbf{P}\{\Theta n^a < \rho_1 < \infty\}, \end{aligned}$$

which, in view of (6.3), yields

$$\mathbf{P}\{A_j(n)\} \ge \frac{C_{18}}{n^{a(d-2)/2}} - \frac{C_{19}}{(\Theta n^a)^{(d-2)/2}} = \frac{C_{20}}{n^{a(d-2)/2}},$$

with $C_{20} := C_{18} - C_{19}/\Theta^{(d-2)/2} \in (0, \infty)$. On the other hand, for all $j \le n - \Theta n^a$,

$$\mathbf{P}\{A_j(n)\} \le \mathbf{P}\{n^a < \rho_1 < \infty\} \le \frac{C_{19}}{n^{a(d-2)/2}},$$

the last inequality following from (6.3). As a consequence,

$$C_{21}\, n^{1-a(d-2)/2} \le \mathbf{E}\left\{ \sum_{j=1}^{n-\Theta n^a} \mathbf{1}_{A_j(n)} \right\} \le C_{19}\, n^{1-a(d-2)/2}. \tag{6.4}$$

Note that $1 - a(d-2)/2 > 0$.

Assume for the moment that we can show that

$$\mathrm{Var}\left(\sum_{j=1}^{n-\Theta n^a} \mathbf{1}_{A_j(n)} \right) = O\left(n^{2-a(d-2)-\delta} \right), \tag{6.5}$$

for some $\delta > 0$. Then by Chebyshev's inequality and (6.4),

$$\mathbf{P}\left\{ \left| \sum_{j=1}^{n-\Theta n^a} \mathbf{1}_{A_j(n)} - \mathbf{E}\left(\sum_{j=1}^{n-\Theta n^a} \mathbf{1}_{A_j(n)} \right) \right| > \frac{1}{2}\mathbf{E}\left(\sum_{j=1}^{n-\Theta n^a} \mathbf{1}_{A_j(n)} \right) \right\} \le$$
$$\le \frac{4\,\mathrm{Var}\left(\sum_{j=1}^{n-\Theta n^a} \mathbf{1}_{A_j(n)} \right)}{(\mathbf{E}(\sum_{j=1}^{n-\Theta n^a} \mathbf{1}_{A_j(n)}))^2} \le \frac{C_{22}}{n^\delta},$$

which, by the Borel–Cantelli lemma, implies that almost surely for all large k,

$$\sum_{j=1}^{n_k-\Theta n_k^a} \mathbf{1}_{A_j(n_k)} \ge \frac{1}{2}\mathbf{E}\left(\sum_{j=1}^{n_k-\Theta n_k^a} \mathbf{1}_{A_j(n_k)} \right),$$

where $n_k := \lfloor k^{2/\delta} \rfloor$. By (6.4), $\mathbf{E}\left(\sum_{j=1}^{n_k-\Theta n_k^a} \mathbf{1}_{A_j(n_k)} \right) \ge 2$, so that for large k, there exists $j \in [1, n_k - \Theta n_k^a]$ such that $\mathbf{1}_{A_j(n_k)} = 1$. This yields $R(n_k) \ge n_k^a$, and will complete the proof of (6.2), using the monotonicity of $n \mapsto R(n)$.

It remains to check (6.5). We have

$$\begin{aligned}
\mathbf{E}\left[\left(\sum_{j=1}^{n-\Theta n^a} \mathbf{1}_{A_j(n)} \right)^2 \right] &= \mathbf{E}\left(\sum_{j=1}^{n-\Theta n^a} \mathbf{1}_{A_j(n)} \right) + 2 \sum_{1 \le i < j \le n - \Theta n^a} \mathbf{P}\{A_i(n) A_j(n)\} = \\
&= \mathbf{E}\left(\sum_{j=1}^{n-\Theta n^a} \mathbf{1}_{A_j(n)} \right) + \\
&\quad +2 \sum_{1 \le i < j \le n - \Theta n^a} \mathbf{P}\{A_i(n),\ A_j(n),\ \tau_i \le j\} + \\
&\quad +2 \sum_{1 \le i < j \le n - \Theta n^a} \mathbf{P}\{A_i(n),\ A_j(n),\ \tau_i > j\}.
\end{aligned}$$

By the Markov property,

$$\mathbf{P}\{A_i(n),\ A_j(n),\ \tau_i \le j\} \le \mathbf{P}\{A_i(n)\}\,\mathbf{P}\{A_j(n)\}.$$

On the other hand, if $A_i(n) \cap \{\tau_i > j\} \ne \emptyset$, then $j < \tau_i \le i + \Theta n^a$, which implies $j - i \le \Theta n^a$. The proof of (6.5) is thus reduced to showing the following: for some $\delta > 0$,

$$\sum_{(i,j) \in \Omega(n)} \mathbf{P}\{A_i(n),\ A_j(n),\ \tau_i > j\} = O\left(\frac{n^{2-a(d-2)}}{n^\delta} \right), \tag{6.6}$$

where $\Omega(n) := \{(i,j) :\ 1 \le i < j \le n - \Theta n^a,\ j - i \le \Theta n^a\}$.

The proof of (6.6) will be carried out in a few steps, namely, we will show that

$$\sum_{(i,j)\in\Omega(n)} \mathbf{P}\left\{A_i(n),\ A_j(n),\ \tau_j > \tau_i > j + \frac{n^a}{2}\right\} = O\left(n^{1-a(d-2)/2} b_n\right), \tag{6.7}$$

$$\sum_{(i,j)\in\Omega(n)} \mathbf{P}\left\{A_i(n),\ A_j(n),\ j < \tau_i \le j + \frac{n^a}{2}\right\} = O\left(\frac{n^{2-a(d-2)/2}}{n^{1-a}}\right), \tag{6.8}$$

$$\sum_{(i,j)\in\Omega(n)} \mathbf{P}\left\{A_i(n),\ A_j(n),\ \tau_j \le \tau_i\right\} = O\left(n^{1-a(d-2)/2} b_n\right), \tag{6.9}$$

where $b_n := \sum_{k=1}^{\Theta n^a} k^{-(d-2)/2}$. Since $b_n = O(n^{1-a(d-2)/2-\delta})$ for some $\delta > 0$ (recalling that $a < \min(\frac{2}{d-2}, 1)$), the estimates (6.7)–(6.9) together suffice to yield (6.6), hence the lower bound in Theorem 5.

To check (6.7), observe that

$$\begin{aligned} I_1 &:= \mathbf{P}\left\{A_i(n),\ A_j(n),\ \tau_j > \tau_i > j + \frac{n^a}{2}\right\} \\ &\le \mathbf{P}\left\{\tau_i \in (j + n^a/2, i + \Theta n^a],\ \tau_j > \tau_i\right\}. \end{aligned}$$

Applying the strong Markov property at τ_i gives that

$$I_1 \le \mathbf{E}\left(\mathbf{1}_{\{\tau_i \in (j+n^a/2, i+\Theta n^a]\}} p_1(S_i, S_j)\right),$$

where $p_1(y,x) := \mathbf{P}^y\{\rho_1(x) < \infty\}$ for $x \in Z^d$ and $y \in Z^d$, and $\mathbf{P}^y$ is the probability under which the random walk starts from y. According to Erdős and Taylor [2], there exists a constant $C_{23} = C_{23}(d)$ such that for all $(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d$,

$$\mathbf{P}^y\left\{\rho_1(x) < \infty\right\} \le \frac{C_{23}}{(\|x-y\|+1)^{d-2}}. \tag{6.10}$$

Therefore

$$I_1 \le C_{23}\,\mathbf{E}\left[\frac{\mathbf{1}_{\{\tau_i \in (j+n^a/2, i+\Theta n^a]\}}}{(\|S_j - S_i\|+1)^{d-2}}\right] = C_{23}\,\mathbf{E}\left[\frac{p_2(S_j, S_i)}{(\|S_j - S_i\|+1)^{d-2}}\right],$$

where $p_2(y,x) := \mathbf{P}^y\{n^a/2 < \rho_1(x) \le i + \Theta n^a - j\}$ (and we have applied the Markov property at j).

We now estimate $p_2(y,x)$. According to [8] p. 184, $\sup_{x\in\mathbb{Z}^d} \mathbf{P}\{S_n = x\} \le C_{24}/n^{d/2}$, so that for $m < n$,

$$\mathbf{P}^y\left\{m \le \rho_1(x) < n\right\} \le \sum_{k=m}^{n-1} \mathbf{P}^y\left\{S_k = x\right\} \le \sum_{k=m}^{n-1} \frac{C_{24}}{k^{d/2}} \le C_{24}\,\frac{n-m}{m^{d/2}}.$$

Therefore, for all $m < n$,

$$\sup_{x\in\mathbb{Z}^d}\ \sup_{y\in\mathbb{Z}^d} \mathbf{P}^y\left\{m \le \rho_1(x) < n\right\} \le C_{24}\,\frac{n}{m^{d/2}}. \tag{6.11}$$

In particular, $p_2(y,x) \le C_{24}\Theta n^a/(n^a/2)^{d/2}$ for all x and y, so that

$$I_1 \le \frac{C_{25}}{n^{a(d-2)/2}}\mathbf{E}\left[\frac{1}{(\|S_j - S_i\| + 1)^{d-2}}\right].$$

Recall Lemma 16.5 of [8] p. 184: $\mathbf{P}\{S_n = x\} \le C_{26}\, n^{-d/2}\exp(-\|x\|^2/2n)$ for all $x \in \mathbb{Z}^d$ and $n \ge 1$, from which it follows that

$$\mathbf{E}\left((\|S_n\| + 1)^{-(d-2)}\right) \le C_{27}\, n^{-(d-2)/2}, \qquad n \ge 1. \tag{6.12}$$

Accordingly,

$$I_1 \le \frac{C_{25}}{n^{a(d-2)/2}}\,\frac{C_{27}}{(j-i)^{(d-2)/2}}.$$

This implies (6.7).

Now let us check (6.8). Clearly,

$$\begin{aligned} I_2 &:= \mathbf{P}\left\{A_i(n),\ A_j(n),\ j < \tau_i \le j + \frac{n^a}{2}\right\} \le \\ &\le \mathbf{P}\left\{\tau_i \in (i+n^a, j+n^a/2],\ j<\tau_i,\ \tau_j \in (j+n^a, j+\Theta n^a]\right\}. \end{aligned}$$

Applying the strong Markov property at τ_i gives

$$I_2 \le \mathbf{E}\left(\mathbf{1}_{\{\tau_i \in (i+n^a, j+n^a/2],\ j<\tau_i\}}\, p_3(S_i, S_j, \tau_i)\right),$$

where $p_3(y,x,\ell) := \mathbf{P}^y\{\rho_1(x) \in (j+n^a-\ell, j+\Theta n^a - \ell]\}$. By (6.11), on the event $\{\tau_i \in (i+n^a, j+n^a/2]\}$, we have $p_3(S_i,S_j,\tau_i) \le C_{24}\Theta n^a/(n^a/2)^{d/2}$, so that

$$\begin{aligned} I_2 &\le \frac{C_{28}}{n^{a(d-2)/2}}\mathbf{P}\{\tau_i \in (i+n^a, j+n^a/2]\} = \\ &= \frac{C_{28}}{n^{a(d-2)/2}}\mathbf{P}\{n^a < \rho_1 \le j + n^a/2 - i]\} \le \\ &\le \frac{C_{28}}{n^{a(d-2)/2}}\,\frac{C_{29}}{n^{a(d-2)/2}}, \end{aligned}$$

the last inequality following from (6.11) (recalling that $j - i \le \Theta n^a$). Therefore,

$$\sum_{(i,j)\in\Omega(n)} I_2 \le n\,\Theta n^a\,\frac{C_{28}C_{29}}{n^{a(d-2)}} = \Theta C_{28}C_{29}\frac{n^{2-a(d-2)}}{n^{1-a}},$$

proving (6.8).

Finally, to verify (6.9), we observe that

$$\begin{aligned} I_3 &:= \mathbf{P}\{A_i(n),\ A_j(n),\ \tau_j \le \tau_i\} \le \\ &\le \mathbf{P}\{\tau_j \in (j+n^a, j+\Theta n^a],\ \tau_i \ge \tau_j\} = \\ &= \mathbf{E}\left(\mathbf{1}_{\{\tau_j \in (j+n^a, j+\Theta n^a]\}}\, p_1(S_j, S_i)\right), \end{aligned}$$

where $p_1(y,x) = \mathbf{P}^y\{\rho_1(x) < \infty\}$ as before, and we have applied in the last equality the strong Markov property at τ_j. By (6.10), $p_1(S_j, S_i) \le C_{23}/(\|S_j - S_i\| + 1)^{d-2}$, so that

$$I_3 \le C_{23}\,\mathbf{E}\left[\frac{\mathbf{1}_{\{\tau_j \in (j+n^a, j+\Theta n^a]\}}}{(\|S_j - S_i\| + 1)^{d-2}}\right].$$

Using the Markov property at j, we get

$$I_3 \le C_{23}\,\mathbf{E}\left[\frac{p_4(S_j, S_j)}{(\|S_j - S_i\| + 1)^{d-2}}\right],$$

where $p_4(y,y) := \mathbf{P}^y\{\rho_1(y) \in (n^a, \Theta n^a]\} = \mathbf{P}\{n^a < \rho_1 \le \Theta n^a\} \le C_{24}\Theta n^{-a(d-2)/2}$ by means of (6.11). Combined with (6.12), it follows that

$$I_3 \le \frac{C_{23} C_{24} \Theta C_{27}}{(j-i)^{(d-2)/2} n^{a(d-2)/2}}.$$

This yields (6.9), and completes the proof of the lower bound in Theorem 5. □

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