A strong invariance principle for two-dimensional random walk in random scenery

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Abstract

We present a strong approximation of two-dimensional Kesten–Spitzer random walk in random scenery by Brownian motion.

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1 Introduction

The following model originates from Kesten and Spitzer (1979): Every lattice site $\mathbf{x} \in \mathbb{Z}^d$ is attached to a price value $Y(\mathbf{x})$, and a random walker moves on $\mathbb{Z}^d$ (in this paper: $d = 2$), whose movement is denoted by $\{\mathbf{S}_n;\ n \geq 0\}$, with say $\mathbf{S}_0 = \mathbf{0}$. Each time the random walker visits $\mathbf{x} \in \mathbb{Z}^d$, he increases (or decreases, if the price is negative) his fortune by $Y(\mathbf{x})$. Thus at step n, the total amount of prices he gets is

$$Z(n) \stackrel{\text{def}}{=} \sum_{j=0}^{n} Y(\mathbf{S}_j). \tag{1.1}$$

Throughout the paper, it is assumed that $\{Y(\mathbf{x});\ \mathbf{x} \in \mathbb{Z}^d\}$ is a collection of independent and identically distributed random variables with $\mathbb{E}(Y(\mathbf{0})) = 0$ and $\sigma^2 \stackrel{\text{def}}{=} \mathbb{E}(Y^2(\mathbf{0})) \in (0, \infty)$. The collection of these variables is referred to as random scenery, and is furthermore supposed to be independent of the random walk $\{\mathbf{S}_n;\ n \geq 0\}$. The process $Z \stackrel{\text{def}}{=} \{Z(n);\ n \geq 0\}$ is the so-called *random walk in random scenery.*

When $d = 1$, Kesten and Spitzer (1979) proved that, under some appropriate regularity conditions upon $Y(\mathbf{0})$, $n^{-3/4} Z(\lfloor nt \rfloor)$ (as a process indexed by $t \in \mathbb{R}_+$) converges weakly in $\mathcal{D}[0, \infty)$ (space of càdlàg functions endowed with the locally uniform convergence topology) to a non-Gaussian process. For $d \geq 3$, it was noted by Bolthausen (1989) that $n^{-1/2} Z(\lfloor nt \rfloor)$ converges weakly to (a constant multiple of) the Wiener process. In Khoshnevisan and Lewis (1998) (for Gaussian sceneries), Csáki *et al.* (1999) and Révész and Shi (2000) these weak limit assertions were strengthened to strong approximation results.

Not surprisingly, the dimension $d = 2$ is critical which separates the asymptotic Gaussian and non-Gaussian behaviours of Z. For this case, Kesten and Spitzer (1979) conjectured that Z still converges weakly to a Wiener process, but with the slightly non-standard normalizer $(n \log n)^{-1/2}$. The conjecture was later proved by Bolthausen (1989) (see also Borodin, 1980): in $\mathcal{D}[0, \infty)$,

$$\left\{ \frac{Z(\lfloor nt \rfloor)}{(n \log n)^{1/2}};\ t \in \mathbb{R}_+ \right\} \text{ converges weakly to } \left\{ \sigma (2/\pi)^{1/2}\, W(t);\ t \in \mathbb{R}_+ \right\}, \tag{1.2}$$

where W denotes a standard one-dimensional Wiener process.

The aim of this paper is to present a version of strong invariance principle for (1.2). Throughout, we assume that $\{\mathbf{S}_n;\ n \geq 0\}$ is a *simple symmetric* random walk on $\mathbb{Z}^2$ (with $\mathbf{S}_0 = \mathbf{0}$), i.e., in each step the walker moves to any of the nearest neighbour sites with equal probability $1/4$.

Theorem 1.1 *Let $d = 2$ and assume that $\mathbb{E}(|Y(\mathbf{0})|^q) < \infty$ for some $q > 2$. Possibly in an enlarged probability space, there exist a version of $\{Z(n);\, n \geq 0\}$ and a standard one-dimensional Wiener process $\{W(t);\, t \geq 0\}$, such that for any $\varepsilon > 0$ as n goes to infinity,*

$$Z(n) - \sigma(2/\pi)^{1/2} W(n \log n) = o\left(n^{1/2} (\log n)^{3/8+\varepsilon}\right), \qquad \text{a.s.} \tag{1.3}$$

Remark. It is important to note that $3/8 < 1/2$. As consequences, Theorem 1.1 implies the weak convergence in (1.2), and also the following iterated logarithm law due to Lewis (1993):

$$\limsup_{n\to\infty} \frac{Z(n)}{n^{1/2}(\log n)^{1/2}(\log\log n)^{1/2}} = \frac{2\sigma}{\pi^{1/2}}, \qquad \text{a.s.} \tag{1.4}$$

There are, however, many other consequences of Theorem 1.1. For example, it follows that Strassen's law holds: let

$$\mathbf{Z}_n(t) \stackrel{\text{def}}{=} \frac{\pi^{1/2} Z(\lfloor nt \rfloor)}{\sigma(n \log n \log\log n)^{1/2}};\ t \in [0,1].$$

Then $\{\mathbf{Z}_n(\cdot)\}_{n\geq 3}$ is almost surely relatively compact in C$[0,1]$ and the set of its limit points consists of all absolutely continuous functions $f(\cdot)$ such that $f(0) = 0$ and $\int_0^1 (f'(u))^2\, \mathrm{d}u \leq 1$.

The Chung-type law of the iterated logarithm

$$\liminf_{n\to\infty} \frac{(\log\log n)^{1/2}}{(n\log n)^{1/2}} \sup_{0\leq k\leq n} |Z(k)| = \frac{\sigma\pi^{1/2}}{2^{1/2}}, \qquad \text{a.s.,} \tag{1.5}$$

is also a consequence of Theorem 1.1. Moreover, (1.4) and (1.5) can be extended to upper–lower class results. □

The rest of the paper is organized as follows. In Section 2, Theorem 1.1 is proved by means of four technical lemmas. The proofs of these lemmas are postponed to Sections 3–6.

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We shall use some technical lemmas (Lemmas 2.1–2.4 below), whose proofs are provided in Sections 3–6, respectively.

Let $\{\mathbf{S}_n;\, n \geq 0\}$ be a simple symmetric random walk on $\mathbb{Z}^2$ as in the Introduction, and let

$$\xi(n, \mathbf{x}) \stackrel{\text{def}}{=} \sum_{i=0}^{n} \mathbf{1}_{\{\mathbf{S}_i = \mathbf{x}\}}, \qquad n \geq 0,\ \mathbf{x} \in \mathbb{Z}^2.$$

The process $\xi(\cdot\,,\,\cdot)$ is often referred to as the local time of the random walk. The random walk in random scenery Z defined in (1.1) can now be written as

$$Z(n) = \sum_{\mathbf{x}\in\mathbb{Z}^2} \xi(n,\mathbf{x})Y(\mathbf{x}).$$

Define the truncated scenery $\{\widehat{Y}(\mathbf{x});\ \mathbf{x}\in\mathbb{Z}^2\}$ and the associated random walk in random scenery $\{\widehat{Z}(n);\ n\geq 0\}$ by

$$\begin{aligned}\widehat{Y}(\mathbf{x}) &\stackrel{\text{def}}{=} Y(\mathbf{x})\mathbf{1}_{\{|Y(\mathbf{x})|\leq\|\mathbf{x}\|\}} - \mathbb{E}(Y(\mathbf{x})\mathbf{1}_{\{|Y(\mathbf{x})|\leq\|\mathbf{x}\|\}}),\\ \widehat{Z}(n) &\stackrel{\text{def}}{=} \sum_{\mathbf{x}\in\mathbb{Z}^2}\xi(n,\mathbf{x})\widehat{Y}(\mathbf{x}).\end{aligned}$$

Our first technical lemma says that $\widehat{Z}(n)$ is reasonably close to $Z(n)$.

Lemma 2.1 *Under the conditions of Theorem* 1.1*, there exists* $\delta>0$ *such that when* $n\to\infty$*,*

$$Z(n)-\widehat{Z}(n) = \mathcal{O}\left(n^{1/2-\delta}\right), \qquad \text{a.s.} \tag{2.1}$$

We now work on the process $\{\widehat{Z}(n);\ n\geq 0\}$. We first look at this process along the subsequence $\{n_k\}$ defined as follows. Fix $\varrho\in(1/2,\,1)$, and let the sequence $\{n_k\}_{k\geq 1}$ of non-decreasing numbers be given by

$$n_k \stackrel{\text{def}}{=} \lfloor \exp(k^{\varrho})\rfloor.$$

We shall frequently use the following relations without further mention: when $k\to\infty$,

$$n_{k+1}\sim n_k, \quad \log(n_{k+1}-n_k)\sim\log n_k, \quad n_{k+1}-n_k\sim\frac{\varrho\, n_k}{(\log n_k)^{(1-\varrho)/\varrho}}, \tag{2.2}$$

where $a_k\sim b_k$ means $\lim_{k\to\infty} a_k/b_k = 1$. Also,

$$\sum_{k=1}^{\ell} n_k^a(\log n_k)^b = \mathcal{O}\left(n_\ell^a(\log n_\ell)^{b-1+1/\varrho}\right), \qquad \ell\to\infty. \tag{2.3}$$

Let us consider

$$U_k \stackrel{\text{def}}{=} \widehat{Z}(n_{k+1}) - \widehat{Z}(n_k).$$

For brevity, we write

$$\xi_k(\mathbf{x}) \stackrel{\text{def}}{=} \xi(n_{k+1},\mathbf{x}) - \xi(n_k,\mathbf{x}), \qquad k\geq 1,\ \mathbf{x}\in\mathbb{Z}^2,$$

so that

$$U_k = \sum_{\mathbf{x}\in\mathbb{Z}^2} \xi_k(\mathbf{x})\widehat{Y}(\mathbf{x}).$$

The random variables U_k, $k = 1, 2, \ldots$ are the increments of $\widehat{Z}(n_k)$. Unfortunately, these are not independent variables given the random walk $\{\mathbf{S}_n;\ n \geq 0\}$. The idea is to replace these variables by another sequence of variables which are conditionally independent given the random walk.

Let $\{Y(\mathbf{x}), Y_1(\mathbf{x}), Y_2(\mathbf{x}), \cdots, \mathbf{x} \in \mathbb{Z}^2\}$ be a collection of iid random variables. (It is always possible to define these on the same probability space by working in a product space). For any $k \geq 1$, let

$$V_k \stackrel{\text{def}}{=} \sum_{\mathbf{x}\notin A_k} \xi_k(\mathbf{x})\widehat{Y}(\mathbf{x}) + \sum_{\mathbf{x}\in A_k} \xi_k(\mathbf{x})\widehat{Y}_k(\mathbf{x}), \tag{2.4}$$

where

$$\widehat{Y}_k(\mathbf{x}) \stackrel{\text{def}}{=} Y_k(\mathbf{x})\mathbf{1}_{\{|Y_k(\mathbf{x})|\leq\|\mathbf{x}\|\}} - \mathbb{E}(Y_k(\mathbf{x})\mathbf{1}_{\{|Y_k(\mathbf{x})|\leq\|\mathbf{x}\|\}}),$$

and

$$A_k \stackrel{\text{def}}{=} \left\{\mathbf{x} \in \mathbb{Z}^2 :\ \xi(n_{k+1}, \mathbf{x}) > \xi(n_k, \mathbf{x}) > 0\right\}.$$

In words, A_k is the set of sites which are visited by the random walk during $[0, n_k]$ and again during $(n_k, n_{k+1}]$.

It turns out that V_k is close to U_k. More precisely, the following estimate holds:

Lemma 2.2 *Under the conditions of Theorem* 1.1*, for any* $\varepsilon > 0$*, we have, when* $k \to \infty$*,*

$$U_k - V_k = o\left(n_k^{1/2}(\log n_k)^{1/2-1/(2\varrho)+\varepsilon}\right), \qquad \text{a.s.} \tag{2.5}$$

By means of Lemma 2.2 (and in light of (2.3)), we can sum over $k \leq \ell - 1$, and use the relation $\widehat{Z}(n_\ell) = \widehat{Z}(n_1) + \sum_{k=1}^{\ell-1} U_k$, to see that for any $\varepsilon > 0$, when $\ell \to \infty$,

$$\widehat{Z}(n_\ell) - \sum_{k=1}^{\ell-1} V_k = o\left(n_\ell^{1/2}(\log n_\ell)^{-1/2+1/(2\varrho)+\varepsilon}\right), \qquad \text{a.s.} \tag{2.6}$$

Since $\{V_k\}_{k\geq1}$ are (conditionally) independent variables (given the random walk), it is possible to embed $\sum_{k=1}^{\ell-1} V_k$ into a Wiener process, via the following lemma.

Lemma 2.3 *Under the conditions of Theorem* 1.1, *possibly in an enlarged probability space, there exists a standard Wiener process* $\{W(t);\, t \ge 0\}$, *such that for any* $\varepsilon > 0$, *when* $\ell \to \infty$,

$$\sum_{k=1}^{\ell-1} V_k - W(b_{n_\ell}) = o\left(n_\ell^{1/2} (\log n_\ell)^{\beta+\varepsilon}\right), \qquad \text{a.s.,} \tag{2.7}$$

where

$$b_n \stackrel{\text{def}}{=} \frac{2\sigma^2}{\pi}\, n \log n, \tag{2.8}$$

$$\beta \stackrel{\text{def}}{=} \max\left(\frac{3}{4} - \frac{1}{4\varrho},\, \frac{1}{4\varrho}\right). \tag{2.9}$$

Assembling (2.1), (2.6) and (2.7), we arrive at: for any $\varepsilon > 0$, when $\ell \to \infty$,

$$Z(n_\ell) - W(b_{n_\ell}) = o\left(n_\ell^{1/2} (\log n_\ell)^{\beta+\varepsilon}\right), \qquad \text{a.s.} \tag{2.10}$$

(We have used the fact that $-1/2 + 1/(2\varrho) \le \beta$). This is a strong approximation for Z along the subsequence $\{n_\ell\}$. To claim that it holds for all large n, we need to control the increments of Z and W.

Lemma 2.4 *Under the conditions of Theorem* 1.1, *with probability one, for any* $\varepsilon > 0$, *as* $\ell \to \infty$,

$$\max_{n_\ell \le n \le n_{\ell+1}} |Z(n) - Z(n_\ell)| = o\left(n_\ell^{1/2} (\log n_\ell)^{1-1/(2\varrho)+\varepsilon}\right), \tag{2.11}$$

$$\sup_{b_{n_\ell} \le t \le b_{n_{\ell+1}}} |W(t) - W(b_{n_\ell})| = \mathcal{O}\left(n_\ell^{1/2} (\log n_\ell)^{1-1/(2\varrho)} (\log\log n_\ell)^{1/2}\right). \tag{2.12}$$

It is now easy to complete the proof of Theorem 1.1. Indeed, since $1 - 1/(2\varrho) < 3/4 - 1/(4\varrho)$, we can bring (2.10), (2.11) and (2.12) together to see that for any $\varepsilon > 0$ and $1/2 < \varrho < 1$,

$$Z(n) - W(b_n) = o\left(n^{1/2} (\log n)^{\beta+\varepsilon}\right), \qquad \text{a.s.}$$

Taking $\varrho = 2/3$ yields $\beta = 3/8$. Theorem 1.1 is proved. □

We prove the four lemmas in the next sections.

3 Proof of Lemma 2.1

Throughout, we assume $q < 3$ without loss of generality.

Since $\mathbb{E}(Y(\mathbf{x})) = 0$ for any $\mathbf{x} \in \mathbb{Z}^2$, we have

$$|Z(n) - \widehat{Z}(n)| \le \sum_{\mathbf{x}\in\mathbb{Z}^2} \xi(n,\mathbf{x}) \left[|Y(\mathbf{x})| \, \mathbf{1}_{\{|Y(\mathbf{x})|>\|\mathbf{x}\|\}} + \mathbb{E}(|Y(\mathbf{x})| \, \mathbf{1}_{\{|Y(\mathbf{x})|>\|\mathbf{x}\|\}}) \right]. \tag{3.1}$$

By Chebyshev's inequality, for $\mathbf{x} \neq \mathbf{0}$,

$$\mathbb{P}(|Y(\mathbf{x})| > \|\mathbf{x}\|) \le \frac{\mathbb{E}(|Y(\mathbf{x})|^q)}{\|\mathbf{x}\|^q} = \frac{\mathbb{E}(|Y(\mathbf{0})|^q)}{\|\mathbf{x}\|^q}.$$

Since $q > 2$, this yields

$$\sum_{\mathbf{x}\in\mathbb{Z}^2} \mathbb{P}(|Y(\mathbf{x})| > \|\mathbf{x}\|) < \infty.$$

Consequently, in the expression $\sum_{\mathbf{x}\in\mathbb{Z}^2} \xi(n,\mathbf{x}) \, |Y(\mathbf{x})| \, \mathbf{1}_{\{|Y(\mathbf{x})|>\|\mathbf{x}\|\}}$ on the right hand side of (3.1), only finitely many terms are different from zero. Moreover, for $\mathbf{x} \neq \mathbf{0}$,

$$\mathbb{E}(|Y(\mathbf{x})| \, \mathbf{1}_{\{|Y(\mathbf{x})|>\|\mathbf{x}\|\}}) \le \frac{\mathbb{E}(|Y(\mathbf{x})|^q)}{\|\mathbf{x}\|^{q-1}} \le \frac{2\,\mathbb{E}(|Y(\mathbf{0})|^q)}{1 + \|\mathbf{x}\|^{q-1}}.$$

Hence, as $n \to \infty$, we have almost surely

$$|Z(n) - \widehat{Z}(n)| \le \mathcal{O}\left(\max_{\mathbf{x}\in\mathbb{Z}^2} \xi(n,\mathbf{x}) \right) + c_1 \sum_{\mathbf{x}\in\mathbb{Z}^2} \frac{\xi(n,\mathbf{x})}{1 + \|\mathbf{x}\|^{q-1}}. \tag{3.2}$$

Observe that

$$\sum_{\mathbf{x}\in\mathbb{Z}^2,\, \|\mathbf{x}\|>n^{1/2}} \frac{\xi(n,\mathbf{x})}{1 + \|\mathbf{x}\|^{q-1}} \le n^{-(q-1)/2} \sum_{\mathbf{x}\in\mathbb{Z}^2} \xi(n,\mathbf{x}) = n^{(3-q)/2},$$

and that

$$\begin{aligned} \sum_{\mathbf{x}\in\mathbb{Z}^2,\, \|\mathbf{x}\|\le n^{1/2}} \frac{\xi(n,\mathbf{x})}{1 + \|\mathbf{x}\|^{q-1}} &\le \left(\max_{\mathbf{x}\in\mathbb{Z}^2} \xi(n,\mathbf{x}) \right) \sum_{\|\mathbf{x}\|\le n^{1/2}} \frac{1}{1 + \|\mathbf{x}\|^{q-1}} \\ &\le c_2 \max_{\mathbf{x}\in\mathbb{Z}^2} \xi(n,\mathbf{x}) \int_0^{n^{1/2}} \frac{r}{1 + r^{q-1}}\, \mathrm{d}r \\ &\le c_3\, n^{(3-q)/2} \max_{\mathbf{x}\in\mathbb{Z}^2} \xi(n,\mathbf{x}). \end{aligned}$$

Plugging these into (3.2) gives that, when $n \to \infty$,

$$Z(n) - \widehat{Z}(n) = \mathcal{O}\left(n^{(3-q)/2} \max_{\mathbf{x}\in\mathbb{Z}^2} \xi(n,\mathbf{x}) \right), \qquad \text{a.s.}$$

According to Erdős and Taylor (1960), $\max_{\mathbf{x}\in\mathbb{Z}^2} \xi(n,\mathbf{x}) = \mathcal{O}(\log^2 n)$ almost surely, and since $q > 2$, this yields Lemma 2.1. □

4 Proof of Lemma 2.2

We start with a preliminary estimate (Lemma 4.1 below), which will be of frequent use later. Recall that $\xi(n, \mathbf{x})$ is the local time of the two-dimensional random walk $\{\mathbf{S}_n;\, n \geq 0\}$. It is well-known (see for example Révész, 1990, p. 183) that

$$\text{(4.1)} \qquad \mathbb{P}(\mathbf{S}_{2k} = \mathbf{0}) = \frac{1}{k\pi} + \mathcal{O}\left(\frac{1}{k^2}\right), \qquad k \to \infty,$$

which implies the existence of a finite and positive constant c_4 such that $\mathbb{P}(\mathbf{S}_n = \mathbf{0}) \leq c_4/n$, for all $n \geq 1$. Since $\xi(n, \mathbf{0}) = \sum_{i=0}^{n} \mathbf{1}_{\{\mathbf{S}_i = \mathbf{0}\}}$, we arrive at: for any integer $m \geq 1$, there exists $c_5 = c_5(m)$ such that

$$\mathbb{E}\left(\xi^m(n, \mathbf{0})\right) \leq c_5 \, (\log n)^m, \qquad n \geq 2.$$

For any fixed $\mathbf{x} \in \mathbb{Z}^2$, $\xi(n, \mathbf{x})$ is stochastically smaller than or equal to $\xi(n, \mathbf{0})$. Accordingly,

$$\text{(4.2)} \qquad \sup_{\mathbf{x} \in \mathbb{Z}^2} \mathbb{E}\left(\xi^m(n, \mathbf{x})\right) \leq c_5 \, (\log n)^m, \qquad n \geq 2.$$

An immediate consequence of (4.2) together with Hölder's inequality is that, for any positive integers ℓ and $m_1, \cdots, m_\ell$,

$$\text{(4.3)} \qquad \sup_{\mathbf{x}_1 \in \mathbb{Z}^2, \cdots, \mathbf{x}_\ell \in \mathbb{Z}^2} \mathbb{E}\left(\xi^{m_1}(n, \mathbf{x}_1) \cdots \xi^{m_\ell}(n, \mathbf{x}_\ell)\right) \leq c_6 \, (\log n)^{m_1 + \cdots + m_\ell}, \qquad n \geq 2,$$

where $c_6 = c_6(\ell, m_1, \cdots, m_\ell)$.

Lemma 4.1 *Let $\{\eta(n, \mathbf{x});\, n \geq 1,\, \mathbf{x} \in \mathbb{Z}^2\}$ be a set of random variables independent of the random walk $\{\mathbf{S}_n\}_{n \geq 0}$, such that for some $\alpha \geq 0$,*

$$\mathbb{P}\left(0 \leq \eta(n, \mathbf{x}) \leq \|\mathbf{x}\|^\alpha;\, n \geq 1,\, \mathbf{x} \in \mathbb{Z}^2\right) = 1.$$

Then for any integers $m \geq 1$ and $\ell \geq 1$, and any $\varepsilon > 0$ and $0 < \nu < \varepsilon/(2\ell)$, there exist $c_7 = c_7(m, \ell, \varepsilon, \nu, \alpha)$ and $c_8 = c_8(\ell, \nu, \alpha)$ such that for all $n \geq 2$,

$$\text{(4.4)} \quad \mathbb{E}\left[\left(\sum_{\mathbf{x} \in \mathbb{Z}^2} \xi^m(n, \mathbf{x}) \eta(n, \mathbf{x})\right)^\ell\right] \leq c_7 \, n^\ell \left(n^{\alpha\ell/2} \mathrm{e}^{-c_8 (\log n)^\nu} + \varphi_{n,\ell,\nu}\right) (\log n)^{(m-1)\ell + \varepsilon},$$

where

$$\text{(4.5)} \qquad \varphi_{n,\ell,\nu} \stackrel{\text{def}}{=} \sup_{\|\mathbf{x}_i\| < n^{1/2} (\log n)^\nu,\, 1 \leq i \leq \ell} \mathbb{E}\left(\prod_{i=1}^{\ell} \eta(n, \mathbf{x}_i)\right).$$

In particular, we have

$$\mathbb{E}\left[\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^m(n,\mathbf{x})\,\|\mathbf{x}\|^\alpha\right)^\ell\right]\le c_9\,n^{\ell+\alpha\ell/2}(\log n)^{(m-1)\ell+\varepsilon},\qquad n\ge 2, \tag{4.6}$$

for some $c_9=c_9(m,\ell,\alpha,\varepsilon)$.

Proof. Write

$$\begin{aligned}\Omega_1 &\overset{\text{def}}{=} \left\{\mathbf{x}\in\mathbb{Z}^2:\ \|\mathbf{x}\|\ge n^{1/2}(\log n)^\nu\right\},\\ \Omega_2 &\overset{\text{def}}{=} \left\{\mathbf{x}\in\mathbb{Z}^2:\ \|\mathbf{x}\|< n^{1/2}(\log n)^\nu\right\}.\end{aligned}$$

Then

$$\begin{aligned}\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^m(n,\mathbf{x})\eta(n,\mathbf{x}) &= \left(\sum_{\mathbf{x}\in\Omega_1}+\sum_{\mathbf{x}\in\Omega_2}\right)\xi^m(n,\mathbf{x})\eta(n,\mathbf{x})\\ &\overset{\text{def}}{=} I_1+I_2,\end{aligned}\tag{4.7}$$

with obvious notation. Observe that for $j=1$ or 2,

$$\mathbb{E}(I_j^\ell)=\sum_{\mathbf{x}_1\in\Omega_j}\cdots\sum_{\mathbf{x}_\ell\in\Omega_j}\mathbb{E}\left(\prod_{i=1}^\ell\xi^m(n,\mathbf{x}_i)\right)\mathbb{E}\left(\prod_{i=1}^\ell\eta(n,\mathbf{x}_i)\right).\tag{4.8}$$

We now estimate $\mathbb{E}(I_1^\ell)$ and $\mathbb{E}(I_2^\ell)$ separately. Define

$$R_n\overset{\text{def}}{=}\{\mathbf{S}_0,\,\mathbf{S}_1,\dots,\mathbf{S}_n\},$$

which is the range of the random walk up to step n. When $\|\mathbf{x}\|\ge n^{1/2}$, we have

$$\mathbb{P}(\mathbf{x}\in R_n)\le\mathbb{P}\left(\max_{0\le k\le n}\|\mathbf{S}_k\|\ge\|\mathbf{x}\|\right)\le c_{10}\,\exp\left(-c_{11}\,\frac{\|\mathbf{x}\|^2}{n}\right),\tag{4.9}$$

for some absolute constants c_{10} and c_{11}.

Since $\xi^m(n,\mathbf{x})=\xi^m(n,\mathbf{x})\mathbf{1}_{\{\mathbf{x}\in R_n\}}$ for any $\mathbf{x}\in\mathbb{Z}^2$, we can apply Hölder's inequality to see that, if $\|\mathbf{x}_i\|\ge n^{1/2}(\log n)^\nu$ for all $1\le i\le\ell$,

$$\begin{aligned}\mathbb{E}\left(\prod_{i=1}^\ell\xi^m(n,\mathbf{x}_i)\right) &\le \left(\prod_{i=1}^\ell\mathbb{E}(\xi^{2\ell m}(n,\mathbf{x}_i))\right)^{1/(2\ell)}\left(\prod_{i=1}^\ell\mathbb{P}(\mathbf{x}_i\in R_n)\right)^{1/(2\ell)}\\ &\le c_{12}\,(\log n)^{m\ell}\exp\left(-\frac{c_{11}}{2\ell}\,\frac{\sum_{i=1}^\ell\|\mathbf{x}_i\|^2}{n}\right),\end{aligned}$$

the last inequality following from (4.2) and (4.9). Therefore, by (4.8) and the assumption $\eta(n,\mathbf{x}) \le \|\mathbf{x}\|^\alpha$,

$$\begin{aligned}\mathbb{E}(I_1^\ell) &\le c_{12}\,(\log n)^{m\ell}\left[\sum_{\|\mathbf{x}\|\ge n^{1/2}(\log n)^\nu}\|\mathbf{x}\|^\alpha \exp\left(-\frac{c_{11}}{2\ell}\,\frac{\|\mathbf{x}\|^2}{n}\right)\right]^\ell \\ &\le c_{13}\,n^{\ell+\alpha\ell/2}\,\exp\left(-c_8\,(\log n)^{2\nu}\right). \qquad (4.10)\end{aligned}$$

In the last inequality, we used the fact that for any fixed constant $c>0$, when $n\to\infty$, $\sum_{\|\mathbf{x}\|\ge n^{1/2}(\log n)^\nu}\|\mathbf{x}\|^\alpha\exp(-c\|\mathbf{x}\|^2/n)=\mathcal{O}(n^{\alpha/2+1}(\log n)^{\alpha\nu}\exp(-c(\log n)^{2\nu}))$.

To estimate $\mathbb{E}(I_2^\ell)$, we recall Lemma 22.5 of Révész (1990, p. 224) (with slightly different notation): for any sites $\mathbf{x}_1,\cdots,\mathbf{x}_\ell$ in $\mathbb{Z}^2$,

$$\mathbb{P}\left(\mathbf{x}_1\in R_n,\ \cdots,\ \mathbf{x}_\ell\in R_n\right)\le(\mu(n))^{\ell-1}\max_{1\le i\le\ell}\mathbb{P}\left(\mathbf{x}_i\in R_n\right), \qquad (4.11)$$

where $\mu(n)\overset{\text{def}}{=}\max_{1\le i<j\le n}\mathbb{P}(\mathbf{x}_i-\mathbf{x}_j\in R_n)$. We take this opportunity to correct a misprint in page 224 of Révész (1990), where the definition of $\mu(n)$ is mistakenly stated as $\max_{1\le i\le n}\mathbb{P}(\mathbf{x}_i\in R_n)$.

We now apply Hölder's inequality. Let $a>1$. It is possible to find $p>1$ such that $1/a+1/p=1$. Then (writing $A\overset{\text{def}}{=}\{\mathbf{x}_1\in R_n,\ \cdots,\ \mathbf{x}_\ell\in R_n\}$ for brevity)

$$\begin{aligned}\mathbb{E}\left(\prod_{i=1}^\ell\xi^m(n,\mathbf{x}_i)\right) &= \mathbb{E}\left(\xi^m(n,\mathbf{x}_1)\cdots\xi^m(n,\mathbf{x}_\ell)\,\mathbf{1}_A\right)\\ &\le \left[\mathbb{E}\left(\xi^{mp}(n,\mathbf{x}_1)\cdots\xi^{mp}(n,\mathbf{x}_\ell)\right)\right]^{1/p}\,\left[\mathbb{P}(A)\right]^{1/a},\end{aligned}$$

which, according to (4.3) and (4.11), is

$$\le c_{14}\,(\log n)^{m\ell}\sum_{1\le i<j\le\ell}\mathbb{P}(\mathbf{x}_i-\mathbf{x}_j\in R_n)^{(\ell-1)/a}\sum_{k=1}^\ell\mathbb{P}(\mathbf{x}_k\in R_n)^{1/a}.$$

Plugging this into (4.8) (and using symmetry) yields that for some $c_{15}=c_{15}(m,\ell,a)$,

$$\begin{aligned}\mathbb{E}(I_2^\ell) &\le c_{15}\,\varphi_{n,\ell,\nu}\,(\log n)^{m\ell}(n(\log n)^{2\nu})^{\ell-2}I_3\\ &\quad +c_{15}\,\varphi_{n,\ell,\nu}\,(\log n)^{m\ell}(n(\log n)^{2\nu})^{\ell-3}I_4, \qquad (4.12)\end{aligned}$$

where

$$I_3\ \overset{\text{def}}{=}\ \sum_{\mathbf{x}_1\in\Omega_2}\sum_{\mathbf{x}_2\in\Omega_2}\mathbb{P}(\mathbf{x}_1-\mathbf{x}_2\in R_n)^{(\ell-1)/a}\,\mathbb{P}(\mathbf{x}_1\in R_n)^{1/a},$$

$$I_4 \stackrel{\text{def}}{=} \sum_{\mathbf{x}_1\in\Omega_2}\sum_{\mathbf{x}_2\in\Omega_2}\sum_{\mathbf{x}_3\in\Omega_2} \mathbb{P}(\mathbf{x}_1-\mathbf{x}_2\in R_n)^{(\ell-1)/a}\,\mathbb{P}(\mathbf{x}_3\in R_n)^{1/a}.$$

(When $\ell=1$, we simply have $\mathbb{E}(I_2^\ell)\le c_{15}\,\varphi_{n,\ell,\nu}\,(\log n)^m\sum_{\mathbf{x}\in\Omega_2}\mathbb{P}(\mathbf{x}\in R_n)^{1/a}$. When $\ell=2$, I_4 should be considered as 0).

If $(\mathbf{x}_1,\mathbf{x}_2)\in\Omega_2\times\Omega_2$, then

$$\mathbf{x}_1-\mathbf{x}_2\in\widetilde{\Omega}_2\stackrel{\text{def}}{=}\left\{\mathbf{x}\in\mathbb{Z}^2:\ \|\mathbf{x}\|<2n^{1/2}(\log n)^\nu\right\}.$$

Thus

$$\begin{aligned} I_3 &\le \sum_{\mathbf{x}_1\in\Omega_2}\mathbb{P}(\mathbf{x}_1\in R_n)^{1/a}\sum_{\mathbf{x}\in\widetilde{\Omega}_2}\mathbb{P}(\mathbf{x}\in R_n)^{(\ell-1)/a}\\ &\le \sum_{\mathbf{x}_1\in\widetilde{\Omega}_2}\mathbb{P}(\mathbf{x}_1\in R_n)^{1/a}\sum_{\mathbf{x}\in\widetilde{\Omega}_2}\mathbb{P}(\mathbf{x}\in R_n)^{(\ell-1)/a}. \end{aligned} \tag{4.13}$$

Similarly,

$$I_4\le c_{16}\,n(\log n)^{2\nu}\sum_{\mathbf{x}_3\in\widetilde{\Omega}_2}\mathbb{P}(\mathbf{x}_3\in R_n)^{1/a}\sum_{\mathbf{x}\in\widetilde{\Omega}_2}\mathbb{P}(\mathbf{x}\in R_n)^{(\ell-1)/a}. \tag{4.14}$$

To see how $\sum_{\mathbf{x}\in\widetilde{\Omega}_2}\mathbb{P}(\mathbf{x}\in R_n)^b$ behaves, we recall the following result of Erdős and Taylor (1960): for $n\ge 2$ and $n^{1/6}<\|\mathbf{x}\|<n^{1/2}/20$,

$$\mathbb{P}\left(\mathbf{x}\in R_n\right)\le c_{17}\frac{\log(n^{1/2}/\|\mathbf{x}\|)}{\log n}.$$

This clearly also yields $\mathbb{P}(\mathbf{x}\in R_n)\le c_{18}/\log n$ for $\|\mathbf{x}\|\ge n^{1/2}/20$. When $\|\mathbf{x}\|\le n^{1/6}$, $\log(n^{1/2}/\|\mathbf{x}\|)/\log n$ is of constant order (except for the special case $\mathbf{x}=\mathbf{0}$). Therefore,

$$\mathbb{P}\left(\mathbf{x}\in R_n\right)\le c_{19}\frac{\log_*(n^{1/2}/(\|\mathbf{x}\|+1))}{\log n},\qquad n\ge 2,\ \mathbf{x}\in\mathbb{Z}^2, \tag{4.15}$$

where $\log_* u\stackrel{\text{def}}{=}\log\max(u,\mathrm{e})$ for all $u\in\mathbb{R}$.

Accordingly, for any $b\ge 0$,

$$\begin{aligned} \sum_{\mathbf{x}\in\widetilde{\Omega}_2}\mathbb{P}(\mathbf{x}\in R_n)^b &\le c_{20}\sum_{\mathbf{x}\in\widetilde{\Omega}_2}\left(\frac{\log_*(n^{1/2}/(\|\mathbf{x}\|+1))}{\log n}\right)^b\\ &\le c_{21}\int_0^{2n^{1/2}(\log n)^\nu}\left(\frac{\log_*(n^{1/2}/(r+1))}{\log n}\right)^b r\,\mathrm{d}r\\ &\le c_{22}\,n(\log n)^{-b+2\nu}, \end{aligned}$$

where $c_{22} = c_{22}(b, \nu)$. Plugging this into (4.13) and (4.14) gives $I_3 \le c_{23}\, n^2 (\log n)^{-\ell/a + 4\nu}$ and $I_4 \le c_{24}\, n^3 (\log n)^{-\ell/a+6\nu}$. Going back to (4.12), we obtain: for some $c_{25} = c_{25}(m, \ell, \nu, a)$,

$$\mathbb{E}(I_2^\ell) \le c_{25}\, \varphi_{n,\ell,\nu}\, n^\ell (\log n)^{m\ell + 2\nu\ell - \ell/a}.$$

Since $\nu < \varepsilon/(2\ell)$ and since $a > 1$ is arbitrary, combining this estimate with (4.7) and (4.10) completes the proof of Lemma 4.1. $\square$

Now we are ready to prove Lemma 2.2.

Proof of Lemma 2.2. By definition,

$$V_k - U_k = \sum_{\mathbf{x} \in A_k} \xi_k(\mathbf{x}) (\widehat{Y}_k(\mathbf{x}) - \widehat{Y}(\mathbf{x})).$$

Let $\mathbb{P}^{\mathbf{S}}(\,\cdot\,) = \mathbb{P}(\,\cdot\,|\{\mathbf{S}_n\}_{n\ge 0})$, the conditional probability given the random walk. We write $\mathbb{E}^{\mathbf{S}}$ for the expectation associated with this conditional probability. Under $\mathbb{P}^{\mathbf{S}}$, for each k, $\{\widehat{Y}_k(\mathbf{x}) - \widehat{Y}(\mathbf{x}),\ \mathbf{x} \in A_k\}$ are independent mean-zero variables.

Recall Rosenthal's inequality (see for example Petrov, 1995, p. 59): if $X_1, \cdots, X_n$ are independent mean-zero variables and if $p \ge 2$, then

$$\mathbb{E}\left(\left| \sum_{i=1}^n X_i \right|^p \right) \le C(p) \left[\sum_{i=1}^n \mathbb{E}(|X_i|^p) + \left(\sum_{i=1}^n \mathbb{E}(X_i^2) \right)^{p/2} \right], \tag{4.16}$$

where $C(p) \in (0, \infty)$ is a constant depending only on p.

Let q be the constant in Theorem 1.1, and let $p > q$ be an even integer. Note that

$$\mathbb{E}^{\mathbf{S}}(|\widehat{Y}(\mathbf{x})|^p) \le c_{26}\, (1 + \|\mathbf{x}\|^{p-q}), \tag{4.17}$$

for some constant $c_{26} = c_{26}(q) > 0$.

Applying (4.16) to our conditional probability $\mathbb{P}^{\mathbf{S}}$ yields

$$\begin{aligned} \mathbb{E}[|V_k - U_k|^p] &= \mathbb{E}\left[\mathbb{E}^{\mathbf{S}} \left(\left| \sum_{\mathbf{x} \in A_k} \xi_k(\mathbf{x}) (\widehat{Y}_k(\mathbf{x}) - \widehat{Y}(\mathbf{x})) \right|^p \right) \right] \\ &\le c_{27}\, \mathbb{E}\left(\sum_{\mathbf{x} \in A_k} \xi_k^p(\mathbf{x})\, \|\mathbf{x}\|^{p-q} \right) + c_{28}\, \mathbb{E}\left[\left(\sum_{\mathbf{x} \in A_k} \xi_k^2(\mathbf{x}) \right)^{p/2} \right]. \end{aligned} \tag{4.18}$$

We write $\|\mathbf{x}\|^{p-q}$ instead of $1 + \|\mathbf{x}\|^{p-q}$ on the right hand side because $\sum_{\mathbf{x} \in A_k} \xi_k^p(\mathbf{x}) \le (\sum_{\mathbf{x} \in A_k} \xi_k^2(\mathbf{x}))^{p/2}$.

We now estimate the two expectation expressions on the right hand side. For the second expression, we note that $\sum_{\mathbf{x}\in A_k}\xi_k^2(\mathbf{x}) = \sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^2(\mathbf{x})\mathbf{1}_{\{\mathbf{x}\in R_{n_k}\}}$, where R_n denotes as before the range of the random walk up to step n. Let $\widetilde{\mathbf{S}}_j \stackrel{\text{def}}{=} \mathbf{S}_{j+n_k} - \mathbf{S}_{n_k}$. Then $\{\widetilde{\mathbf{S}}_j\}_{j\geq 0}$ is again a simple symmetric random walk on $\mathbb{Z}^2$, independent of $\{\mathbf{S}_n\}_{0\leq n\leq n_k}$. If we define $\widetilde{\xi}(j,\mathbf{y}) \stackrel{\text{def}}{=} \sum_{i=0}^{j}\mathbf{1}_{\{\widetilde{\mathbf{S}}_i=\mathbf{y}\}}$, the local time of the new random walk, then $\xi_k(\mathbf{x}) = \widetilde{\xi}(n_{k+1}-n_k, \mathbf{x}-\mathbf{S}_{n_k})$. By a change of variables $\mathbf{y} = \mathbf{x}-\mathbf{S}_{n_k}$,

$$\sum_{\mathbf{x}\in A_k}\xi_k^2(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{Z}^2}\widetilde{\xi}^{\,2}(n_{k+1}-n_k,\mathbf{y})\mathbf{1}_{\{\mathbf{y}+\mathbf{S}_{n_k}\in R_{n_k}\}}.$$

Note that $\{\mathbf{1}_{\{\mathbf{y}+\mathbf{S}_{n_k}\in R_{n_k}\}};\, \mathbf{y}\in\mathbb{Z}^2\}$ is independent of $\{\widetilde{\mathbf{S}}_j\}_{j\geq 0}$ (thus of its local times), and is distributed as $\{\mathbf{1}_{\{\mathbf{y}\in R_{n_k}\}};\, \mathbf{y}\in\mathbb{Z}^2\}$ (this is easily seen using time reversal). As a consequence,

$$\mathbb{E}\left[\left(\sum_{\mathbf{x}\in A_k}\xi_k^2(\mathbf{x})\right)^{p/2}\right] = \mathbb{E}\left[\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\widetilde{\xi}^{\,2}(n_{k+1}-n_k,\mathbf{x})\mathbf{1}_{\{\mathbf{x}\in R_{n_k}\}}\right)^{p/2}\right].$$

For the expression on the right hand side, we can apply Lemma 4.1 to $n = n_{k+1}-n_k$ and $\nu = \min((1-\varrho)/(3\varrho),\, \varepsilon/(2p))$. To see how $\varphi_{n,\ell,\nu}$ (defined in (4.5)) behaves in this setting, we observe that by (4.11), if $\|\mathbf{x}_i\| < n^{1/2}(\log n)^\nu$ (for all $1\leq i\leq \ell$), then

$$\mathbb{P}\left(\mathbf{x}_1\in R_{n_k},\ \cdots\ \mathbf{x}_\ell\in R_{n_k}\right) \leq \sup_{\|\mathbf{x}\|<2n^{1/2}(\log n)^\nu}\left(\mathbb{P}(\mathbf{x}\in R_{n_k})\right)^\ell,$$

which, according to (4.15), is

$$\leq c_{29}\left(\frac{\log\{n_k^{1/2}/(2n^{1/2}(\log n)^\nu)\}}{\log n_k}\right)^\ell \leq c_{30}\left(\frac{\log\log n_k}{\log n_k}\right)^\ell.$$

(We have used (2.2) and the fact that $(1-\varrho)/(2\varrho) > \nu$). Therefore, by (4.4) (taking $m=2$ and $\ell = p/2$ there; this is the place where we need p to be an even integer),

$$\mathbb{E}\left[\left(\sum_{\mathbf{x}\in A_k}\xi_k^2(\mathbf{x})\right)^{p/2}\right] \leq c_{31}\, n_k^{p/2}(\log n_k)^{p/2-p/(2\varrho)+\varepsilon}(\log\log n_k)^{p/2}. \tag{4.19}$$

We now estimate the expression $\mathbb{E}(\sum_{\mathbf{x}\in A_k}\xi_k^p(\mathbf{x})\|\mathbf{x}\|^{p-q})$ on the right hand side of (4.18). For further applications in Section 5, we estimate $\mathbb{E}\{(\sum_{\mathbf{x}\in A_k}\xi_k^p(\mathbf{x})\|\mathbf{x}\|^\alpha)^\ell\}$ for $\alpha\geq 0$ and integer $\ell\geq 1$.

By the same argument as before, we see that the random variable $\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^p(\mathbf{x})\|\mathbf{x}\|^\alpha$ is distributed as $\sum_{\mathbf{x}\in\mathbb{Z}^2}\widetilde{\xi}^{\,p}(n_{k+1}-n_k,\mathbf{x})\,\|\mathbf{x}-\mathbf{S}_{n_k}\|^\alpha$, where $\widetilde{\xi}$ is independent of the variable $\mathbf{S}_{n_k}$. Thus,

$$\begin{aligned}
\mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^p(\mathbf{x})\|\mathbf{x}\|^\alpha\right)^\ell &= \mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\widetilde{\xi}^{\,p}(n_{k+1}-n_k,\mathbf{x})\,\|\mathbf{x}-\mathbf{S}_{n_k}\|^\alpha\right)^\ell\\
&\le c_{32}\,\mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\widetilde{\xi}^{\,p}(n_{k+1}-n_k,\mathbf{x})\,\|\mathbf{x}\|^\alpha\right)^\ell\\
&\quad +c_{32}\,\mathbb{E}\left\{\|\mathbf{S}_{n_k}\|^{\alpha\ell}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\widetilde{\xi}^{\,p}(n_{k+1}-n_k,\mathbf{x})\right)^\ell\right\}\\
&\le c_{32}\,\mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^p(n_{k+1}-n_k,\mathbf{x})\,\|\mathbf{x}\|^\alpha\right)^\ell\\
&\quad +c_{33}\,n_k^{\alpha\ell/2}\,\mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^p(n_{k+1}-n_k,\mathbf{x})\right)^\ell.
\end{aligned}$$

We can apply (4.6) to see that for any $\varepsilon>0$,

$$\begin{aligned}
\mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^p(n_{k+1}-n_k,\mathbf{x})\,\|\mathbf{x}\|^\alpha\right)^\ell &\le c_{34}\,n_k^{\ell+\alpha\ell/2}(\log n_k)^{-c_{35}\,\ell(1-\varrho)/\varrho+(p-1)\ell+\varepsilon},\\
n_k^{\alpha\ell/2}\,\mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^p(n_{k+1}-n_k,\mathbf{x})\right)^\ell &\le c_{36}\,n_k^{\ell+\alpha\ell/2}(\log n_k)^{-\ell(1-\varrho)/\varrho+(p-1)\ell+\varepsilon},
\end{aligned}$$

where $c_{35}\stackrel{\text{def}}{=}1+\alpha/2\ge 1$. Note that $-\ell(1-\varrho)/\varrho+(p-1)\ell=-\ell/\varrho+p\ell$. Consequently, for any $\varepsilon>0$,

$$\mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^p(\mathbf{x})\,\|\mathbf{x}\|^\alpha\right)^\ell\le c_{37}\,n_k^{\ell+\alpha\ell/2}(\log n_k)^{-\ell/\varrho+p\ell+\varepsilon}, \tag{4.20}$$

for some $c_{37}=c_{37}(p,\ell,\alpha,\varrho,\varepsilon)$. This is a general estimate which we shall use for several times in Section 5.

Take $\alpha=p-q$ and $\ell=1$, and since $\sum_{\mathbf{x}\in A_k}\xi_k^p(\mathbf{x})\|\mathbf{x}\|^{p-q}\le\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^p(\mathbf{x})\|\mathbf{x}\|^{p-q}$, we obtain:

$$\mathbb{E}\left(\sum_{\mathbf{x}\in A_k}\xi_k^p(\mathbf{x})\,\|\mathbf{x}\|^{p-q}\right)\le c_{38}\,n_k^{1+(p-q)/2}(\log n_k)^{-1/\varrho+p+\varepsilon}.$$

Plugging this into (4.19) and (4.18) yields that, for any $\varepsilon>0$,

$$\mathbb{E}[\,|V_k-U_k|^p]\le c_{39}\,n_k^{p/2}(\log n_k)^{p/2-p/(2\varrho)+\varepsilon}.$$

Lemma 2.2 now follows by means of an application of Chebyshev's inequality and the Borel–Cantelli lemma. $\square$

5 Proof of Lemma 2.3

We use the Skorokhod embedding schema (for more details, see Skorokhod, 1965) summarized as follows. Let X be a random variable with $\mathbb{E}(X) = 0$ and $\mathbb{E}(|X|^p) < \infty$ for some $p \geq 2$, and let $\{W(t);\, t \geq 0\}$ be any given Wiener process starting from 0. The Skorokhod embedding ensures the existence of (finite) stopping time τ such that $W(\tau)$ is distributed as X, and that $\mathbb{E}(\tau) = \mathbb{E}(X^2)$. Moreover, for any $a \in [1,\, p/2]$,

$$\mathbb{E}(\tau^a) \leq C(p)\, \mathbb{E}(|X|^{2a}),$$

where $C(p) \in (0, \infty)$ is a constant whose value depends only on p. By iterating the construction and using the strong Markov property, this yields an embedding of independent but not necessarily identically distributed variables into a Wiener process: if $\{X_k\}_{k\geq 1}$ is a sequence of independent random variables, with $\mathbb{E}(X_k) = 0$ and $\mathbb{E}(|X_k|^p) < \infty$ for some $p \geq 2$ and all $k \geq 1$, then there exists a non-decreasing sequence of finite stopping times $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots$ with $\mathbb{E}(\tau_k - \tau_{k-1}) = \mathbb{E}(X_k^2)$ for any $k \geq 1$, such that

$$\{X_k\}_{k\geq 1} \stackrel{\text{(law)}}{=} \{W(\tau_k) - W(\tau_{k-1})\}_{k\geq 1}\,,$$

where "$\stackrel{\text{(law)}}{=}$" stands for identity in law. Moreover, for any $k \geq 1$ and any $1 \leq a \leq p/2$,

$$\mathbb{E}\left((\tau_k - \tau_{k-1})^a\right) \leq C(p)\, \mathbb{E}(|X_k|^{2a}).$$

Proof of Lemma 2.3. Let $\mathbb{P}^{\mathbf{S}}(\,\cdot\,) = \mathbb{P}(\,\cdot\,|\{\mathbf{S}_n\}_{n\geq 0})$ as before, and let $\{V_k\}_{k\geq 1}$ be the sequence of random variables defined in (2.4). Under $\mathbb{P}^{\mathbf{S}}$, these are mean-zero independent variables with $\mathbb{E}^{\mathbf{S}}(|V_k|^p) < \infty$ (for all $p \geq 0$), so that by the aforementioned Skorokhod-type embedding, there exist finite stopping times $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots$ satisfying $\mathbb{E}^{\mathbf{S}}(\tau_k - \tau_{k-1}) = \mathbb{E}^{\mathbf{S}}(V_k^2)$ and $\mathbb{E}^{\mathbf{S}}((\tau_k - \tau_{k-1})^p) \leq c_{40}\, \mathbb{E}^{\mathbf{S}}(|V_k|^{2p})$ for any $k \geq 1$ and $p \geq 1$, such that $\{V_k\}_{k\geq 1} \stackrel{\text{(law)}}{=} \{W(\tau_k) - W(\tau_{k-1})\}_{k\geq 1}$.

Without loss of generality, we assume that $\{V_k\}_{k\geq 1} = \{W(\tau_k) - W(\tau_{k-1})\}_{k\geq 1}$ (otherwise, by a usual coupling argument, we can work in an enlarged probability space, with redefined variables and processes; see for example page 53 of Berkes and Philipp, 1979 for more details). Therefore, for any $\ell \geq 1$,

$$W(\tau_{\ell-1}) = \sum_{k=1}^{\ell-1} V_k. \tag{5.1}$$

In order to show Lemma 2.3 we state and prove several lemmas.

Let

$$T_k \stackrel{\text{def}}{=} \mathbb{E}^{\mathbf{S}}(\tau_k - \tau_{k-1}) = \mathbb{E}^{\mathbf{S}}(V_k^2) = \sum_{\mathbf{x}\in\mathbb{Z}^2} \xi_k^2(\mathbf{x})\mathbb{E}(\widehat{Y}^2(\mathbf{x})).$$

Lemma 5.1 *For any $\varepsilon > 0$, as $\ell \to \infty$ we have*

$$\tau_{\ell-1} - \sum_{k=1}^{\ell-1} T_k = o\left(n_\ell(\log n_\ell)^{3/2-1/(2\varrho)+\varepsilon}\right), \qquad \text{a.s.} \tag{5.2}$$

Proof. We can write

$$\tau_{\ell-1} - \sum_{k=1}^{\ell-1} T_k = \sum_{k=1}^{\ell-1} ((\tau_k - \tau_{k-1}) - T_k) \stackrel{\text{def}}{=} \sum_{k=1}^{\ell-1} \Delta_k,$$

and note that $\{\Delta_k\}_{k\geq 1}$ is a sequence of independent mean-zero variables under $\mathbb{P}^{\mathbf{S}}$. By Rosenthal's inequality recalled in (4.16), for any $p \geq 4$,

$$\mathbb{E}^{\mathbf{S}}\left(\left|\tau_{\ell-1} - \sum_{k=1}^{\ell-1} T_k\right|^{p/2}\right) \leq c_{41}\left[\sum_{k=1}^{\ell-1} \mathbb{E}^{\mathbf{S}}(|\Delta_k|^{p/2}) + \left(\sum_{k=1}^{\ell-1}\mathbb{E}^{\mathbf{S}}(\Delta_k^2)\right)^{p/4}\right].$$

Since $\mathbb{E}^{\mathbf{S}}(|\Delta_k|^{p/2}) \leq c_{42}\,\mathbb{E}^{\mathbf{S}}((\tau_k - \tau_{k-1})^{p/2}) \leq c_{42}c_{40}\,\mathbb{E}^{\mathbf{S}}(|V_k|^p)$, and $\mathbb{E}^{\mathbf{S}}(\Delta_k^2) \leq \mathbb{E}^{\mathbf{S}}((\tau_k - \tau_{k-1})^2) \leq c_{40}\,\mathbb{E}^{\mathbf{S}}(V_k^4)$, this leads to:

$$\mathbb{E}^{\mathbf{S}}\left(\left|\tau_{\ell-1} - \sum_{k=1}^{\ell-1} T_k\right|^{p/2}\right) \leq c_{43}\sum_{k=1}^{\ell-1} \mathbb{E}^{\mathbf{S}}(|V_k|^p) + c_{44}\left(\sum_{k=1}^{\ell-1}\mathbb{E}^{\mathbf{S}}(V_k^4)\right)^{p/4}. \tag{5.3}$$

At this stage, we need to estimate $\mathbb{E}^{\mathbf{S}}(|V_k|^p)$. This can be done by another application of Rosenthal's inequality in (4.16), for V_k is sum of independent mean-zero variables under $\mathbb{P}^{\mathbf{S}}$:

$$\mathbb{E}^{\mathbf{S}}(|V_k|^p) \leq c_{45}\sum_{\mathbf{x}\in\mathbb{Z}^2} \xi_k^p(\mathbf{x})\|\mathbf{x}\|^{p-q} + c_{46}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^2(\mathbf{x})\right)^{p/2}.$$

We have used (4.17) and the fact that $\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^p(\mathbf{x}) \leq (\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^2(\mathbf{x}))^{p/2}$.

Plugging this into (5.3), and taking expectation (with respect to $\mathbb{E}$) on both sides, we obtain:

$$\begin{aligned}
\mathbb{E}\left(\left|\tau_{\ell-1}-\sum_{k=1}^{\ell-1} T_k\right|^{p/2}\right) &\leq c_{47}\sum_{k=1}^{\ell-1}\mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^p(\mathbf{x})\|\mathbf{x}\|^{p-q}\right)+c_{48}\sum_{k=1}^{\ell-1}\mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^2(\mathbf{x})\right)^{p/2} \\
&\quad +c_{49}\,\mathbb{E}\left(\sum_{k=1}^{\ell-1}\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^4(\mathbf{x})\,\|\mathbf{x}\|^{4-q}+\sum_{k=1}^{\ell-1}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^2(\mathbf{x})\right)^2\right)^{p/4} \\
&\stackrel{\text{def}}{=} I_5+I_6+I_7. \qquad (5.4)
\end{aligned}$$

We now assume that $p>4$ is an even integer. By (4.20),

$$\begin{aligned}
\mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^p(\mathbf{x})\,\|\mathbf{x}\|^{p-q}\right) &\leq c_{50}\,n_k^{1+(p-q)/2}(\log n_k)^{-1/\varrho+p+\varepsilon}, \\
\mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^2(\mathbf{x})\right)^{p/2} &\leq c_{51}\,n_k^{p/2}(\log n_k)^{-p/(2\varrho)+p+\varepsilon},
\end{aligned}$$

which yields

$$\begin{aligned}
I_5 &\leq c_{52}\,I_6\leq c_{53}\sum_{k=1}^{\ell-1} n_k^{p/2}(\log n_k)^{-p/(2\varrho)+p+\varepsilon} \\
&\leq c_{54}\,n_\ell^{p/2}(\log n_\ell)^{1/\varrho-p/(2\varrho)+p-1+\varepsilon}. \qquad (5.5)
\end{aligned}$$

To estimate I_7, we first note that

$$I_7\leq c_{55}\,\mathbb{E}\left(\sum_{k=1}^{\ell-1}\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^4(\mathbf{x})\,\|\mathbf{x}\|^{4-q}\right)^{p/4}+c_{56}\,\mathbb{E}\left(\sum_{k=1}^{\ell-1}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^2(\mathbf{x})\right)^2\right)^{p/4}.$$

Observe that for any b and α, $\{\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^b(\mathbf{x})\,\|\mathbf{x}\|^\alpha\}_{k\geq 1}$ is a sequence of independent random variables. Now we make use of another inequality of Rosenthal, which can be found in Petrov (1995, p. 63): let $p>1$ and let $X_1, X_2, \cdots$ be independent variables with $\mathbb{E}(|X_k|^p)<\infty$ for all $k\geq 1$. Then there exists a constant $c(p)$ depending only on p, such that for all $n\geq 1$,

$$\mathbb{E}\left(\left|\sum_{k=1}^n X_k\right|^p\right)\leq c(p)\left[\sum_{k=1}^n\mathbb{E}(|X_k|^p)+\left(\sum_{k=1}^n\mathbb{E}(|X_k|)\right)^p\right].$$

Applying this inequality to $X_k = \sum_{\mathbf{x}\in\mathbb{Z}^2} \xi_k^4(\mathbf{x})\,\|\mathbf{x}\|^{4-q}$ and to $X_k = (\sum_{\mathbf{x}\in\mathbb{Z}^2} \xi_k^2(\mathbf{x}))^2$, respectively, and we obtain:

$$\begin{aligned} I_7 &\le c_{57} \sum_{k=1}^{\ell-1} \mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2} \xi_k^4(\mathbf{x})\,\|\mathbf{x}\|^{4-q}\right)^{p/4} + c_{58} \left(\sum_{k=1}^{\ell-1} \mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2} \xi_k^4(\mathbf{x})\,\|\mathbf{x}\|^{4-q}\right)\right)^{p/4} \\ &\quad + c_{59} \sum_{k=1}^{\ell-1} \mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2} \xi_k^2(\mathbf{x})\right)^{p/2} + c_{60} \left(\sum_{k=1}^{\ell-1} \mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2} \xi_k^2(\mathbf{x})\right)^2\right)^{p/4}. \end{aligned}$$

By applying (4.20) we can see that the dominating term is

$$\begin{aligned} \left(\sum_{k=1}^{\ell-1} \mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2} \xi_k^2(\mathbf{x})\right)^2\right)^{p/4} &\le c_{61} \left(\sum_{k=1}^{\ell-1} (n_{k+1}-n_k)^2 (\log n_k)^{2+\varepsilon}\right)^{p/4} \\ &\le c_{62}\, n_\ell^{p/2} (\log n_\ell)^{3p/4 - p/(4\varrho) + p\varepsilon/4}, \end{aligned}$$

which means

$$I_7 = \mathcal{O}\left(n_\ell^{p/2} (\log n_\ell)^{3p/4 - p/(4\varrho) + p\varepsilon/4}\right), \qquad \ell\to\infty.$$

Combining this with (5.4) and (5.5) yields that, for any $\varepsilon > 0$,

$$\mathbb{E}\left(\left| \tau_{\ell-1} - \sum_{k=1}^{\ell-1} T_k \right|^{p/2}\right) \le c_{63}\, n_\ell^{p/2} (\log n_\ell)^{3p/4 - p/(4\varrho)+\varepsilon}.$$

By choosing p sufficiently large and applying the Borel–Cantelli lemma, we obtain (5.2). $\square$

The next lemma says that T_k is close to H_k defined by

$$H_k \stackrel{\text{def}}{=} \sigma^2 \sum_{\mathbf{x}\in\mathbb{Z}^2} \xi_k^2(\mathbf{x}). \tag{5.6}$$

Lemma 5.2 *As $\ell\to\infty$,*

$$\sum_{k=1}^{\ell-1} (H_k - T_k) = \mathcal{O}(n_\ell), \qquad \text{a.s.} \tag{5.7}$$

Proof. By definition,

$$\mathbb{E}|H_k - T_k| = \mathbb{E}\left| \sum_{\mathbf{x}\in\mathbb{Z}^2} \xi_k^2(\mathbf{x})(\sigma^2 - \mathbb{E}(\widehat{Y}^2(\mathbf{x})))\right| = \sum_{\mathbf{x}\in\mathbb{Z}^2} \mathbb{E}\left(\xi_k^2(\mathbf{x})\right)\,(\sigma^2 - \mathbb{E}(\widehat{Y}^2(\mathbf{x}))).$$

For each $\mathbf{x} \in \mathbb{Z}^2$,

$$\begin{aligned}\sigma^2 - \mathbb{E}(\widehat{Y}^2(\mathbf{x})) &= \mathbb{E}\left(Y^2(\mathbf{x})\mathbf{1}_{\{Y(\mathbf{x})>\|\mathbf{x}\|\}}\right) + \left[\mathbb{E}\left(Y(\mathbf{x})\mathbf{1}_{\{Y(\mathbf{x})>\|\mathbf{x}\|\}}\right)\right]^2 \\ &\leq \frac{c_{64}}{1+\|\mathbf{x}\|^{q-2}},\end{aligned}$$

whereas $\mathbb{E}\left(\xi_k^2(\mathbf{x})\right) \leq \mathbb{E}\left(\xi^2(n_{k+1},\mathbf{x})\right)$. Therefore,

$$\mathbb{E}|H_k - T_k| \leq c_{64}\,\mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2} \frac{\xi^2(n_{k+1},\mathbf{x})}{1+\|\mathbf{x}\|^{q-2}}\right). \tag{5.8}$$

It is easy to estimate the expression on the right hand side. Indeed, $\sup_{\mathbf{x}\in\mathbb{Z}^2}\mathbb{E}(\xi^2(n,\mathbf{x})) \leq c_{65}\,(\log n)^2$, cf. (4.2). On the other hand, by (4.6), for any $\varepsilon > 0$, there exists $c_{66} = c_{66}(\varepsilon)$ such that $\mathbb{E}[\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^2(n,\mathbf{x})] \leq c_{66}\,n(\log n)^{1+\varepsilon}$. Accordingly,

$$\begin{aligned}\mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\frac{\xi^2(n,\mathbf{x})}{1+\|\mathbf{x}\|^{q-2}}\right) &= \sum_{\|\mathbf{x}\|\leq n^{1/2}} \frac{\mathbb{E}(\xi^2(n,\mathbf{x}))}{1+\|\mathbf{x}\|^{q-2}} + \mathbb{E}\left(\sum_{\|\mathbf{x}\|>n^{1/2}}\frac{\xi^2(n,\mathbf{x})}{1+\|\mathbf{x}\|^{q-2}}\right) \\ &\leq \sum_{\|\mathbf{x}\|\leq n^{1/2}} \frac{c_{65}\,(\log n)^2}{1+\|\mathbf{x}\|^{q-2}} + n^{1-q/2}\,\mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^2(n,\mathbf{x})\right) \\ &\leq c_{67}\,n^{2-q/2}(\log n)^2 + c_{66}\,n^{2-q/2}(\log n)^{1+\varepsilon} \\ &\leq c_{68}\,n^{2-q/2}(\log n)^2. \end{aligned} \tag{5.9}$$

Plugging this into (5.8) yields that

$$\mathbb{E}|H_k - T_k| = \mathcal{O}\left(n_k^{2-q/2}(\log n_k)^2\right), \qquad k\to\infty.$$

By Chebyshev's inequality,

$$\mathbb{P}(|H_k - T_k| \geq n_{k+1} - n_k) \leq c_{69}\,n_k^{1-q/2}(\log n_k)^{1+1/\varrho},$$

which is summable for k. Hence, by the Borel–Cantelli lemma, when $k\to\infty$, $H_k - T_k = \mathcal{O}(n_{k+1}-n_k)$, a.s. This immediately yields (5.7). □

Finally, we need the following lemma to estimate $\sum_{k=1}^{\ell-1} H_k$.

Lemma 5.3 *For any $\varepsilon > 0$, as $\ell\to\infty$,*

$$\sum_{k=1}^{\ell-1} H_k - \frac{2\sigma^2}{\pi}\,n_\ell \log n_\ell = o\left(n_\ell(\log n_\ell)^{1/(2\varrho)+\varepsilon}\right), \qquad \text{a.s.} \tag{5.10}$$

Proof. We now estimate the first two moments of H_k. First, by writing $\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^2(n,\mathbf{x}) = \sum_{i=0}^{n}\sum_{j=0}^{n}\mathbf{1}_{\{\mathbf{S}_i=\mathbf{S}_j\}}$, we have $\mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^2(n,\mathbf{x})\right) = n+1+2\sum_{i=0}^{n-1}\sum_{m=1}^{n-i}\mathbb{P}(\mathbf{S}_m=\mathbf{0})$, which, in view of (4.1), yields that

$$\mathbb{E}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^2(n,\mathbf{x})\right) = \frac{2}{\pi}\, n\log n + \mathcal{O}(n), \qquad n\to\infty. \tag{5.11}$$

For the second moment of $\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^2(n,\mathbf{x})$, Bolthausen (1989) proved (cf. also Lewis, 1993) that

$$\mathrm{Var}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^2(n,\mathbf{x})\right) = \mathcal{O}(n^2), \qquad n\to\infty. \tag{5.12}$$

Recall from (5.6) that $H_k = \sigma^2\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi_k^2(\mathbf{x})$, which is distributed as $\sigma^2\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^2(n_{k+1}-n_k,\mathbf{x})$. Therefore, by (5.11),

$$\begin{aligned}\sum_{k=1}^{\ell-1}\mathbb{E}(H_k) &= \frac{2\sigma^2}{\pi}\sum_{k=1}^{\ell-1}(n_{k+1}-n_k)\log(n_{k+1}-n_k) + \mathcal{O}\left(n_\ell\right)\\ &= \frac{2\sigma^2}{\pi}\, n_\ell\log n_\ell + \mathcal{O}\left(n_\ell\log\log n_\ell\right),\end{aligned}$$

whereas according to (5.12),

$$\mathbb{E}\left(\sum_{k=1}^{\ell-1}(H_k-\mathbb{E}(H_k))\right)^2 = \sum_{k=1}^{\ell-1}\mathrm{Var}(H_k) = \mathcal{O}\left(n_\ell^2(\log n_\ell)^{1-1/\varrho}\right).$$

Consequently,

$$\mathbb{E}\left(\sum_{k=1}^{\ell-1}H_k - \frac{2\sigma^2}{\pi}\, n_\ell\log n_\ell\right)^2 = \mathcal{O}\left(n_\ell^2(\log\log n_\ell)^2\right), \qquad \ell\to\infty.$$

Now (5.10) follows by means of the Borel–Cantelli lemma. □

We are now ready to complete the proof of Lemma 2.3. Indeed, Lemmas 5.1, 5.2 and 5.3 together imply that for any $\varepsilon>0$, almost surely when $\ell\to\infty$,

$$\tau_{\ell-1} - \frac{2\sigma^2}{\pi}\, n_\ell\log n_\ell = o\left(n_\ell(\log n_\ell)^{2\beta+\varepsilon}\right), \tag{5.13}$$

where β is as in (2.9). Note that $\beta<1/2$ and $2\varrho>1$.

Let us recall the following result in Csörgő and Révész (1981), p. 30: let $t \mapsto a_t$ be a non-decreasing function on $\mathbb{R}_+$ such that $0 < a_t \le t$ and that $t \mapsto t/a_t$ is non-decreasing. Then

$$\limsup_{t\to\infty} \frac{\sup_{0\le u\le a_t} \sup_{0\le s\le t-a_t} |W(s+u) - W(s)|}{\sqrt{2a_t(\log(t/a_t) + \log\log t)}} = 1, \qquad \text{a.s.} \tag{5.14}$$

Applying (5.14) to $t = (3\sigma^2/\pi) n_\ell \log n_\ell$ and $a_t = c_{70}\, t/(\log t)^{1-2\beta-\varepsilon}$ (for $\varepsilon \in (0, 1-2\beta)$, of course), and in view of (5.13), we obtain: for any $\varepsilon > 0$,

$$W(\tau_{\ell-1}) - W\left(\frac{2\sigma^2}{\pi} n_\ell \log n_\ell\right) = o\left(n_\ell^{1/2} (\log n_\ell)^{\beta+\varepsilon}\right), \qquad \text{a.s.}$$

In light of (5.1), this yields Lemma 2.3. □

6 Proof of Lemma 2.4

We start with two moment estimates for $Z(n)$ and $\widetilde{Z}(n)$. Recall that $2 < q < 3$.

Lemma 6.1 *There exists a finite and positive constant c_{71} such that*

$$\mathbb{E}\left[(Z(n) - \widehat{Z}(n))^2\right] \le c_{71}\, n^{2-q/2} (\log n)^2, \qquad n \ge 2.$$

Proof. By definition,

$$Z(n) - \widehat{Z}(n) = \sum_{\mathbf{x}\in\mathbb{Z}^2} \xi(n, \mathbf{x}) \ \left(Y(\mathbf{x}) \mathbf{1}_{\{|Y(\mathbf{x})| > \|\mathbf{x}\|\}} - \mathbb{E}\left[Y(\mathbf{x}) \mathbf{1}_{\{|Y(\mathbf{x})| > \|\mathbf{x}\|\}}\right]\right).$$

Let $\mathbb{P}^{\mathbf{S}}(\cdot) \stackrel{\text{def}}{=} \mathbb{P}(\,\cdot\,|\{\mathbf{S}_n\}_{n\ge 0})$ be as before the conditional probability given the random walk. Then

$$\begin{aligned} & \mathbb{E}^{\mathbf{S}}\left[(Z(n) - \widehat{Z}(n))^2\right] \\ = & \sum_{\mathbf{x}\in\mathbb{Z}^2} \xi^2(n, \mathbf{x}) \left\{\mathbb{E}\left[Y^2(\mathbf{x}) \mathbf{1}_{\{|Y(\mathbf{x})| > \|\mathbf{x}\|\}}\right] - \left[\mathbb{E}\left(Y(\mathbf{x}) \mathbf{1}_{\{|Y(\mathbf{x})| > \|\mathbf{x}\|\}}\right)\right]^2\right\} \\ \le & \ c_{72} \sum_{\mathbf{x}\in\mathbb{Z}^2} \frac{\xi^2(n, \mathbf{x})}{1 + \|\mathbf{x}\|^{q-2}}. \end{aligned}$$

Taking expectation (with respect to $\mathbb{E}$) on both sides, and the lemma follows from (5.9). □

Lemma 6.2 *For any $p \geq q$ and $\varepsilon > 0$, there exists a finite and positive constant c_{73} satisfying*

$$\mathbb{E}\left(|\widehat{Z}(n)|^p\right) \leq c_{73}\, n^{p/2}(\log n)^{p/2+\varepsilon}, \qquad n \geq 2.$$

Proof. By Rosenthal's inequality (cf. (4.16)), for any $p \geq q$,

$$\begin{aligned}\mathbb{E}^{\mathbf{S}}\left(|\widehat{Z}(n)|^p\right) &\leq c_{74}\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^p(n,\mathbf{x})\,\mathbb{E}(|\widehat{Y}(\mathbf{x})|^p) + c_{74}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^2(n,\mathbf{x})\,\mathbb{E}(\widehat{Y}^2(\mathbf{x}))\right)^{p/2}\\ &\leq c_{75}\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^p(n,\mathbf{x})\,\|\mathbf{x}\|^{p-q} + c_{76}\left(\sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^2(n,\mathbf{x})\right)^{p/2}.\end{aligned}$$

Taking expectation (with respect to $\mathbb{E}$) on both sides, and applying (4.6) to $\alpha = p - q$ and $\alpha = 0$ respectively, we obtain: for any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{E}\left(|\widehat{Z}(n)|^p\right) &\leq c_{77}\, n^{1+(p-q)/2}(\log n)^{p-1+\varepsilon} + c_{78}\, n^{p/2}(\log n)^{p/2+\varepsilon}\\ &\leq c_{79}\, n^{p/2}(\log n)^{p/2+\varepsilon},\end{aligned}$$

as desired. □

We have now all the ingredients to prove Lemma 2.4.

Proof of Lemma 2.4. Taking $t = b_{n_{\ell+1}}$ and $a_t = c_{80}\, t/(\log t)^{(1-\varrho)/\varrho}$ in (5.14) yields the estimate (2.12). So we only have to check (2.11). We use the following maximal inequality due to Bolthausen (1989): let $X_m \stackrel{\text{def}}{=} \sum_{\mathbf{x}\in\mathbb{Z}^2}\xi^2(m,\mathbf{x})$, then for any $a > \sqrt{2}\,\sigma$ and any $m \geq 1$,

$$\mathbb{P}\left(\max_{0\leq i\leq m} Z(i) \geq a\sqrt{X_m}\right) \leq 2\mathbb{P}\left(Z(m) \geq (a - \sqrt{2}\,\sigma)\sqrt{X_m}\right).$$

Therefore, writing $n \stackrel{\text{def}}{=} n_{\ell+1} - n_\ell$ for brevity,

$$\begin{aligned}I_8 &\stackrel{\text{def}}{=} \mathbb{P}\left(\max_{n_\ell\leq j\leq n_{\ell+1}}(Z(j) - Z(n_\ell)) > (n_{\ell+1} - n_\ell)^{1/2}(\log n_\ell)^{1/2+\varepsilon}\right)\\ &\leq \mathbb{P}\left(\max_{0\leq i\leq n} Z(i) > n^{1/2}(\log n)^{1/2+\varepsilon}\right)\\ &\leq \mathbb{P}\left(\max_{0\leq i\leq n} Z(i) > (\log n)^{\varepsilon}\sqrt{\frac{\pi}{3}X_n}\right) + \mathbb{P}\left(X_n > \frac{3}{\pi}n\log n\right)\\ &\leq 2\mathbb{P}\left(Z(n) > (\sqrt{\pi/3}\,(\log n)^{\varepsilon} - \sqrt{2}\,\sigma)\sqrt{X_n}\right) + \mathbb{P}\left(X_n > \frac{3}{\pi}n\log n\right).\end{aligned}$$

When n is sufficiently large, $\sqrt{\pi/3}\,(\log n)^\varepsilon - \sqrt{2}\,\sigma \geq \sqrt{\pi}\,(\log n)^\varepsilon/2$. Thus

$$\begin{aligned}
I_8 &\leq 2\mathbb{P}\left(Z(n) > \frac{\sqrt{\pi}\,(\log n)^\varepsilon}{2}\sqrt{X_n}\right) + \mathbb{P}\left(X_n > \frac{3}{\pi}n\log n\right) \\
&\leq 2\mathbb{P}\left(Z(n) > \frac{1}{2}n^{1/2}(\log n)^{1/2+\varepsilon}\right) \\
&\qquad +2\mathbb{P}\left(X_n < \frac{n\log n}{\pi}\right) + \mathbb{P}\left(X_n > \frac{3}{\pi}n\log n\right) \\
&\stackrel{\text{def}}{=} 2I_9 + 2I_{10} + I_{11}. \qquad (6.1)
\end{aligned}$$

Observe that by Lemmas 6.1 and 6.2, for any $p \geq 2$ and $\varepsilon > 0$,

$$\begin{aligned}
I_9 &\leq \mathbb{P}\left(Z(n) - \widehat{Z}(n) > \frac{1}{4}n^{1/2}(\log n)^{1/2+\varepsilon}\right) \\
&\qquad +\mathbb{P}\left(\widehat{Z}(n) > \frac{1}{4}n^{1/2}(\log n)^{1/2+\varepsilon}\right) \\
&\leq 16c_{71}\, n^{1-q/2}(\log n)^{1-2\varepsilon} + c_{81}\,(\log n)^{-p\varepsilon/2} \\
&\leq c_{82}\,(\log n)^{-p\varepsilon/2},
\end{aligned}$$

whereas according to (5.11) and (5.12),

$$I_{10} + I_{11} = \mathbb{P}\left(\left|X_n - \frac{2}{\pi}n\log n\right| > \frac{n\log n}{\pi}\right) \leq \frac{c_{83}\,n^2}{[(n\log n)/\pi]^2} = \frac{\pi^2 c_{83}}{(\log n)^2}.$$

We can choose p sufficiently large such that $p\varepsilon/2 \geq 2$. Plugging these estimates into (6.1) yields

$$\mathbb{P}\left(\max_{n_\ell \leq j \leq n_{\ell+1}} (Z(j) - Z(n_\ell)) > (n_{\ell+1} - n_\ell)^{1/2}(\log n_\ell)^{1/2+\varepsilon}\right) \leq \frac{c_{84}}{(\log(n_{\ell+1} - n_\ell))^2},$$

which is summable for ℓ. By the Borel–Cantelli lemma, and since $\varepsilon > 0$ is arbitrary, we have, almost surely for $\ell \to \infty$,

$$\begin{aligned}
\max_{n_\ell \leq j \leq n_{\ell+1}} (Z(j) - Z(n_\ell)) &= o\left((n_{\ell+1} - n_\ell)^{1/2}(\log n_\ell)^{1/2+\varepsilon}\right) \\
&= o\left(n_\ell^{1/2}(\log n_\ell)^{1-1/(2\varrho)+\varepsilon}\right).
\end{aligned}$$

The same estimate holds for $(-Z)$ in place of Z. This yields (2.11), and completes the proof of Lemma 2.4. $\square$

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