

Pointwise and uniform asymptotics of the Vervaat error process

Dedicated to the memory of Arthur Hing-Chiu Chan (1946–1999), PhD 1977, Carleton University

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Abstract. It is well known that, asymptotically, the appropriately normalized uniform Vervaat process, i.e., the integrated uniform Bahadur–Kiefer process properly normalized, behaves like the square of the uniform empirical process. We give a complete description of the strong and weak asymptotic behaviour in sup-norm of this representation of the Vervaat process and, likewise, we also study its pointwise asymptotic behaviour.

Keywords. Empirical process, quantile process, Bahadur–Kiefer process, Vervaat process, Vervaat error process, Kiefer process, Brownian bridge, Wiener process, strong approximation, law of the iterated logarithm, convergence in distribution.

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1 Introduction and preliminary results

Let U_1, U_2, \dots be independent copies of a random variable U uniformly distributed over the interval $[0, 1]$. Let

$$F_n(t) := \frac{1}{n} \sum_{k=1}^n \mathbf{1}\{U_k \leq t\}, \quad 0 \leq t \leq 1,$$

denote the empirical distribution function based on U_1, U_2, \dots, U_n , where $\mathbf{1}$ is the indicator function. Let F_n^{-1} be the left-continuous inverse of F_n . We denote the empirical and quantile processes over the interval $[0, 1]$ by

$$\begin{aligned} \alpha_n(t) &:= n^{1/2}(F_n(t) - t), \quad 0 \leq t \leq 1, \\ \beta_n(t) &:= n^{1/2}(F_n^{-1}(t) - t), \quad 0 \leq t \leq 1, \end{aligned}$$

respectively. The sum

$$R_n(t) := \alpha_n(t) + \beta_n(t), \quad 0 \leq t \leq 1,$$

of the empirical and quantile processes is known in the literature as the Bahadur–Kiefer process (cf. Bahadur, 1966, Kiefer, 1967, 1970).

The Bahadur–Kiefer process enjoys some remarkable asymptotic properties, which are of interest in statistical quantile data analysis (cf., e.g., Csörgő, 1983, Shorack and Wellner, 1986). We summarize the most relevant results of Kiefer (1967, 1970) in this regard in the following theorem. Throughout the paper, we use the notation $\log_2 x := \log \log x$ for $x > 3$.

Theorem A (Kiefer, 1967, 1970) *For every fixed $t \in (0, 1)$, we have*

$$n^{1/4} R_n(t) \rightarrow_d (t(1-t))^{1/4} \mathcal{N}(|\widetilde{\mathcal{N}}|)^{1/2}, \quad n \rightarrow \infty, \quad (1.1)$$

$$\limsup_{n \rightarrow \infty} \frac{n^{1/4} |R_n(t)|}{(\log_2 n)^{3/4}} = (t(1-t))^{1/4} \frac{2^{5/4}}{3^{3/4}} \quad \text{a.s.}, \quad (1.2)$$

where \mathcal{N} and $\widetilde{\mathcal{N}}$ are independent standard normal variables and \rightarrow_d denotes convergence in distribution. Also,

$$\lim_{n \rightarrow \infty} n^{1/4} (\log n)^{-1/2} \frac{\|R_n\|}{(\|\alpha_n\|)^{1/2}} = 1 \quad \text{a.s.}, \quad (1.3)$$

where $\|f\| := \sup_{0 \leq t \leq 1} |f(t)|$ denotes the uniform sup-norm of f .

Concerning the a.s. result in (1.3), Kiefer announced it, but did not publish his proof. However, he proved convergence in probability (cf. Theorem 1A, and the two sentences right after, in Kiefer, 1970). Eventually, the upper bound for the almost sure convergence in (1.3) was established by Shorack (1982), and the lower bound by Deheuvels and Mason (1990). For a review of these results and for further developments along these lines we refer to Shorack and Wellner (1986), Deheuvels and Mason (1992, 1994), Einmahl (1996), Csörgő and Szyszkowicz (1998), Deheuvels (2000).

Via using the usual and the other laws of the iterated logarithm for α_n (cf. Theorem G below), (1.3) immediately implies

$$\limsup_{n \rightarrow \infty} n^{1/4} (\log n)^{-1/2} (\log_2 n)^{-1/4} \|R_n\| = 2^{-1/4} \quad \text{a.s.}, \quad (1.4)$$

$$\liminf_{n \rightarrow \infty} n^{1/4} (\log n)^{-1/2} (\log_2 n)^{1/4} \|R_n\| = \frac{\pi^{1/2}}{8^{1/4}} \quad \text{a.s.}, \quad (1.5)$$

while a direct application of (1.3) together with the weak convergence of α_n to a Brownian bridge B gives

$$n^{1/4}(\log n)^{-1/2}\|R_n\| \rightarrow_d (\|B\|)^{1/2}, \quad n \rightarrow \infty, \quad (1.6)$$

(cf. Kiefer, 1970, Deheuvels and Mason, 1990).

Moreover, initiated by Shorack and Wellner (1986, p. 593), and pursued by Deheuvels and Mason (1990, Theorem 2A), the result in (1.6) was extended to convergence in distribution in weighted sup-norm metric $\|\cdot/q\|$ under appropriate conditions on the weight function $q(\cdot)$ on $(0, 1)$, concluding in

$$n^{1/4}\|R_n/(2q \log(\max(n^{1/2}/q, e)))^{1/2}\|^{(n)} \rightarrow_d (\|B/q\|)^{1/2}, \quad n \rightarrow \infty$$

where

$$\|f\|^{(n)} = \sup_{1/(n+1) \leq x \leq 1-1/(n+1)} |f(x)|.$$

The asymptotic behaviour of R_n in L_p norms was in turn established as follows.

Theorem B (Csörgő and Shi, 1998, 2001) *For any $p \in [2, \infty)$, we have*

$$\lim_{n \rightarrow \infty} n^{1/4} \frac{\|R_n\|_p}{(\|\alpha_n\|_{p/2})^{1/2}} = c_0(p) \quad \text{a.s.}, \quad (1.7)$$

where

$$c_0(p) := (\mathbf{E}|\mathcal{N}|^p)^{1/p} = \sqrt{2} \left(\frac{\Gamma((p+1)/2)}{\sqrt{\pi}} \right)^{1/p}, \quad (1.8)$$

\mathcal{N} stands, as before, for a standard normal variable, and $\|f\|_p := \left(\int_0^1 |f(t)|^p dt \right)^{1/p}$, the L_p norm of f .

In particular, (1.7) yields L_p versions of the laws of the iterated logarithm (LIL's) for the Bahadur–Kiefer process R_n . We also note that (1.7) combined with the weak convergence of α_n to a Brownian bridge B implies that, for $p \in [2, \infty)$,

$$n^{1/4}\|R_n\|_p \rightarrow_d c_0(p)(\|B\|_{p/2})^{1/2}, \quad n \rightarrow \infty. \quad (1.9)$$

Nevertheless, in spite of all these convergence in distribution results of R_n in various metrics, the following result, due to Vervaat (1972a, b), is also true.

Theorem C (Vervaat, 1972a, b) *The statement*

$$a_n R_n \rightarrow_d Y, \quad n \rightarrow \infty,$$

cannot hold true in the space $D[0, 1]$ (endowed with the Skorohod topology) for any sequence $\{a_n\}$ of positive real numbers and any non-degenerate random element Y of $D[0, 1]$.

Vervaat's (1972a, b) proof of Theorem C is based, in a most crucial and elegant way, on the following integrated Bahadur–Kiefer process

$$I_n(t) := \int_0^t R_n(s) ds, \quad 0 \leq t \leq 1.$$

Concerning the latter, he established the weak convergence of

$$V_n(t) := 2n^{1/2}I_n(t) \quad (1.10)$$

to B^2 , the square of a Brownian bridge, via proving the following theorem (for a discussion of related details we refer to Csörgő and Zitikis, 1999, 2001).

Theorem D (Vervaat, 1972a, b) *We have*

$$\lim_{n \rightarrow \infty} \|V_n - \alpha_n^2\| = 0 \quad \text{in probability.} \quad (1.11)$$

In particular, in the space $C[0, 1]$ (endowed with the uniform topology),

$$V_n \rightarrow_d B^2, \quad n \rightarrow \infty. \quad (1.12)$$

For use of terminology, we call the process V_n of (1.10) the uniform Vervaat process, that coincides with the integrated empirical difference process in Shorack and Wellner (1986, p. 594). For further references and elaboration on this terminology we refer to Section 1 of Zitikis (1998).

As a consequence of (1.12), Vervaat (1972a, b) concluded Theorem C on account of a Brownian bridge B being nondifferentiable. In retrospect we note that Theorem C can also be concluded from a combination of (1.1) and (1.6), or from that of (1.9) with (1.6).

2 The Vervaat error process: definition and main results

Bahadur (1966) introduced R_n as the remainder term in the representation

$$\beta_n = -\alpha_n + R_n$$

of the quantile process β_n in terms of the empirical process α_n . As we have seen in Theorems A and B, the remainder term R_n , i.e., the Bahadur–Kiefer process, is asymptotically smaller than the main term α_n , i.e., the empirical process, in both the L_p and sup-norm topologies.

In a similar vein, we can think of the process

$$Q_n(t) := V_n(t) - \alpha_n^2(t), \quad 0 \leq t \leq 1, \quad (2.1)$$

that appears in the statement (1.11) of Theorem D as the remainder term Q_n in the following representation

$$V_n = \alpha_n^2 + Q_n \quad (2.2)$$

of the uniform Vervaat process V_n in terms of the square of the empirical process. It is well-known (cf. Zitikis, 1998, for details and references) that the remainder term Q_n in (2.2) is asymptotically smaller than the main term α_n^2 . Thus, just like in the case of R_n , one may like to know how small the remainder term Q_n is. We call this remainder term Q_n the Vervaat error process.

In view of Theorems A and B, one suspects that there should be substantial differences between the asymptotic pointwise, sup- and L_p -norms behaviour of the process Q_n . Indeed, Csörgő and Zitikis (2001) established the following strong convergence result for $\|Q_n\|_p$.

Theorem E (Csörgő and Zitikis, 2001) *For any $p \in [1, \infty)$, we have*

$$\lim_{n \rightarrow \infty} n^{1/4} \frac{\|Q_n\|_p}{(\|\alpha_n\|_{3p/2})^{3/2}} = \frac{1}{\sqrt{3}} c_0(p) \quad \text{a.s.,} \quad (2.3)$$

where $c_0(p)$ is defined in (1.8).

For a comparison of this result to that of Theorem B, as well as for that of their consequences, we refer to Csörgő and Zitikis (2001), who have also conjectured that in sup-norm the analogue statement of (2.3) should be of the following form:

$$\lim_{n \rightarrow \infty} b_n n^{1/4} \frac{\|Q_n\|}{\|\alpha_n\|^{3/2}} = c \quad \text{a.s.}, \quad (2.4)$$

where b_n is a slowly varying function converging to 0 and c is a positive constant.

One of the aims of this exposition is to prove that this conjecture is true with $b_n = (\log n)^{-1/2}$. In addition, we also study the pointwise behaviour of the Vervaat error process Q_n . We summarize our results in the following theorem, which parallels Theorem A of Kiefer (1967, 1970) concerning the process R_n .

Theorem 2.1 *For every fixed $t \in (0, 1)$, we have*

$$n^{1/4} Q_n(t) \rightarrow_d (4/3)^{1/2} (t(1-t))^{3/4} \mathcal{N}(|\tilde{\mathcal{N}}|)^{3/2}, \quad n \rightarrow \infty, \quad (2.5)$$

$$\limsup_{n \rightarrow \infty} \frac{n^{1/4} |Q_n(t)|}{(\log_2 n)^{5/4}} = (t(1-t))^{3/4} \frac{2^{11/4} 3^{1/4}}{5^{5/4}} \quad \text{a.s.}, \quad (2.6)$$

where \mathcal{N} and $\tilde{\mathcal{N}}$ are independent standard normal variables. Also,

$$\lim_{n \rightarrow \infty} n^{1/4} (\log n)^{-1/2} \frac{\|Q_n\|}{(\|\alpha_n\|)^{3/2}} = (4/3)^{1/2} \quad \text{a.s.} \quad (2.7)$$

As a consequence of this theorem, as well as that of Theorem E combined with (2.7), we have the following corollary, which confirms Conjecture 2.1 of Csörgő and Zitikis (2001).

Corollary 2.1 *The statement*

$$a_n Q_n \rightarrow_d Y, \quad n \rightarrow \infty,$$

cannot hold true in the space $D[0, 1]$ (endowed with the Skorohod topology) for any sequence $\{a_n\}$ of positive real numbers and for any non-degenerate random element Y of the space $D[0, 1]$.

Another consequence via (2.7) is the following corollary.

Corollary 2.2 *We have*

$$\limsup_{n \rightarrow \infty} n^{1/4} (\log n)^{-1/2} (\log_2 n)^{-3/4} \|Q_n\| = \frac{2^{1/4}}{3^{1/2}} \quad \text{a.s.}, \quad (2.8)$$

$$\liminf_{n \rightarrow \infty} n^{1/4} (\log n)^{-1/2} (\log_2 n)^{3/4} \|Q_n\| = \frac{\pi^{3/2}}{3^{1/2} 2^{5/4}} \quad \text{a.s.}, \quad (2.9)$$

$$n^{1/4} (\log n)^{-1/2} \|Q_n\| \rightarrow_d (4/3)^{1/2} \|B\|^{3/2}, \quad n \rightarrow \infty, \quad (2.10)$$

where B is a standard Brownian bridge.

We note that (2.8) and (2.9) follow from (2.7) by means of the usual and the other LIL's for α_n (cf. Theorem G below). As to (2.10), it results from a direct application of (2.7) together with the weak convergence of α_n to a Brownian bridge B .

As an application of (2.8) of Corollary 2.2 one easily concludes the (functional) LIL and the other LIL for the Vervaat process V_n itself via those for α_n . Concerning the functional LIL, we also refer to Vervaat (1972a, b).

Remark 2.1 In the literature we also find general forms of the Vervaat process that are based on random variables X_1, X_2, \dots replacing the uniform $[0,1]$ random variables U_1, U_2, \dots . In particular, such a general Vervaat process first appeared and was put to good use in Csörgő and Zitikis (1996). We refer to Zitikis (1998) for a detailed survey on this subject. For related though rather different limit theorems for the general Vervaat process, we refer to Csörgő and Zitikis (1998). It is obvious that the results of this paper can be generalized in such a way that they would cover general forms of the Vervaat process as well. However, a solution of this problem under reasonably optimal assumptions may constitute a rather challenging mathematical task which is definitely not within the scope of the present paper. For a recent review of Vervaat and Vervaat error processes we refer to Csörgő and Zitikis (1999). \square

3 The Vervaat error process in terms of a Kiefer process

We introduce some two-parameter Gaussian processes. Let $\{W(x, y), x \geq 0, y \geq 0\}$ be a Wiener (Brownian) sheet, i.e., a two-parameter Gaussian process with $\mathbf{E}W(x, y) = 0$ and covariance function

$$\mathbf{E}W(x_1, y_1)W(x_2, y_2) = (x_1 \wedge x_2)(y_1 \wedge y_2).$$

Next we define a Kiefer process $\{K(x, y), 0 \leq x \leq 1, y \geq 0\}$ by

$$K(x, y) := W(x, y) - xW(1, y),$$

where $W(x, y)$ is a Wiener sheet. A Kiefer process $K(x, y)$ can be characterized as a mean zero Gaussian process with covariance function

$$\mathbf{E}K(x_1, y_1)K(x_2, y_2) = (x_1 \wedge x_2 - x_1x_2)(y_1 \wedge y_2).$$

Remark 3.1 In this paper we define, without loss of generality, all Kiefer processes $K(x, y)$ and Brownian bridges $B(x)$ to be equal to zero if $x < 0$ or $x > 1$. \square

Concerning the uniform empirical process α_n , Komlós *et al.* (1975) established the following fundamental embedding theorem.

Theorem F (Komlós, Major and Tusnády, 1975) *On a suitable probability space, the uniform empirical process $\{\alpha_k(x), 0 \leq x \leq 1, k = 1, 2, \dots\}$ and a Kiefer process $\{K(x, k), 0 \leq x \leq 1, k = 1, 2, \dots\}$ can be so constructed that, as $n \rightarrow \infty$,*

$$\max_{1 \leq k \leq n} \sup_{0 \leq x \leq 1} |\alpha_k(x) - k^{-1/2}K(x, k)| = \mathcal{O}\left(\frac{(\log n)^2}{n^{1/2}}\right) \quad \text{a.s.}$$

Combining (1.4) with Theorem F, we arrive at (cf. Csörgő and Révész, 1975):

Proposition A *On the probability space of Theorem F for the uniform quantile process $\{\beta_k(x), 0 \leq x \leq 1, k = 1, 2, \dots\}$ with the Kiefer process $\{K(x, k), 0 \leq x \leq 1, k = 1, 2, \dots\}$ of Theorem F, as $n \rightarrow \infty$, we have*

$$\max_{1 \leq k \leq n} \sup_{0 \leq x \leq 1} |\beta_k(x) + k^{-1/2}K(x, k)| = \mathcal{O}\left(\frac{(\log n)^{1/2}(\log_2 n)^{1/4}}{n^{1/4}}\right) \quad \text{a.s.} \quad (3.1)$$

We note in passing (cf. Deheuvels, 1998a,b, 2000 and references therein) that the almost sure rate of convergence in (3.1) cannot be improved even when approximating the uniform quantile process by any other Kiefer process.

In the sequel we will frequently make use of the following classical results.

Theorem G For any uniform empirical process $\alpha_n(x)$ and Kiefer process $K(x, n)$ we have

$$\limsup_{n \rightarrow \infty} \frac{\sup_{0 \leq x \leq 1} |K(x, n)|}{(n \log_2 n)^{1/2}} = \limsup_{n \rightarrow \infty} \frac{\sup_{0 \leq x \leq 1} |\alpha_n(x)|}{(\log_2 n)^{1/2}} = \frac{1}{2^{1/2}}, \quad (3.2)$$

$$\liminf_{n \rightarrow \infty} \frac{(\log_2 n)^{1/2} \sup_{0 \leq x \leq 1} |K(x, n)|}{n^{1/2}} = \liminf_{n \rightarrow \infty} (\log_2 n)^{1/2} \sup_{0 \leq x \leq 1} |\alpha_n(x)| = \frac{\pi}{8^{1/2}}, \quad (3.3)$$

almost surely.

The usual LIL (3.2) for α_n was proved by Smirnov (1944) and Chung (1949) and that for K was first established by Kiefer (1972), via his fundamental strong invariance principle. The other LIL (3.3) was independently established by Kuelbs (1979) and Mogulskii (1979).

For further use we quote the following two results of A.H.C. Chan (1977) (cf. Theorem 1.15.2, and Theorems S.1.14.2 and S.1.15.1, respectively, in Csörgő and Révész, 1981 and Theorem 14.3.1 in Shorack and Wellner, 1986).

Theorem H (Chan, 1977) Let $K(\cdot, \cdot)$ be a Kiefer process, and let $\{h_n\}$ be a non-increasing sequence of positive numbers such that $\{nh_n\}$ is non-decreasing,

$$\lim_{n \rightarrow \infty} nh_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\log(1/h_n)}{\log_2 n} = \infty.$$

Then

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq 1-h_n} \sup_{0 \leq s \leq h_n} \frac{|K(t+s, n) - K(t, n)|}{(2nh_n \log(1/h_n))^{1/2}} = 1 \quad \text{a.s.}$$

Remark 3.2 The referee has called our attention to an oversight in stating the conditions of Theorems 1.14.2 and 1.15.2 in Csörgő and Révész (1981). Namely, there the essential conditions in Theorem H that nh_n is non-decreasing and $\lim_{n \rightarrow \infty} nh_n = \infty$ were omitted. We also note in passing that the just mentioned Theorem 14.3.1 in Shorack and Wellner, i.e., Theorem H, is correctly stated under its full set of conditions as it is also given here now. \square

Theorem I (Chan, 1977) Let $0 < \varepsilon_T \leq \frac{1}{2}$, $0 < a_T \leq T$ be functions of T such that ε_T and a_T/T are non-increasing, a_T and $T\varepsilon_T$ are non-decreasing and $\lim_{T \rightarrow \infty} T\varepsilon_T = \infty$. Define $K((x_1, x_2], t) = K(x_2, t) - K(x_1, t)$ ($0 \leq x_1 < x_2 \leq 1$). Then almost surely,

$$\begin{aligned} & \limsup_{T \rightarrow \infty} \sup_{0 \leq t \leq T-a_T} \sup_{0 \leq z \leq a_T} \sup_{0 \leq x \leq 1-\varepsilon_T} \sup_{0 \leq s \leq \varepsilon_T} \beta_T |K((x, x+s], t+z) - K((x, x+s], t)| \\ &= \limsup_{T \rightarrow \infty} \sup_{0 \leq t \leq T-a_T} \sup_{0 \leq x \leq 1-\varepsilon_T} \beta_T |K((x, x+\varepsilon_T], t+a_T) - K((x, x+\varepsilon_T], t)| = 1, \end{aligned}$$

where

$$\beta_T = \left(2a_T \varepsilon_T (1 - \varepsilon_T) \left(\log \frac{T}{\varepsilon_T a_T} + \log_2 T \right) \right)^{-1/2}.$$

If in addition, $\lim_{T \rightarrow \infty} (\log \frac{T}{\varepsilon_T a_T}) / \log_2 T = \infty$, then \limsup can be replaced by $\lim_{T \rightarrow \infty}$.

The main result of this section is the following strong approximation of the Vervaat error process $Q_n(t)$ defined in (2.1) via a Kiefer process.

Theorem 3.1 *On the probability space of Theorem F, using the there constructed Kiefer process $K(\cdot, \cdot)$, $Q_n(\cdot)$ can be approximated as follows. As $n \rightarrow \infty$ we have*

$$\sup_{0 < t < 1} |Q_n(t) - Z_n(t)| = \mathcal{O}(n^{-3/8}(\log n)^{3/4}(\log_2 n)^{5/8}) \quad \text{a.s.}, \quad (3.4)$$

where $\{Z_n(t), 0 < t < 1, n = 1, 2, \dots\}$ is defined by

$$Z_n(t) := 2 \frac{K(t, n)}{n} \int_0^1 \left(K \left(t - s \frac{K(t, n)}{n}, n \right) - K(t, n) \right) ds. \quad (3.5)$$

The proof of this theorem is based on the next two lemmas, which are of interest on their own.

Lemma 3.1 *Let*

$$A_n(t) := 2n^{1/2} \int_{F_n^{-1}(t)}^t (\alpha_n(u) - \alpha_n(t)) du, \quad 0 < t < 1, n = 1, 2, \dots \quad (3.6)$$

Then

$$Q_n(t) = A_n(t) - R_n^2(t), \quad (3.7)$$

and, consequently, as $n \rightarrow \infty$,

$$\sup_{0 < t < 1} |Q_n(t) - A_n(t)| = \mathcal{O}(n^{-1/2}(\log n)(\log_2 n)^{1/2}) \quad \text{a.s.} \quad (3.8)$$

Proof. We have the following easy-to-check representation

$$\begin{aligned} V_n(t) &= 2n^{1/2} \int_0^t R_n(s) ds = 2n \int_{F_n^{-1}(t)}^t (F_n(s) - t) ds \\ &= -\alpha_n(t)\beta_n(t) + A_n(t) - (\alpha_n(t) + \beta_n(t))\beta_n(t) \end{aligned} \quad (3.9)$$

for all $t \in (0, 1)$ and $n = 1, 2, \dots$. The crucial identity in the first line of (3.9) is due to Vervaat (1972a, p. 248). For details and further discussions we refer to Shorack and Wellner (1986, p. 594) and Zitikis (1998, pp. 670–671). By (2.1), $Q_n(t) = V_n(t) - \alpha_n^2(t) = A_n(t) - R_n^2(t)$, yielding (3.7). The conclusion (3.8) follows from (3.7) and (1.4). \square

Remark 3.3 While in this paper we are yet to study the stochastic process $Q_n(t)$ in terms of $A_n(t)$, it is interesting to note that, as a result of the representation (3.7), we already know the best possible stochastic behaviour of their difference $A_n(t) - Q_n(t) = R_n^2(t)$. Moreover, on account of having (1.1)–(1.8), we can immediately write down exact analogues for the difference process $A_n(t) - Q_n(t)$ via $R_n^2(t)$. \square

Lemma 3.2 *On the probability space of Theorem F, using the there constructed Kiefer process $K(\cdot, \cdot)$, the stochastic process $A_n(t)$ of Lemma 3.1 can be approximated such that when $n \rightarrow \infty$,*

$$\sup_{0 < t < 1} |A_n(t) - Z_n(t)| = \mathcal{O}(n^{-3/8}(\log n)^{3/4}(\log_2 n)^{5/8}) \quad \text{a.s.},$$

where $Z_n(t)$ is as in Theorem 3.1.

Proof. By a change of variables $u = t - s(t - F_n^{-1}(t)) = t + sn^{-1/2}\beta_n(t)$ in (3.6), we have

$$A_n(t) = -2\beta_n(t) \int_0^1 \left(\alpha_n \left(t + s \frac{\beta_n(t)}{n^{1/2}} \right) - \alpha_n(t) \right) ds.$$

The usual LIL for β_n confirms that $\|\beta_n\| = \mathcal{O}((\log_2 n)^{1/2})$ almost surely (when $n \rightarrow \infty$). Therefore, by Theorem F, when $n \rightarrow \infty$,

$$\begin{aligned} A_n(t) &= -2n^{-1/2}\beta_n(t) \int_0^1 \left(K \left(t + s \frac{\beta_n(t)}{n^{1/2}}, n \right) - K(t, n) \right) ds \\ &\quad + \mathcal{O} \left(n^{-1/2} (\log n)^2 (\log_2 n)^{1/2} \right), \quad \text{a.s.}, \end{aligned} \quad (3.10)$$

uniformly in $t \in (0, 1)$.

For all $t, s \in (0, 1)$, we have

$$\begin{aligned} &\left| K \left(t + s \frac{\beta_n(t)}{n^{1/2}}, n \right) - K \left(t - s \frac{K(t, n)}{n}, n \right) \right| \\ &:= |K(u + v, n) - K(u, n)| \\ &\leq \sup_{0 \leq u \leq 1 - h_n} \sup_{0 \leq v \leq h_n} |K(u + v, n) - K(u, n)|, \end{aligned}$$

where

$$u = t - s \frac{K(t, n)}{n}, \quad v = s \left(\frac{\beta_n(t)}{n^{1/2}} + \frac{K(t, n)}{n} \right)$$

and, on account of Proposition A, $h_n = \mathcal{O}(n^{-3/4}(\log n)^{1/2}(\log_2 n)^{1/4})$ ($n \rightarrow \infty$), almost surely. Thus, according to Theorem H, almost surely, when $n \rightarrow \infty$,

$$K \left(t + s \frac{\beta_n(t)}{n^{1/2}}, n \right) - K \left(t - s \frac{K(t, n)}{n}, n \right) = \mathcal{O} \left(n^{1/8} (\log n)^{3/4} (\log_2 n)^{1/8} \right),$$

uniformly in $t, s \in (0, 1)$. Inserting this into (3.10) and using again the LIL for β_n , we arrive at:

$$\begin{aligned} A_n(t) &= -2n^{-1/2}\beta_n(t) \int_0^1 \left(K \left(t - s \frac{K(t, n)}{n}, n \right) - K(t, n) \right) ds \\ &\quad + \mathcal{O}(n^{-3/8}(\log n)^{3/4}(\log_2 n)^{5/8}) \quad \text{a.s.} \end{aligned} \quad (3.11)$$

According to Proposition A,

$$-2n^{-1/2}\beta_n(t) = 2 \frac{K(t, n)}{n} + \mathcal{O} \left(\frac{(\log n)^{1/2} (\log_2 n)^{1/4}}{n^{3/4}} \right) \quad (3.12)$$

almost surely and uniformly in $t \in (0, 1)$. On the other hand, applying Theorem H to the integrand in (3.11) with $h_n = \mathcal{O}(n^{-1/2}(\log_2 n)^{1/2})$, we obtain:

$$\int_0^1 \left(K \left(t - s \frac{K(t, n)}{n}, n \right) - K(t, n) \right) ds = \mathcal{O} \left(n^{1/4} (\log n)^{1/2} (\log_2 n)^{1/4} \right),$$

almost surely and uniformly in $t \in (0, 1)$. Plugging this and (3.12) into (3.11) yields that almost surely, when $n \rightarrow \infty$,

$$A_n(t) = Z_n(t) + \mathcal{O} \left(\frac{(\log n)(\log_2 n)^{1/2}}{n^{1/2}} \right) + \mathcal{O}(n^{-3/8}(\log n)^{3/4}(\log_2 n)^{5/8}),$$

uniformly in $t \in (0, 1)$. This yields Lemma 3.2. \square

Proof of Theorem 3.1. Follows from Lemmas 3.1 and 3.2. \square

We mention that the process $Z_n(t)$ was put to use in Csörgő and Zitikis (2001) in their study of $Q_n(t)$ in L_p norm. Moreover, they also remarked that their conjecture as stated in (2.4) is equivalent to stating it in terms of Z_n instead of Q_n . Likewise, in view of Theorem 3.1 now, the proof of Theorem 2.1 can, and will, be based on proving the statements (2.5)–(2.7) with the process $Z_n(t)$ replacing $Q_n(t)$ in all of them. The latter goal in turn will be achieved via using the next simple, though essential, observation. A discrete time version of this was stated, somewhat implicitly, in Einmahl (1996, p. 530) (cf. also Csörgő and Shi, 1998).

Proposition B *Let $\{K(x, y), 0 \leq x \leq 1, y \geq 0\}$ be a Kiefer process. For any fixed $0 \leq u < v \leq 1$, the process*

$$\left\{ \frac{K(u + (v - u)x, y) - xK(v, y) - (1 - x)K(u, y)}{\sqrt{v - u}}, x \in [0, 1], y \geq 0 \right\}$$

is a Kiefer process that is independent of $\{K(s, y), s \in [0, u], y \geq 0\}$ and $\{K(s, y), s \in [v, 1], y \geq 0\}$.

Based on Theorem 3.1 and this crucial observation, the respective proofs of the *pointwise* statements of (2.5)–(2.6) and the proof of the *uniform* property as in (2.7) of the process Q_n will take different routes. Hence, Section 4 is devoted to proving (2.5)–(2.6), while Section 5 will be on establishing the almost sure ratio statement of (2.7).

4 Pointwise behaviour of the Vervaat error process

In this section we are to prove (2.5) and (2.6) of Theorem 2.1, that are concerned with the pointwise asymptotics of $Q_n(t)$. The intriguing pointwise behaviour of the Bahadur–Kiefer process as in (1.1) and (1.2), respectively, was first established by Kiefer (1967). Moreover, the result of (1.2) was also proved, explained and extended via strong approximations and functional laws of the iterated logarithm by Deheuvels and Mason (1992, 1994), Deheuvels, Einmahl and Mason (2000). One of the key elements of the proofs in these papers is the establishment of local (pointwise) strong approximations of the empirical process that are independent of its global (uniform) approximation. Our own approach to proving (2.5) and (2.6) is similar and connects also the two approaches when combining it with that of Deheuvels and Mason (1992) in our proof of (2.6).

Our approach is accomplished via first establishing a strong approximation of $Z_n(t)$ of (3.5) for any fixed $t \in (0, 1)$ by a process in n , which is an integral of a Wiener sheet in its first parameter (the local approximation) over a random interval determined by the global approximation that is independent of this Wiener sheet (the integrand) (cf. Lemma 4.1). Moreover, we also believe that our Lemma 4.1 is of independent interest in the context of studying the pointwise behaviour of the Vervaat error process.

Lemma 4.1 *Given $Z_n(t)$ as in (3.5), then for any fixed $t \in (0, 1)$ one can define a Wiener sheet $W^*(\cdot, \cdot)$ such that, as $n \rightarrow \infty$, we have*

$$Z_n(t) = 2 \int_0^{|K(t, n)|/n} W^*(y, n) dy + \mathcal{O} \left(\frac{(\log n)^{1/2} \log_2 n}{n^{1/2}} \right) \quad \text{a.s.}, \quad (4.1)$$

where $\{W^(y, n), y \geq 0, n = 1, 2, \dots\}$ is independent of $\{K(t, n), n = 1, 2, \dots\}$.*

Proof. Using Proposition B with $u = 0$, $v = t \in (0, 1)$ fixed, and writing $1 - x$ instead of x , we see that

$$K_1^*(x, n) := \frac{K(t(1-x), n) - (1-x)K(t, n)}{t^{1/2}}, \quad 0 \leq x \leq 1, \quad (4.2)$$

is a Kiefer process, independent of $K(t, \cdot)$. Likewise, letting $u = t$, $v = 1$, we see that

$$K_2^*(x, n) := \frac{K(t + (1-t)x, n) - (1-x)K(t, n)}{(1-t)^{1/2}}, \quad 0 \leq x \leq 1,$$

is also a Kiefer process, independent of $K(t, \cdot)$. Moreover, $K_1^*(\cdot, \cdot)$ and $K_2^*(\cdot, \cdot)$ are independent. Consequently, the components of the vector

$$(K(t, \cdot), K_1^*(\cdot, \cdot), K_2^*(\cdot, \cdot)) \quad (4.3)$$

are independent processes.

In case $K(t, n) = 0$, we have $Z_n(t) = 0$ by (3.5). Assuming $K(t, n) \neq 0$, we introduce a substitution in the integral in the definition of $Z_n(t)$ of (3.5) on defining x via

$$\begin{aligned} (1-x)t &= t - s \frac{K(t, n)}{n} & \text{if } K(t, n) > 0, \\ t + (1-t)x &= t - s \frac{K(t, n)}{n} & \text{if } K(t, n) < 0. \end{aligned}$$

Hence we arrive at the following representation of the latter process for each fixed t :

$$Z_n(t) = \mathcal{I}_1(n) \mathbf{1}\{K(t, n) > 0\} - \mathcal{I}_2(n) \mathbf{1}\{K(t, n) \leq 0\}, \quad (4.4)$$

where

$$\begin{aligned} \mathcal{I}_1(n) &:= 2t \int_0^{K(t, n)/(nt)} (K(t(1-x), n) - K(t, n)) dx, \\ \mathcal{I}_2(n) &:= 2(1-t) \int_0^{-K(t, n)/(n(1-t))} (K(t + (1-t)x, n) - K(t, n)) dx. \end{aligned}$$

Considering $\mathcal{I}_1(n)$, and remembering that this is the case when $K(t, n) > 0$, using the definition (4.2) of $K_1^*(\cdot, \cdot)$, we obtain: almost surely as $n \rightarrow \infty$,

$$\begin{aligned} \mathcal{I}_1(n) &= 2t^{3/2} \int_0^{K(t, n)/(nt)} K_1^*(x, n) dx - 2tK(t, n) \int_0^{K(t, n)/(nt)} x dx \\ &= 2t^{3/2} \int_0^{K(t, n)/(nt)} K_1^*(x, n) dx - \frac{t(K(t, n))^3}{n^2 t^2} \\ &= 2t^{3/2} \int_0^{K(t, n)/(nt)} K_1^*(x, n) dx + \mathcal{O}\left(\frac{(\log_2 n)^{3/2}}{n^{1/2}}\right), \end{aligned} \quad (4.5)$$

the last line following from the LIL for the Kiefer process with fixed t . Since $K_1^*(\cdot, \cdot)$ is a Kiefer process independent of $K(t, \cdot)$, we can use the representation

$$K_1^*(x, n) = W_1^*(x, n) - xW_1^*(1, n),$$

where $W_1^*(\cdot, \cdot)$ is a Wiener sheet, independent of $K(t, \cdot)$. This independence property will be crucial in our use later on.

Observe that

$$\int_0^{K(t,n)/(nt)} K_1^*(x, n) dx = \int_0^{K(t,n)/(nt)} W_1^*(x, n) dx - \frac{K^2(t, n) W_1^*(1, n)}{2t^2 n^2}.$$

By the LIL for the Kiefer process K and the Wiener process $W_1^*(1, \cdot)$, we have, when $n \rightarrow \infty$, $K^2(t, n) W_1^*(1, n) = \mathcal{O}(n^{3/2}(\log_2 n)^{3/2})$ almost surely. Going back to (4.5), we obtain: as $n \rightarrow \infty$, almost surely

$$\begin{aligned} \mathcal{I}_1(n) &= 2t^{3/2} \int_0^{K(t,n)/(nt)} W_1^*(x, n) dx + \mathcal{O}\left(\frac{(\log_2 n)^{3/2}}{n^{1/2}}\right) \\ &= 2 \int_0^{K(t,n)/n} t^{1/2} W_1^*\left(\frac{y}{t}, n\right) dy + \mathcal{O}\left(\frac{(\log_2 n)^{3/2}}{n^{1/2}}\right). \end{aligned} \quad (4.6)$$

Similarly, in the case $K(t, n) < 0$, we can show that as $n \rightarrow \infty$, for $\mathcal{I}_2(n)$ of (4.4) we have, almost surely,

$$\mathcal{I}_2(n) = 2 \int_0^{-K(t,n)/n} (1-t)^{1/2} W_2^*\left(\frac{y}{1-t}, n\right) dy + \mathcal{O}\left(\frac{(\log_2 n)^{3/2}}{n^{1/2}}\right), \quad (4.7)$$

where, just like $W_1^*(\cdot, \cdot)$ of (4.6), the Wiener sheet $W_2^*(\cdot, \cdot)$ of (4.7) is also independent of $K(t, \cdot)$ (cf. (4.3)).

Combining (4.6) and (4.7) with (4.4) yields that, for each fixed $t \in (0, 1)$, as $n \rightarrow \infty$,

$$Z_n(t) = 2 \int_0^{|K(t,n)|/n} W^*(y, n) dy + \mathcal{O}\left(\frac{(\log_2 n)(\log n)^{1/2}}{n^{1/2}}\right) \quad \text{a.s.},$$

where

$$W^*(y, n) := t^{1/2} W_1^*\left(\frac{y}{t}, n\right) \mathbf{1}\{K(t, n) > 0\} - (1-t)^{1/2} W_2^*\left(\frac{y}{1-t}, n\right) \mathbf{1}\{K(t, n) \leq 0\}.$$

Since $W^*(\cdot, \cdot)$ is a Wiener sheet, independent of $K(t, \cdot)$, this yields Lemma 4.1. \square

The rest of this section is devoted to the proof of (2.5) and (2.6) in Theorem 2.1. For the sake of clarity, they are proved separately.

Proof of (2.5). For each fixed n and T ,

$$\Gamma_n := \frac{3^{1/2}}{n^{1/2} T^{3/2}} \int_0^T W^*(y, n) dy$$

is a standard normal variable. Thus, by conditioning, if T is a random variable independent of $W^*(\cdot, n)$, then Γ_n is also a standard normal variable, independent of T . In view of the independence of $K(t, \cdot)$ and $W^*(\cdot, \cdot)$, we have, for each fixed n and t , (taking $T := |K(t, n)|/n$)

$$\begin{aligned} & 2 \int_0^{|K(t,n)|/n} W^*(y, n) dy \\ &= \left(\frac{4}{3}\right)^{1/2} \frac{(t(1-t))^{3/4}}{n^{1/4}} \left(\frac{n^{1/2}|T|}{(t(1-t))^{1/2}}\right)^{3/2} \left(\frac{3^{1/2}}{n^{1/2} T^{3/2}} \int_0^T W^*(y, n) dy\right) \\ &=_d \left(\frac{4}{3}\right)^{1/2} (t(1-t))^{3/4} n^{-1/4} (|\widetilde{\mathcal{N}}|)^{3/2} \mathcal{N}, \end{aligned}$$

where “ $=_d$ ” denotes identity in distribution, and \mathcal{N} and $\widetilde{\mathcal{N}}$ are independent standard normal random variables. This, in the light of (3.4) and (4.1), yields (2.5). \square

Proof of (2.6). Fix again $t \in (0, 1)$. Define

$$\begin{aligned} W_t(n) &:= \frac{K(t, n)}{(t(1-t))^{1/2}}, \quad n \geq 1, \\ W_t^*(x, y) &:= (t(1-t))^{1/4} W^* \left(\frac{x}{(t(1-t))^{1/2}}, y \right), \quad x \geq 0, y \geq 0. \end{aligned}$$

Clearly, $W_t(\cdot)$ is a Wiener process, and $W_t^*(\cdot, \cdot)$ is a Wiener sheet, independent of $\{K(t, n), n = 1, 2, \dots\}$, and thus of $W_t(\cdot)$ as well. We also observe that

$$\begin{aligned} &\int_0^{|K(t, n)|/n} W^*(y, n) dy \\ &= (t(1-t))^{3/4} \frac{(2 \log_2 n)^{5/4}}{n^{1/4}} \int_0^{|W_t(n)|/(2n \log_2 n)^{1/2}} \frac{W_t^*(u(\frac{2 \log_2 n}{n})^{1/2}, n)}{n^{1/4} (2 \log_2 n)^{3/4}} du. \end{aligned}$$

By Theorem 1.1 of Deheuvels and Mason (1992), as $n \rightarrow \infty$, the set of limit points of

$$\left\{ \left(\frac{W_t(n)}{(2n \log_2 n)^{1/2}}, \frac{W_t^*(u(\frac{2 \log_2 n}{n})^{1/2}, n)}{n^{1/4} (2 \log_2 n)^{3/4}} \right), u \in [0, 1] \right\}$$

is almost surely equal to

$$\begin{aligned} \mathcal{D} := \{ (c, f) : f \text{ absolutely continuous with respect to the Lebesgue measure,} \\ |c| \in (0, 1), f(0) = 0, c^2 + \int_0^{|c|} (f'(u))^2 du \leq 1 \}. \end{aligned} \quad (4.8)$$

Consequently, with probability one,

$$\limsup_{n \rightarrow \infty} \frac{n^{1/4}}{(\log_2 n)^{5/4}} \int_0^{|K(t, n)|/n} W^*(y, n) dy = 2^{5/4} (t(1-t))^{3/4} \sup_{(c, f) \in \mathcal{D}} \int_0^{|c|} f(u) du.$$

We now determine the value of the “sup” expression on the right hand side. Integrating by parts, using the Cauchy–Schwarz inequality and (4.8), we get, for each $(c, f) \in \mathcal{D}$,

$$\begin{aligned} &\int_0^{|c|} f(u) du = \int_0^{|c|} (|c| - u) f'(u) du \\ &\leq \left(\int_0^{|c|} (|c| - u)^2 du \right)^{1/2} \left(\int_0^{|c|} (f'(u))^2 du \right)^{1/2} \leq \left(\frac{|c|^3 (1 - c^2)}{3} \right)^{1/2}. \end{aligned}$$

Since $|c|^3 (1 - c^2) \leq 2 \times 3^{3/2} / 5^{5/2}$ for any $c \in [-1, 1]$, this yields

$$\sup_{(c, f) \in \mathcal{D}} \int_0^{|c|} f(u) du \leq \frac{2^{1/2} 3^{1/4}}{5^{5/4}}. \quad (4.9)$$

Choosing

$$c = (3/5)^{1/2}, \quad f(u) = \frac{2^{1/2} 3^{1/4}}{5^{1/4}} u - \frac{5^{1/4}}{2^{1/2} 3^{1/4}} u^2,$$

it is seen that in (4.9) we have, in fact, an equality. Accordingly, with probability one,

$$\limsup_{n \rightarrow \infty} \frac{n^{1/4}}{(\log_2 n)^{5/4}} \int_0^{|K(t, n)|/n} W^*(y, n) dy = (t(1-t))^{3/4} \frac{2^{7/4} 3^{1/4}}{5^{5/4}}.$$

This yields (2.6) in view of (4.1) and (3.4). \square

5 Uniform behaviour of the Vervaat error process

This section is devoted to establishing the *uniform* property (2.7) in Theorem 2.1. In view of Theorems 3.1 and F it suffices to show that

$$\limsup_{n \rightarrow \infty} \frac{n}{(\log n)^{1/2}} \frac{\|Z_n\|}{\|K(\cdot, n)\|^{3/2}} \leq \left(\frac{4}{3}\right)^{1/2} \quad \text{a.s.}, \quad (5.1)$$

$$\liminf_{n \rightarrow \infty} \frac{n}{(\log n)^{1/2}} \frac{\|Z_n\|}{\|K(\cdot, n)\|^{3/2}} \geq \left(\frac{4}{3}\right)^{1/2} \quad \text{a.s.}, \quad (5.2)$$

where $\|Z_n\| := \sup_{0 \leq t \leq 1} |Z_n(t)|$ and $\|K(\cdot, n)\| := \sup_{0 \leq t \leq 1} |K(t, n)|$.

In our proof of (5.1) and (5.2), globally speaking, we make use of some ideas of Deheuvels and Mason (1990). Moreover, here too we continue using strong approximation methods and combine them with the approach that was established by Einmahl (1996) for proving the lower bound part of (1.3).

Recall that (cf. (3.5)), by definition,

$$\begin{aligned} Z_n(t) &= 2 \frac{K(t, n)}{n} \int_0^1 \left(K\left(t - s \frac{K(t, n)}{n}, n\right) - K(t, n) \right) ds \\ &= 2 \int_0^{K(t, n)/n} (K(t - z, n) - K(t, n)) dz. \end{aligned} \quad (5.3)$$

Let $\{h_n\}$ be a sequence satisfying the conditions of Theorem H, $0 < h_n \leq 1$ and put $N = N(n) := [1/h_n]$, where $[\cdot]$ denotes integral part. Let $t_i = t_i(n) := i/N$ (for $0 \leq i \leq N$).

Lemma 5.1 *There exist Wiener processes $\{W_{i,n}(\cdot), i = 0, 1, \dots, N-1\}$ that are independent of $\{K(t_i, n), i = 0, 1, \dots, N-1\}$, and are such that, as $n \rightarrow \infty$, for $t = t_i + v/N$, $v \in [0, 1]$, we have*

$$Z_n(t) = \frac{2n^{1/2}}{N^{3/2}} \int_0^{A_i} (W_{i,n}(v+y) - W_{i,n}(v)) dy + \mathcal{O}(\psi(n, N)), \quad \text{a.s.}, \quad (5.4)$$

where

$$A_i := \frac{N |K(t_i, n)|}{n}, \quad (5.5)$$

$$\psi(n, N) := \frac{(\log n)(\log_2 n)^{1/4}}{N^{1/2} n^{1/4}} + \frac{N^{1/2}(\log n)^{1/2} \log_2 n}{n^{1/2}} + \frac{N^2(\log N)^{1/2} \log_2 n}{n} \quad (5.6)$$

and \mathcal{O} is uniform in $i = 0, 1, \dots, N-1$ and $v \in [0, 1]$.

Proof. Let $t \in [t_i, t_{i+1}]$. Then

$$Z_n(t) = 2 \left(\int_0^{K(t_i, n)/n} + \int_{K(t_i, n)/n}^{K(t, n)/n} \right) (K(t - z, n) - K(t, n)) dz.$$

By Theorem H, when $n \rightarrow \infty$,

$$\max_{0 \leq i \leq N-1} \sup_{t_i \leq t \leq t_{i+1}} \frac{|K(t, n) - K(t_i, n)|}{n} = \mathcal{O} \left(\frac{(\log n)^{1/2}}{N^{1/2} n^{1/2}} \right) \quad \text{a.s.}$$

On the other hand, by the LIL (3.2) for the Kiefer process and Theorem H with $h_n = (\frac{\log_2 n}{n})^{1/2}$,

$$K(t - z, n) - K(t, n) = \mathcal{O} \left(n^{1/4} (\log n)^{1/2} (\log_2 n)^{1/4} \right) \quad \text{a.s.},$$

uniformly in $0 \leq i \leq N - 1$, $t \in [t_i, t_{i+1}]$ and $z \in [K(t_i, n)/n, K(t, n)/n]$. Therefore, on using Theorem H again,

$$\int_{K(t_i, n)/n}^{K(t, n)/n} (K(t - z, n) - K(t, n)) dz = \mathcal{O} \left(\frac{(\log n)(\log_2 n)^{1/4}}{N^{1/2}n^{1/4}} \right) \quad \text{a.s.}$$

As a consequence, for $t \in [t_i, t_{i+1}]$, we have, almost surely when $n \rightarrow \infty$,

$$Z_n(t) = 2 \int_0^{K(t_i, n)/n} (K(t - z, n) - K(t, n)) dz + \mathcal{O} \left(\frac{(\log n)(\log_2 n)^{1/4}}{N^{1/2}n^{1/4}} \right), \quad (5.7)$$

where \mathcal{O} is uniform in $i = 0, 1, \dots, N - 1$ and $t \in [t_i, t_{i+1}]$.

Let i be such that $K(t_i, n) \leq 0$, and for $u \in [0, 1]$ define

$$B_{i,n}^*(u) := \frac{N^{1/2}}{n^{1/2}} \left(K \left(t_i + \frac{u}{N}, n \right) - uK(t_{i+1}, n) - (1 - u)K(t_i, n) \right).$$

It follows from Proposition B that for each fixed n ,

$$\{B_{i,n}^*(\cdot), i = 0, 1, \dots, N - 1\}$$

are independent Brownian bridges, and independent of $\{K(t_i, n), i = 0, 1, \dots, N - 1\}$.

Now, if $t \in [t_i, t_{i+1}]$, then $t = t_i + v/N$ for some $v \in [0, 1]$.

$$\begin{aligned} & \int_0^{K(t_i, n)/n} (K(t - z, n) - K(t, n)) dz \\ &= -\frac{n^{1/2}}{N^{3/2}} \int_0^{A_i} (B_{i,n}^*(v + y) - B_{i,n}^*(v)) dy - \frac{K(t_{i+1}, n) - K(t_i, n)}{N} \int_0^{A_i} y dy, \end{aligned}$$

where A_i is defined by (5.5). Since $\int_0^{A_i} y dy = N^2 K^2(t_i, n)/2n^2$, an application of the LIL (3.2) for the Kiefer process and Theorem H yields that with probability one,

$$\begin{aligned} & \int_0^{K(t_i, n)/n} (K(t - z, n) - K(t, n)) dz \\ &= -\frac{n^{1/2}}{N^{3/2}} \int_0^{A_i} (B_{i,n}^*(v + y) - B_{i,n}^*(v)) dy + \mathcal{O} \left(\frac{N^{1/2}(\log n)^{1/2} \log_2 n}{n^{1/2}} \right), \quad (5.8) \end{aligned}$$

uniformly in $i = 0, 1, \dots, N - 1$ and $v \in [0, 1]$.

Note that $B_{i,n}^*$ can be represented as

$$B_{i,n}^*(u) := -W_{i,n}^*(u) + uW_{i,n}^*(1),$$

where (for each fixed n) $\{W_{i,n}^*, i = 0, 1, \dots, N - 1\}$ are independent Wiener processes which are also independent of $\{K(t_i, n), i = 0, 1, \dots, N - 1\}$. Hence, almost surely,

$$\begin{aligned} & \int_0^{A_i} (B_{i,n}^*(v + y) - B_{i,n}^*(v)) dy \\ &= -\int_0^{A_i} (W_{i,n}^*(v + y) - W_{i,n}^*(v)) dy + W_{i,n}^*(1) \int_0^{A_i} y dy \\ &= -\int_0^{A_i} (W_{i,n}^*(v + y) - W_{i,n}^*(v)) dy + \mathcal{O} \left(\frac{N^2(\log N)^{1/2} \log_2 n}{n} \right), \quad (5.9) \end{aligned}$$

the last equality following from the LIL (3.2) for the Kiefer process and the fact that almost surely, $\sup_{0 \leq i \leq N-1} |W_{i,n}^*(1)| = \mathcal{O}((\log N)^{1/2})$. This last fact can be easily checked using the usual estimate for Gaussian tails and the Borel–Cantelli lemma, regardless of the dependency structure of the standard normal variables $\{W_{i,n}^*(1), i = 0, 1, \dots, N-1; n = 1, 2, \dots\}$.

Putting (5.7), (5.8) and (5.9) together, we obtain:

$$Z_n(t) = \frac{2n^{1/2}}{N^{3/2}} \int_0^{A_i} (W_{i,n}^*(v+y) - W_{i,n}^*(v)) dy + \mathcal{O}(\psi(n, N)), \quad (5.10)$$

for $t = t_i + v/N$, $0 \leq v \leq 1$, $K(t_i, n) \leq 0$, where $\psi(n, N)$ is defined by (5.6).

If $K(t_i, n) > 0$, then by introducing the Kiefer process $K^\leftarrow(t, n) := K(1-t, n)$, we can write

$$\begin{aligned} & 2 \int_0^{K(t_i, n)/n} (K(t-z, n) - K(t, n)) dz \\ &= 2 \int_0^{K(t_i, n)/n} (K^\leftarrow(1-t+z, n) - K^\leftarrow(1-t, n)) dz, \end{aligned}$$

and, similarly to the argument leading to (5.10), we can find independent Wiener processes $\{W_{i,n}^\leftarrow, i = 0, 1, \dots, N-1\}$ which are also independent of $\{K(t_i, n), i = 0, 1, \dots, N-1\}$, such that almost surely,

$$Z_n(t) = \frac{2n^{1/2}}{N^{3/2}} \int_0^{A_i} (W_{i,n}^\leftarrow(v+y) - W_{i,n}^\leftarrow(v)) dy + \mathcal{O}(\psi(n, N)), \quad (5.11)$$

for $t = t_i + v/N$, $0 \leq v \leq 1$, $K(t_i, n) > 0$.

Let

$$W_{i,n}(\cdot) := W_{i,n}^*(\cdot) \mathbf{1}\{K(t_i, n) \leq 0\} + W_{i,n}^\leftarrow(\cdot) \mathbf{1}\{K(t_i, n) > 0\},$$

$i = 0, 1, \dots, N-1$. These Wiener processes $\{W_{i,n}, i = 0, 1, \dots, N-1\}$ are independent of $\{K(t_i, n), i = 0, 1, \dots, N-1\}$, however we note that they are *not* claimed to be independent between themselves.

Lemma 5.1 now follows from combining (5.10) and (5.11). \square

In our next lemma we are to study the process in the main term of (5.4). To this end, we fix a positive real number A , and introduce the mean zero Gaussian process

$$X(v) := \frac{3^{1/2}}{A^{3/2}} \int_0^A (W(v+y) - W(v)) dy, \quad v \in [0, 1]. \quad (5.12)$$

It is straightforward to compute its covariance:

$$\mathbf{E}X(u)X(v) = \begin{cases} 1 - \frac{3|v-u|}{2A} + \frac{|v-u|^3}{2A^3} & \text{if } |v-u| \leq A, \\ 0 & \text{if } |v-u| > A, \end{cases} \quad (5.13)$$

and therefore

$$\mathbf{E}(X(v) - X(u))^2 = \begin{cases} \frac{3|v-u|}{A} - \frac{|v-u|^3}{A^3} & \text{if } |v-u| \leq A, \\ 2 & \text{if } |v-u| > A. \end{cases} \quad (5.14)$$

Lemma 5.2 *For any $x > 0$, and arbitrary $p \geq 2$, we have*

$$\mathbf{P} \left(\sup_{0 \leq v \leq 1} |X(v)| \geq x \left(1 + \frac{8\sqrt{3}}{\sqrt{pA} \log p} \right) \right) \leq 4p^2 \int_x^\infty e^{-s^2/2} ds \quad (5.15)$$

and

$$\mathbf{P} \left(\sup_{0 \leq v \leq 1} |X(v)| \leq x \right) \leq (\Phi(x))^{1/A}, \quad (5.16)$$

where $\Phi(x)$ is the standard normal distribution function.

Proof. First we prove (5.15). According to a well-known inequality of Fernique (1964) for Gaussian processes, for any $x > 0$,

$$\mathbf{P} \left(\sup_{0 \leq v \leq 1} |X(v)| \geq x \left(\sigma^2 + 4 \int_1^\infty \varphi(p^{-s^2}) ds \right) \right) \leq 4p^2 \int_x^\infty e^{-s^2/2} ds,$$

where $p \geq 2$ is arbitrary, and σ and φ are such that

$$\mathbf{E}(X(v))^2 \leq \sigma^2, \quad \mathbf{E}(X(v) - X(u))^2 \leq \varphi^2(v - u).$$

In view of (5.13) and (5.14), we can choose $\sigma = 1$ and $\varphi(h) = (3h/A)^{1/2}$. Since

$$\int_1^\infty \varphi(p^{-s^2}) ds \leq \frac{\sqrt{3}}{\sqrt{A}} \int_1^\infty e^{-(s \log p)/2} ds = \frac{2\sqrt{3}}{\sqrt{pA} \log p},$$

we conclude (5.15).

Concerning (5.16), we note that

$$\sup_{0 \leq v \leq 1} |X(v)| \geq \max_{0 \leq i \leq k} X(iA),$$

where $k = [1/A]$ and, due to (5.13), the random variables $\{X(0), X(A), \dots, X(kA)\}$ are seen to be independent standard normal ones. Hence (5.16) follows as well. \square

Now we are ready to prove (5.1) and (5.2).

Proof of (5.1). Let

$$N = N(n) := \left[n^{1/2} (\log_2 n)^{-4} \right], \quad (5.17)$$

For each n , we split $\{0, 1, \dots, N - 1\}$ into two (random) parts:

$$\begin{aligned} \mathcal{J}_1 = \mathcal{J}_1(n) &:= \{i : |K(t_i, n)| \leq n^{1/2} (\log_2 n)^{-1}, 0 \leq i \leq N - 1\}, \\ \mathcal{J}_2 = \mathcal{J}_2(n) &:= \{i : |K(t_i, n)| > n^{1/2} (\log_2 n)^{-1}, 0 \leq i \leq N - 1\}. \end{aligned}$$

If $i \in \mathcal{J}_1$, then on applying the LIL (3.2) for the Kiefer process and Theorem H with $h_n = n^{-1/2} (\log_2 n)^{-1}$, we conclude

$$\int_0^{K(t_i, n)/n} (K(t - z, n) - K(t, n)) dz = \mathcal{O} \left(\frac{(\log n)^{1/2}}{n^{1/4} (\log_2 n)^{3/2}} \right) \quad \text{a.s.}$$

uniformly in $i \in \mathcal{J}_1$. In view of (5.7), we obtain: when $n \rightarrow \infty$,

$$\max_{i \in \mathcal{J}_1} \sup_{t \in [t_i, t_{i+1}]} |Z_n(t)| = \mathcal{O} \left(\frac{(\log n)^{1/2}}{n^{1/4} (\log_2 n)^{3/2}} \right) \quad \text{a.s.} \quad (5.18)$$

For $i \in \mathcal{J}_2$, we consider the variables

$$X_{i,n} := \frac{3^{1/2}}{A_i^{3/2}} \sup_{0 \leq v \leq 1} \left| \int_0^{A_i} (W_{i,n}(v+y) - W_{i,n}(v)) dy \right|.$$

According to (5.4)–(5.6) of Lemma 5.1, when $n \rightarrow \infty$, we have almost surely

$$\max_{i \in \mathcal{J}_2} \sup_{t \in [t_i, t_{i+1}]} |Z_n(t)| \leq \frac{2\|K(\cdot, n)\|^{3/2}}{3^{1/2} n} \max_{i \in \mathcal{J}_2} X_{i,n} + \mathcal{O}\left(\frac{(\log n)^{1/2}}{n^{1/4} \log_2 n}\right),$$

which, combined with (5.18), yields

$$\|Z_n\| \leq \frac{2\|K(\cdot, n)\|^{3/2}}{3^{1/2} n} \max_{i \in \mathcal{J}_2} X_{i,n} + \mathcal{O}\left(\frac{(\log n)^{1/2}}{n^{1/4} \log_2 n}\right) \quad \text{a.s.} \quad (5.19)$$

We now show that the variables $X_{i,n}$, $i \in \mathcal{J}_2$ have Gaussian-like tails. Recall that the vector $\{W_{i,n}, i = 0, 1, \dots, N-1\}$ is independent of the vector $\{K(t_i, n), i = 0, 1, \dots, N-1\}$. Hence, given $A_i = A$, the conditional distribution of $X_{i,n}$ coincides with the distribution of $\sup_{0 \leq v \leq 1} |X(v)|$ of Lemma 5.2. Therefore, on applying (5.15) and using the fact that $A_i \geq (\log_2 n)^{-6}$ for all $i \in \mathcal{J}_2$, we obtain

$$\mathbf{P}\left(X_{i,n} \geq x \left(1 + \frac{8\sqrt{3}(\log_2 n)^3}{\sqrt{p} \log p}\right)\right) \leq 4p^2 \int_x^\infty e^{-s^2/2} ds. \quad (5.20)$$

Let $\varepsilon \in (0, 1)$. We choose $p = \varepsilon^{-2}(\log_2 n)^6$ and $n_0 = n_0(\varepsilon)$ such that for all $n \geq n_0$, $8\sqrt{3}(\log_2 n)^3/(\sqrt{p} \log p) \leq \varepsilon$. Thus, for any $x > 0$ and $n \geq n_0$, (5.20) yields

$$\mathbf{P}(X_{i,n} \geq (1 + \varepsilon)x) \leq \frac{4(\log_2 n)^{12}}{\varepsilon^4} \int_x^\infty e^{-s^2/2} ds \leq \frac{4(\log_2 n)^{12} e^{-x^2/2}}{\varepsilon^4 x},$$

as well as

$$\mathbf{P}\left(\max_{i \in \mathcal{J}_2} X_{i,n} \geq (1 + \varepsilon)x\right) \leq \frac{4N(\log_2 n)^{12} e^{-x^2/2}}{\varepsilon^4 x} \leq \frac{4n^{1/2}(\log_2 n)^8 e^{-x^2/2}}{\varepsilon^4 x}.$$

Taking $x := (1 + \varepsilon)(\log n)^{1/2}$, we obtain:

$$\mathbf{P}\left(\max_{i \in \mathcal{J}_2} X_{i,n} \geq (1 + \varepsilon)^2(\log n)^{1/2}\right) \leq \frac{4(\log_2 n)^8}{\varepsilon^4(1 + \varepsilon)(\log n)^{1/2} n^\varepsilon}. \quad (5.21)$$

With (5.19) and (5.21) in mind, let $n_k = \lceil k^{2/\varepsilon} \rceil$. Consequently, by the Borel–Cantelli lemma, we have, almost surely when $k \rightarrow \infty$,

$$\|Z_{n_k}\| \leq \frac{2(1 + \varepsilon)^2(\log n_k)^{1/2} \|K(\cdot, n_k)\|^{3/2}}{3^{1/2} n_k} + \mathcal{O}\left(\frac{(\log n_k)^{1/2}}{n_k^{1/4} \log_2 n_k}\right). \quad (5.22)$$

Now let $n_k \leq n < n_{k+1}$, and note that $n_{k+1} - n_k = \mathcal{O}(n_k^{1-\varepsilon/2})$, $k \rightarrow \infty$. By (5.3),

$$\begin{aligned} |Z_n(t) - Z_{n_k}(t)| &\leq 2 \left| \int_0^{K(t,n)/n} \Delta_{k,n,t}(z) dz \right| \\ &\quad + 2 \left| \int_{K(t,n_k)/n_k}^{K(t,n)/n} (K(t-z, n_k) - K(t, n_k)) dz \right|, \end{aligned}$$

where $\Delta_{k,n,t}(z) := K(t-z, n) - K(t-z, n_k) - K(t, n) + K(t, n_k)$. According to Theorem I and the LIL (3.2) for the Kiefer process, when $k \rightarrow \infty$,

$$\Delta_{k,n,t}(z) = \mathcal{O}\left(n_k^{(1-\varepsilon)/4}(\log n_k)^{1/2}(\log_2 n_k)^{1/4}\right) \quad \text{a.s.},$$

uniformly in $t \in (0, 1)$, $z \in [0, K(t, n)/n]$ and $n_k \leq n < n_{k+1}$. Thus, by the just mentioned LIL (3.2) for the Kiefer process,

$$\int_0^{K(t,n)/n} \Delta_{k,n,t}(z) dz = \mathcal{O}\left(n_k^{-(1+\varepsilon)/4}(\log n_k)^{1/2}(\log_2 n_k)^{3/4}\right) \quad \text{a.s.},$$

uniformly in $t \in (0, 1)$ and $n_k \leq n < n_{k+1}$.

On the other hand, by the same LIL (3.2) for the Kiefer process and Theorem H, as $k \rightarrow \infty$, we have

$$K(t-z, n_k) - K(t, n_k) = \mathcal{O}\left(n_k^{1/4}(\log n_k)^{1/2}(\log_2 n_k)^{1/4}\right) \quad \text{a.s.},$$

uniformly in $t \in (0, 1)$, $z \in [K(t, n_k)/n_k, K(t, n)/n]$ and $n_k \leq n < n_{k+1}$. Moreover, since Corollary 1.12.4 of Csörgő and Révész (1981) implies that, as $k \rightarrow \infty$,

$$\sup_{0 \leq t \leq 1} |K(t, n) - K(t, n_k)| = \mathcal{O}\left(n_k^{1/2-\varepsilon/4}(\log n_k)^{1/2}\right) \quad \text{a.s.}, \quad (5.23)$$

uniformly in $n_k \leq n < n_{k+1}$, it follows that

$$\max_{n_k \leq n < n_{k+1}} \sup_{t \in [0, 1]} \left| \frac{K(t, n)}{n} - \frac{K(t, n_k)}{n_k} \right| = \mathcal{O}\left(n_k^{-1/2-\varepsilon/4}(\log n_k)^{1/2}\right) \quad \text{a.s.}$$

Therefore, almost surely when $k \rightarrow \infty$,

$$\int_{K(t, n_k)/n_k}^{K(t, n)/n} (K(t-z, n_k) - K(t, n_k)) dz = \mathcal{O}\left(n_k^{-(1+\varepsilon)/4}(\log n_k)(\log_2 n_k)^{1/4}\right),$$

uniformly in $t \in [0, 1]$ and $n_k \leq n < n_{k+1}$. Thus we have proved that, as $k \rightarrow \infty$,

$$\max_{n_k \leq n < n_{k+1}} \|Z_n - Z_{n_k}\| = \mathcal{O}\left(n_k^{-(1+\varepsilon)/4}(\log n_k)(\log_2 n_k)^{1/4}\right), \quad \text{a.s.}$$

Going back to (5.22) and taking (5.23) into account, when $n \rightarrow \infty$, we obtain

$$\begin{aligned} \|Z_n\| &\leq \frac{2(1+\varepsilon)^2(\log n)^{1/2}}{3^{1/2}n} \|K(\cdot, n)\|^{3/2} + \mathcal{O}\left(\frac{(\log n)^{1/2}}{n^{1/4}\log_2 n}\right) \\ &= (1+o(1)) \frac{2(1+\varepsilon)^2(\log n)^{1/2}}{3^{1/2}n} \|K(\cdot, n)\|^{3/2} \quad \text{a.s.}, \end{aligned}$$

where in the last line we used the other LIL (3.3) for the Kiefer process. This also completes the proof of (5.1). \square

Proof of (5.2). Here we are to show that for any given $\varepsilon \in (0, 1)$

$$\liminf_{n \rightarrow \infty} \frac{n}{(\log n)^{1/2}} \frac{\|Z_n\|}{\|K(\cdot, n)\|^{3/2}} \geq (1-\varepsilon) \left(\frac{4}{3}\right)^{1/2} \quad \text{a.s.} \quad (5.24)$$

Let $N := \lceil n^{\varepsilon/2} \rceil$ and $t_i = i/N$, $i = 0, 1, \dots, N$. Put

$$\tilde{A}_n := \frac{N}{n} \max_{0 \leq i \leq N} |K(t_i, n)|$$

and let i_0 be the almost surely unique index where the latter maximum is attained. Using this index, and \tilde{A}_n as just defined, via Lemma 5.1 we let

$$\tilde{X}_n(v) := \frac{3^{1/2}}{\tilde{A}_n^{3/2}} \int_0^{\tilde{A}_n} (W_{i_0, n}(v+y) - W_{i_0, n}(v)) dy,$$

and conclude that, given $\tilde{A}_n = A$, the distribution of the process $\{\tilde{X}_n(v), v \in [0, 1]\}$ is equal to that of $\{X(v), v \in [0, 1]\}$ in (5.12).

Using now the almost sure representation of Z_n as in Lemma 5.1, on recalling that $N = \lceil n^{\varepsilon/2} \rceil$ and taking $\varepsilon \in (0, 1)$ small enough, via (5.6) we conclude

$$\begin{aligned} \|Z_n\| &\geq \frac{2n^{1/2} \tilde{A}_n^{3/2}}{3^{1/2} N^{3/2}} \sup_{0 \leq v \leq 1} |\tilde{X}_n(v)| + \mathcal{O}\left(\frac{(\log n)(\log_2 n)^{1/4}}{n^{(1+\varepsilon)/4}}\right) \\ &= \frac{2 \max_{0 \leq i \leq N} |K(t_i, n)|^{3/2}}{3^{1/2} n} \sup_{0 \leq v \leq 1} |\tilde{X}_n(v)| + \mathcal{O}\left(\frac{(\log n)(\log_2 n)^{1/4}}{n^{(1+\varepsilon)/4}}\right). \end{aligned} \quad (5.25)$$

By the inequalities $1 - \Phi(x) \geq (c/x)e^{-x^2/2}$ for $x > 1$ with some constant $c > 0$ and $1 - u \leq e^{-u}$ for $u > 0$, it follows from Lemma 5.2 that for all large n

$$\begin{aligned} \mathbf{P}\left(\sup_{0 \leq v \leq 1} |\tilde{X}_n(v)| < (1 - \varepsilon)(\log n)^{1/2} \middle| \tilde{A}_n\right) &\leq \left(\Phi((1 - \varepsilon)(\log n)^{1/2})\right)^{1/\tilde{A}_n} \\ &\leq \left(1 - \frac{c}{(1 - \varepsilon)(\log n)^{1/2}} n^{-(1-\varepsilon)^2/2}\right)^{1/\tilde{A}_n} \\ &\leq \exp\left(-\frac{c}{(1 - \varepsilon)(\log n)^{1/2} \tilde{A}_n} n^{-(1-\varepsilon)^2/2}\right). \end{aligned}$$

Taking expectations on both sides, we arrive at

$$\begin{aligned} &\mathbf{P}\left(\sup_{0 \leq v \leq 1} |\tilde{X}_n(v)| < (1 - \varepsilon)(\log n)^{1/2}\right) \\ &\leq \mathbf{E} \exp\left(-\frac{c}{(1 - \varepsilon)(\log n)^{1/2} \tilde{A}_n} n^{-(1-\varepsilon)^2/2}\right) \\ &\leq \mathbf{P}(\tilde{A}_n > n^{(\varepsilon-1)/2} \log n) + \exp\left(-\frac{cn^{\varepsilon(1-\varepsilon)/2}}{(1 - \varepsilon)(\log n)^{3/2}}\right). \end{aligned} \quad (5.26)$$

Estimating now the first term on the right hand side of the inequality (5.26), via the Kolmogorov–Smirnov distribution (cf., e.g., Shorack and Wellner, 1986, p. 34) we get

$$\begin{aligned} \mathbf{P}(\tilde{A}_n > n^{(\varepsilon-1)/2} \log n) &\leq \mathbf{P}\left(\sup_{0 \leq t \leq 1} |K(t, n)| > n^{1/2} \log n\right) \\ &= \mathbf{P}(\|B\| > \log n) \leq 2 \exp(-2(\log n)^2), \end{aligned}$$

where we used the fact that $K(\cdot, n)/n^{1/2} = B(\cdot)$, a Brownian bridge for each n . Combining now the latter inequality with that of (5.26), and then using the Borel–Cantelli lemma, we obtain with probability one that we have for all large enough n

$$\sup_{0 \leq v \leq 1} |\tilde{X}_n(v)| \geq (1 - \varepsilon)(\log n)^{1/2}.$$

This in turn, in combination with (5.25), results in concluding that with probability one we have for all large n

$$\|Z_n\| \geq \frac{(1-\varepsilon)2(\log n)^{1/2} \max_{1 \leq i \leq N} |K(t_i, n)|^{3/2}}{3^{1/2}n} + \mathcal{O}\left(\frac{(\log n)(\log_2 n)^{1/4}}{n^{(1+\varepsilon)/4}}\right),$$

or, equivalently,

$$\begin{aligned} \frac{n\|Z_n\|}{(\log n)^{1/2}\|K(\cdot, n)\|^{3/2}} &\geq \frac{2(1-\varepsilon) \max_{1 \leq i \leq N} |K(t_i, n)|^{3/2}}{3^{1/2} \|K(\cdot, n)\|^{3/2}} \\ &+ \mathcal{O}\left(\frac{n^{3/4}}{(\log_2 n)^{3/4} \|K(\cdot, n)\|^{3/2}} \frac{(\log n)^{1/2} \log_2 n}{n^{\varepsilon/4}}\right). \end{aligned} \quad (5.27)$$

Consider next the obvious inequality

$$\|K(\cdot, n)\| \leq \max_{1 \leq i \leq N} |K(t_i, n)| + \max_{1 \leq i \leq N} \sup_{t_{i-1} \leq t \leq t_i} |K(t, n) - K(t_{i-1}, n)|.$$

Consequently, on applying Theorem H and the other law of the iterated logarithm for a Kiefer process (cf. (3.3)), we obtain

$$\lim_{n \rightarrow \infty} \frac{\max_{1 \leq i \leq N} |K(t_i, n)|}{\|K(\cdot, n)\|} = 1 \quad \text{a.s.} \quad (5.28)$$

It remains to show that the \mathcal{O} term of (5.27) goes to zero almost surely as $n \rightarrow \infty$, which in turn follows by the other law of the iterated logarithm for a Kiefer process (cf. (3.3)). The latter in combination with (5.27) and (5.28) results in (5.24). Since ε can be taken arbitrarily small, this also concludes the proof of (5.2). Having now (5.1) and (5.2) verified, the proof of the *uniform* property (2.7) in Theorem 2.1 is also complete. \square

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