Small ball estimates for Brownian motion under a weighted sup-norm

by

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Summary. Let $\{W(t);\ 0 \le t \le 1\}$ be a real-valued Wiener process, starting from 0. For a large class of functions f which may vanish at 0, we obtain the exact asymptotics, as ε goes to 0, of $\log \mathbb{P}(\sup_{0<t\le 1} |W(t)/f(t)| < \varepsilon)$.

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1. Introduction

Let $\{X(t);\ 0 \le t \le 1\}$ be a real-valued mean-zero Gaussian process, and let "$\|\cdot\|$" be a semi-norm in the space of real functions on $[0,1]$. In the literature, asymptotic behaviours of

$$\log \mathbb{P}\Big(\|X\| > \lambda\Big), \qquad \text{for } \lambda \to \infty, \tag{1.1}$$

$$\log \mathbb{P}\Big(\|X\| < \varepsilon\Big), \qquad \text{for } \varepsilon \to 0, \tag{1.2}$$

are studied due to various motivations. Despite their apparent resemblance, they are really very *different* types of estimates. Problems related to (1.1) are usually called "large deviations", which are studied by a large number of mathematicians and found many applications in mathematics and physics. Those related to (1.2), often referred to as "small ball estimates", receive also much research interest, yet relatively little is known. As is demonstrated by a recent paper of Kuelbs and Li [14], one should *not* expect to see a general result for (1.2) for an arbitrary Gaussian process X. It would therefore be preferable to study some "well-behaved" processes.

Not surprisingly, the example of Brownian motion (as well as related processes, such as Brownian bridge and sheet) attracts the most contribution. Let us cite the recent works [3], [11], [14]–[17], [19]–[20], [23]–[26], as well as the survey articles [18] and [21].

In the rest of the paper, $\{W(t);\ 0 \le t \le 1\}$ denotes a one-dimensional Wiener process, starting from 0, and we look at the asymptotic behaviour of $\log \mathbb{P}(\|W\| < \varepsilon)$ for small ε. A typical situation is:

$$\log \mathbb{P}\Big(\|W\| < \varepsilon\Big) \sim -\frac{c}{\varepsilon^b}, \qquad \varepsilon \to 0, \tag{1.3}$$

for finite constants $b > 0$ and $c > 0$. Below are some known examples:

(i) uniform sup-norm $\|\phi\|_\infty \stackrel{\text{def}}{=} \sup_{0\le t\le 1} |\phi(t)|$: $b = 2$ and $c = \pi^2/8$;

(ii) L^2–norm $\|\phi\|_2 \stackrel{\text{def}}{=} (\int_0^1 \phi^2(t)\, dt)^{1/2}$: $b = 2$ and $c = 1/8$;

(iii) Hölder norm $\|\phi\|_{\mathcal{H},\alpha} \stackrel{\text{def}}{=} \sup_{0\le s<t\le 1} |\phi(t) - \phi(s)|/(t-s)^\alpha$ (for $0 < \alpha < 1/2$): $b = 2/(1-2\alpha)$, but the value of c is unknown.

(See, Chung [6], Cameron and Martin [5], Baldi and Roynette [3], and Kuelbs and Li [15]). We mention that the L^p–norm (for $1 \le p < \infty$) is treated by Donsker and Varadhan [12] who get $b = 2$ and an "exotic" value of c when $p \ne 2$. More general norms, such as Sobolev or Besov norms, are also studied, and perhaps understandably, results are obtained with less accuracy concerning the values of b and — especially — c.

We intend to study the log-probability in (1.3) for W under the weighted sup-norm:

$$\|W\|_f \stackrel{\text{def}}{=} \sup_{0<t\le 1} \frac{|W(t)|}{f(t)}, \tag{1.4}$$

for a large class of functions $f \ge 0$. First, recall the following known results.

Theorem A (Mogulskii [22]). *If f is a piece-wise continuous function such that*

$$\inf_{0\le t\le 1} f(t) > 0, \tag{H1}$$

then

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\Big(\|W\|_f < \varepsilon\Big) = -\frac{\pi^2}{8}\int_0^1 \frac{dt}{f^2(t)}. \tag{1.5}$$

Theorem B ([24]). *If $f(t) = t^\alpha$ for some $0 \le \alpha < 1/2$, then*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\Big(\|W\|_f < \varepsilon\Big) = -\frac{\pi^2}{8(1-2\alpha)}.$$

Observe that condition (H1) is not satisfied by $f(t) = t^\alpha$ $(\alpha > 0)$, nonetheless the constants agree in the theorems. Our aim is to characterize functions f for which (H1) fails, but for which (1.3) still holds with $b = 2$. Part of the motivations is that weight functions vanishing at 0 are of particular interest in limit theorems in probability and statistics, cf. the book of Csörgő and Horváth [9].

Let us give some precisions about what kind of functions f is treated in the paper. In order that (1.4) be well-defined, we clearly have to limit ourselves to those functions f such that

$$\inf_{\delta\le t\le 1} f(t) > 0, \qquad \text{for all } 0 < \delta \le 1. \tag{1.6}$$

In the literature, a function f satisfying (1.6) is called *positive* (cf. for example Csörgő et al. [10]).

There is no problem for the definition of $\|W\|_f$ under (H1). However, when this condition fails, the situation is somewhat delicate. Of course, $\|W\|_f < \infty$ implies

$$\limsup_{t\to 0} \frac{|W(t)|}{f(t)} < \infty \qquad \text{a.s.} \tag{1.7}$$

According to a well-known integral test (cf. Csörgő and Horváth [9, Corollary 4.1.1]), when a positive function f is nondecreasing in the neighbourhood of 0, (1.7) holds if and only if

$$\int_0^1 \frac{dt}{t} \exp\left(-c\,\frac{f^2(t)}{t}\right) < \infty, \qquad \text{for some } c > 0. \tag{1.8}$$

(A typical such function is $\nu\sqrt{t\log\log(1/t)}$, with $\nu > 0$, which is not surprising in view of the usual law of the iterated logarithm). By an abuse of notation, from now on, we call f a "weight function" if it satisfies (1.7) and (1.6).

Here is the main result of the paper.

Theorem 1.1. *If a positive function f satisfies either* (H1) *or the following condition:*

(H2) *f is nondecreasing in a neighbourhood of* 0,

then

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\Big(\|W\|_f < \varepsilon\Big) = -\frac{\pi^2}{8}\int_0^1 \frac{dt}{f^2(t)}.$$

Remark 1.2. Condition (H2) is of no surprise. If (H1) fails, i.e. if $\inf_{0<t\le 1} f(t) = 0$, one has to use the test (1.8) to decide whether $\|W\|_f$ is well-defined, and this is where the monotonicity of f is needed. The problem of whether the monotonicity is necessary in this kind of test remains open (cf. Csörgő et al. [10, p. 243]), to the best of our knowledge.

Remark 1.3. An immediate consequence of Theorem 1.1 is that under the sup-norm $\|\cdot\|_f$, the decay rate (1.3) is with $b = 2$ if and only if

$$\int_0^1 \frac{dt}{f^2(t)} < \infty. \tag{1.9}$$

Remark 1.4. Since condition (1.8) is implicitly imposed in our study, compared with (1.8), it is easily seen that, from a practical point of view, (1.9) is verified by most "usual" weight functions of W, such as $f(t) \approx t^\alpha$ (for $-\infty < \alpha < 1/2$), or $f(t) \approx t^{1/2}(\log(1/t))^\beta$ (for $\beta > 1/2$), when t is in the neighbourhood of 0. A typical weight function of W which disobeys (1.9) is $f(t) \approx \sqrt{t\log\log(1/t)}$.

Theorem 1.1 is proved in Section 2. Section 3 is devoted to some discussions of the rate of decay in (1.3) when (1.9) fails. In Section 4, we present some applications of Theorem

1.1 by giving Chung's functional iterated logarithm law for W and small ball estimates in higher dimensions. The situation for the Brownian bridge is also treated.

2. Proof of Theorem 1.1

Notation: for any deterministic function or stochastic process ϕ, we write, for brevity,

$$\phi^{\uparrow}(s,t) \stackrel{\text{def}}{=} \sup_{s\le u\le t} \phi(u), \tag{2.1}$$

$$\phi_{\downarrow}(s,t) \stackrel{\text{def}}{=} \inf_{s\le u\le t} \phi(u), \tag{2.2}$$

$$\phi^{*}(s,t) \stackrel{\text{def}}{=} \sup_{s\le u\le t} |\phi(u)|, \qquad t > s \ge 0. \tag{2.3}$$

and moreover,

$$\phi^{*}(t) \stackrel{\text{def}}{=} \phi^{*}(0,t) = \sup_{0\le u\le t} |\phi(u)|, \qquad t > 0. \tag{2.4}$$

Let $\{\Xi(t);\, 0 \le t \le 1\}$ be a standard one-dimensional Brownian bridge, which can be realized as $\{W(t) - tW(1);\, 0 \le t \le 1\}$. Recall the well-known small ball probabilities for W and Ξ under the uniform sup-norm:

$$\log \mathbb{P}\Big(W^{*}(1) < \varepsilon\Big) \sim -\frac{\pi^2}{8\varepsilon^2}, \tag{2.5}$$

$$\log \mathbb{P}\Big(\Xi^{*}(1) < \varepsilon\Big) \sim -\frac{\pi^2}{8\varepsilon^2}, \qquad \varepsilon \to 0. \tag{2.6}$$

Here is a basic heuristic explanation to the similarity between (2.5) and (2.6): when W is constrained to stay in a small ball, $W(1)$ becomes very close to 0, which indicates that the Wiener process behaves somewhat like a Brownian bridge (note that $\log \mathbb{P}(|W(1)| < \varepsilon)$ is negligible compared with $1/\varepsilon^2$).

Proof of Theorem 1.1: the upper bound. Pick $n \ge 2$ and let $t_k = t_{k,n} \stackrel{\text{def}}{=} k/n$ for $0 \le k \le n$. According to the notation in (2.3) and (2.1),

$$\begin{aligned}\mathbb{P}\Big(\|W\|_f < \varepsilon\Big) &\le \mathbb{P}\Big(W^{*}(t_{k-1}, t_k) < \varepsilon f^{\uparrow}(t_{k-1}, t_k), \text{ for all } 1 \le k \le n\Big)\\ &\stackrel{\text{def}}{=} \mathbb{P}\Big(\bigcap_{k=1}^{n} A_k\Big),\end{aligned}$$

with obvious notation. Write $\mathcal{F}$ for the natural completed filtration of W. By conditioning on $\mathcal{F}_{t_{n-1}}$,

$$\mathbb{P}\Big(\bigcap_{k=1}^{n} A_k\Big) = \mathbb{E}\Big[\,\mathbb{1}_{\{\cap_{k=1}^{n-1} A_k\}}\,\mathbb{P}\Big(A_n\,\Big|\,\mathcal{F}_{t_{n-1}}\Big)\Big].$$

Consider the conditional probability $\mathbb{P}(A_n \,|\, \mathcal{F}_{t_{n-1}})$. Since $t \mapsto W(t+t_{n-1}) - W(t_{n-1})$ is again a Wiener process, independent of $\mathcal{F}_{t_{n-1}}$, by a well-known inequality of Anderson [2] for Gaussian shifted balls,

$$\begin{aligned}
\mathbb{P}\Big(A_n\,\Big|\,\mathcal{F}_{t_{n-1}}\Big) &\le \sup_{x\in\mathbb{R}} \mathbb{P}\Big(\sup_{0\le t\le t_n - t_{n-1}} |W(t)+x| < \varepsilon f^{\uparrow}(t_{n-1},t_n)\Big)\\
&= \mathbb{P}\Big(W^*(t_n - t_{n-1}) < \varepsilon f^{\uparrow}(t_{n-1},t_n)\Big)\\
&= \mathbb{P}\Big(W^*(1) < \frac{\varepsilon f^{\uparrow}(t_{n-1},t_n)}{(t_n - t_{n-1})^{1/2}}\Big),
\end{aligned}$$

the last identity following from the Brownian scaling property. Accordingly,

$$\mathbb{P}\Big(\|W\|_f < \varepsilon\Big) \le \mathbb{P}\Big(\bigcap_{k=1}^{n-1} A_k\Big)\,\mathbb{P}\Big(W^*(1) < \frac{\varepsilon f^{\uparrow}(t_{n-1},t_n)}{(t_n - t_{n-1})^{1/2}}\Big).$$

Iterating the same procedure,

$$\mathbb{P}\Big(\|W\|_f < \varepsilon\Big) \le \prod_{k=1}^{n} \mathbb{P}\Big(W^*(1) < \frac{\varepsilon f^{\uparrow}(t_{k-1},t_k)}{(t_k - t_{k-1})^{1/2}}\Big). \tag{2.7}$$

Hence, in view of (2.5),

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\Big(\|W\|_f < \varepsilon\Big) \le -\frac{\pi^2}{8}\sum_{k=1}^{n} \frac{t_k - t_{k-1}}{(f^{\uparrow}(t_{k-1},t_k))^2}.$$

This yields the upper bound in Theorem 1.1 by sending n to infinity. $\square$

The proof of the lower bound is based on a preliminary estimate and on Khatri's classical inequality.

Lemma 2.1. *If f satisfies* (H1),

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\Big(\sup_{0\le t\le 1} \frac{|W(t)|}{f(t)} < \varepsilon\Big) \ge -\frac{\pi^2}{8}\int_0^1 \frac{dt}{f^2(t)}.$$

Fact 2.2 (Khatri [13]). *Let $(Y_1, \ldots, Y_n)$ be a mean-zero Gaussian vector. For all positive numbers a_k $(1 \le k \le n)$,*

$$\mathbb{P}\Big(\bigcap_{k=1}^{n} (\,|Y_k| < a_k) \Big) \ge \prod_{k=1}^{n} \mathbb{P}(\,|Y_k| < a_k).$$

By assuming Lemma 2.1 for the moment, we can prove the lower bound in Theorem 1.1. Observe that Lemma 2.1 is basically Mogulskii's Theorem A, except that we do not assume piece-wise continuity. To illustrate our approach, we include the proof of the lemma, which is postponed to the end of the section.

Proof of Theorem 1.1: the lower bound. Let f be a positive function, in the sense of (1.6). Of course, we only have to treat the case $\int_0^1 dt/f^2(t) < \infty$ (otherwise the upper bound suffices for the conclusion). If (H1) holds, Lemma 2.1 is nothing else but the lower bound in Theorem 1.1. So the remaining situation to be discussed is: f nondecreasing in a neighbourhood of 0, say $(0, \delta)$.

Let $0 < a < \delta$. Define $g(t) \stackrel{\text{def}}{=} f(at)/3\sqrt{a}$ for $t \in [0,1]$. Then g is nondecreasing over $[0,1]$. Let us bound below the probability

$$\Delta_1 \stackrel{\text{def}}{=} \mathbb{P}\Big(\|W\|_g < 3\varepsilon;\ |W(1)| < \mu\varepsilon \Big),$$

where $\mu > 0$ is a fixed constant. For $j \ge 0$, write $s_j \stackrel{\text{def}}{=} 2^{-j}$ and

$$\begin{aligned} F_j &\stackrel{\text{def}}{=} \Big\{ |W(s_{j+1})| < g(s_{j+1})\varepsilon \Big\}, \\ F_{-1} &\stackrel{\text{def}}{=} \Big\{ |W(1)| < \mu\,\varepsilon \Big\}. \end{aligned}$$

Observe that

$$\begin{aligned} \Delta_1 &\ge \mathbb{P}\Big[\bigcap_{j=0}^{\infty} \Big(W^*(s_{j+1}, s_j) < 3g(s_{j+1})\varepsilon \Big) \cap \bigcap_{j=-1}^{\infty} F_j \Big] \\ &\ge \mathbb{P}\Big[\bigcap_{j=0}^{\infty} \Big(\Xi_j^*(s_{j+1}, s_j) < g(s_{j+1})\varepsilon \Big) \cap \bigcap_{j=-1}^{\infty} F_j \Big) \Big], \end{aligned} \tag{2.8}$$

where

$$\Xi_j(t) \stackrel{\text{def}}{=} W(t) - W(s_{j+1}) - \frac{t - s_{j+1}}{s_j - s_{j+1}} \Big(W(s_j) - W(s_{j+1}) \Big).$$

Since $\{\Xi_j(s_{j+1}+(s_j-s_{j+1})t)/\sqrt{s_j-s_{j+1}};\ 0\le t\le 1\}_{j\ge 0}$ is a sequence of independent Brownian bridges, and independent of $(F_j)_{j\ge 0}$, the term on the right hand side of (2.8) equals

$$\Big[\prod_{j=0}^{\infty}\mathbb{P}\Big(\Xi^*(1)<\frac{g(s_{j+1})\varepsilon}{\sqrt{s_j-s_{j+1}}}\Big)\Big]\,\mathbb{P}\Big(\bigcap_{j=-1}^{\infty}F_j\Big).$$

Going back to (2.8), and applying Khatri's inequality (cf. Fact 2.2), we obtain:

$$\Delta_1\ge\Big[\prod_{j=0}^{\infty}\mathbb{P}\Big(\Xi^*(1)<\frac{g(s_{j+1})\varepsilon}{\sqrt{s_j-s_{j+1}}}\Big)\Big]\prod_{j=-1}^{\infty}\mathbb{P}(F_j).$$

Since $F_j\supset\{W^*(s_{j+1})<g(s_{j+1})\varepsilon\}$ (for $j\ge 0$) and $\mathbb{P}(F_{-1})\ge\mu\varepsilon/2$ (for $\varepsilon<\varepsilon_0(\mu)$, where $\varepsilon_0(\mu)$ is a small constant depending only on μ), it follows from the Brownian scaling property that

$$\prod_{j=-1}^{\infty}\mathbb{P}(F_j)\ge\frac{\mu\varepsilon}{2}\prod_{j=0}^{\infty}\mathbb{P}\Big(W^*(1)<\frac{g(s_{j+1})\varepsilon}{\sqrt{s_{j+1}}}\Big).$$

By (2.5)–(2.6), there exists a finite universal constant $\kappa>0$ such that

$$\mathbb{P}\Big(W^*(1)<x\Big)\ge\exp\big(-\frac{\kappa}{x^2}\big),\qquad\text{and}\qquad\mathbb{P}\Big(\Xi^*(1)<x\Big)\ge\exp\big(-\frac{\kappa}{x^2}\big),$$

for all $x>0$. Hence, for $0<\varepsilon<\varepsilon_0(\mu)$,

$$\begin{aligned}\Delta_1&\ge\frac{\mu\varepsilon}{2}\exp\Big(-\frac{\kappa}{\varepsilon^2}\sum_{j=0}^{\infty}\frac{1}{2^{j+1}g^2(2^{-j-1})}-\frac{\kappa}{\varepsilon^2}\sum_{j=0}^{\infty}\frac{1}{2^{j+1}g^2(2^{-j-1})}\Big)\\&\ge\frac{\mu\varepsilon}{2}\exp\Big(-\frac{4\kappa}{\varepsilon^2}\int_0^{1/2}\frac{dt}{g^2(t)}\Big)\\&\ge\frac{\mu\varepsilon}{2}\exp\Big(-\frac{4\kappa}{\varepsilon^2}\int_0^{1}\frac{dt}{g^2(t)}\Big),\end{aligned}$$

which, by the definition of g, means

$$\mathbb{P}\Big(\sup_{0\le t\le a}\frac{|W(t)|}{f(t)}<\varepsilon;\ |W(a)|<\sqrt{a}\,\mu\varepsilon\Big)\ge\frac{\mu\varepsilon}{2}\exp\Big(-36\kappa\int_0^a\frac{dt}{f^2(t)}\Big),\tag{2.9}$$

for all $0<a<\delta$ and all $0<\varepsilon<\varepsilon_0(\mu)$.

Now write $\lambda\overset{\text{def}}{=}\inf_{a\le t\le 1}f(t)>0$ and fix $0<\nu<1$. It is noted that

$$\begin{aligned}\mathbb{P}\Big(\|W\|_f<\varepsilon\Big)&=\mathbb{P}\Big(\sup_{0\le t\le a}\frac{|W(t)|}{f(t)}<\varepsilon;\ \sup_{a\le t\le 1}\frac{|W(t)|}{f(t)}<\varepsilon\Big)\\&\ge\mathbb{P}\Big(\sup_{0\le t\le a}\frac{|W(t)|}{f(t)}<\varepsilon;\ |W(a)|<\lambda\nu\varepsilon;\\&\qquad\qquad\sup_{a\le t\le 1}\frac{|W(t)-W(a)|}{f(t)}<(1-\nu)\varepsilon\Big).\end{aligned}$$

Since the Wiener process has independent increments, by scaling, the probability on the right hand side is equal to

$$\mathbb{P}\Big(\sup_{0\le t\le a}\frac{|W(t)|}{f(t)}<\varepsilon;\ |W(a)|<\lambda\nu\varepsilon\Big)\,\mathbb{P}\Big(\sup_{0\le t\le 1}\frac{|W(t)|}{f((1-a)t+a)}<\frac{(1-\nu)\varepsilon}{\sqrt{1-a}}\Big).$$

Applying, respectively, (2.9) to $\mu\stackrel{\text{def}}{=}\lambda\nu/\sqrt{a}$, and Lemma 2.1 to $t\mapsto f((1-a)t+a)$, we have

$$\liminf_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}\Big(\|W\|_f<\varepsilon\Big)\ge-\frac{36\kappa}{\varepsilon^2}\int_0^a\frac{dt}{f^2(t)}-\frac{\pi^2}{8(1-\nu)^2}\int_a^1\frac{dt}{f^2(t)}.$$

Letting a and ν tend to 0 gives the desired lower bound in Theorem 1.1. $\square$

The rest of the section is devoted to the proof of Lemma 2.1.

Proof of Lemma 2.1. Let $n\ge 2$ and $t_k\stackrel{\text{def}}{=}k/n$ (for $0\le k\le n$). Pick a small number $\nu>0$ such that $\inf_{0<t\le 1}f(t)\ge 3n\nu$ (which is possible, in light of (H1)). Then

$$\begin{aligned}\mathbb{P}\Big(\|W\|_f<\varepsilon\Big)&\ge\mathbb{P}\Big(W^*(t_{k-1},t_k)<\varepsilon f_\downarrow(t_{k-1},t_k),\\&\qquad\qquad|W(t_k)-W(t_{k-1})|<\nu\varepsilon,\ \text{for all } 1\le k\le n\Big)\\&\ge\mathbb{P}\Big(\Theta_k^*(t_{k-1},t_k)<(f_\downarrow(t_{k-1},t_k)-2n\nu)\varepsilon,\\&\qquad\qquad|W(t_k)-W(t_{k-1})|<\nu\varepsilon,\ \text{for all } 1\le k\le n\Big),\end{aligned}$$

where

$$\Theta_k(t)\stackrel{\text{def}}{=}W(t)-W(t_{k-1})-\frac{t-t_{k-1}}{t_k-t_{k-1}}\Big(W(t_k)-W(t_{k-1})\Big).$$

Observe that $\{\Theta^*(t_{k-1},t_k)\}_{1\le k\le n}$ and $\{W(t_k)-W(t_{k-1})\}_{1\le k\le n}$ are mutually independent variables, and that for each k, $\Theta^*(t_{k-1},t_k)$ is distributed as $\sqrt{t_k-t_{k-1}}\,\Xi^*(1)$. Accordingly,

$$\begin{aligned}\mathbb{P}\Big(\|W\|_f<\varepsilon\Big)\ge\Big[\prod_{k=1}^n\mathbb{P}\Big(\Xi^*(1)<\frac{(f_\downarrow(t_{k-1},t_k)-2n\nu)\varepsilon}{(t_k-t_{k-1})^{1/2}}\Big)\Big]\\\times\prod_{k=1}^n\mathbb{P}\Big(|W(1)|<\frac{\nu\varepsilon}{(t_k-t_{k-1})^{1/2}}\Big).\end{aligned}$$

Since $\lim_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}(|W(1)|<\varepsilon)=0$, by (2.6), this implies

$$\liminf_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}\Big(\|W\|_f<\varepsilon\Big)\ge-\frac{\pi^2}{8}\sum_{k=1}^n\frac{t_k-t_{k-1}}{(f_\downarrow(t_{k-1},t_k)-2n\nu)^2}.$$

Lemma 2.1 is proved by first sending ν to 0 and then letting n tend to infinity. $\square$

3. Critical case

Theorem 1.1 confirms that (1.3) holds for $\|W\|_f$ with $b = 2$ for most of the weight functions f of W (this means that the decay of the small ball probability is of the same rate for many f's). However, if condition (1.9) fails, we should no longer expect to see (1.3) in general: instead of a power of ε, additional terms (typically: logarithms) appear. If $\int_0^1 dt/f^2(t) = \infty$, we call it the "critical" case, since this ruins the rate of decay $b = 2$ in (1.3).

Theorem 3.1. *For any positive function f satisfying* (H2), *such that* $\int_0^1 dt/f^2(t) = \infty$,

$$\limsup_{\varepsilon\to 0} \frac{\varepsilon^2}{F(\varepsilon)} \log \mathbb{P}\Big(\|W\|_f < \varepsilon \Big) \le -\frac{\pi^2}{8},$$

where

$$F(\varepsilon) \stackrel{\text{def}}{=} \int_\varepsilon^1 \frac{dt}{f^2(t)}.$$

Proof. First, we replace (2.5) by a refined estimate (cf. Chung [6]): $\mathbb{P}(W^*(1) < x) \le 2\exp(-\pi^2/8x^2)$ for $x > 0$. By (2.7), for all $\varepsilon \in (0, 1/2)$, $\delta > \varepsilon$ such that f is nondecreasing on $(0, \delta)$, and $n > 2/\delta\varepsilon$,

$$\begin{aligned}
\mathbb{P}\Big(\|W\|_f < \varepsilon \Big) &\le \prod_{k=1}^{n} \Big[2\exp\Big(-\frac{\pi^2 (t_k - t_{k-1})}{8\varepsilon^2 \,(f^{\uparrow}(t_{k-1}, t_k))^2} \Big) \Big] \\
&\le \prod_{k=1}^{[\delta n]} \Big[2\exp\Big(-\frac{\pi^2 (t_k - t_{k-1})}{8\varepsilon^2 \,(f^{\uparrow}(t_{k-1}, t_k))^2} \Big) \Big] \\
&= \exp\Big([\delta n] \log 2 - \frac{\pi^2}{8\varepsilon^2} \sum_{k=1}^{[\delta n]} \frac{t_k - t_{k-1}}{(f^{\uparrow}(t_{k-1}, t_k))^2} \Big).
\end{aligned}$$

Recall that $t_k = k/n$. Since f is nondecreasing on $(0, \delta)$,

$$\sum_{k=1}^{[\delta n]} \frac{t_k - t_{k-1}}{(f^{\uparrow}(t_{k-1}, t_k))^2} \ge \int_{1/[\delta n]}^{\delta} \frac{dt}{f^2(t)} = \int_\varepsilon^\delta \frac{dt}{f^2(t)} + \int_{1/[\delta n]}^{\varepsilon} \frac{dt}{f^2(t)}.$$

Take $n \stackrel{\text{def}}{=} [3/\delta\varepsilon]$, then

$$[\delta n] \log 2 \le \frac{3}{\varepsilon} \log 2 \le \frac{\pi^2}{8\varepsilon^2} \int_{\varepsilon/2}^{\varepsilon} \frac{dt}{f^2(t)} \le \frac{\pi^2}{8\varepsilon^2} \int_{1/[\delta n]}^{\varepsilon} \frac{dt}{f^2(t)},$$

where in the second inequality, we have used the monotonicity of f together with the fact $\lim_{t\to 0^+} f(t) = 0$. Consequently,

$$\mathbb{P}\Big(\|W\|_f < \varepsilon\Big) \le \exp\Big(-\frac{\pi^2}{8\varepsilon^2}\int_\varepsilon^\delta \frac{dt}{f^2(t)}\Big),$$

proving Theorem 3.1. □

Remark 3.2. The choice of n in the proof of Theorem 3.1 is not optimal if in addition, $f(t)$ is greater than (a multiple of) $t^{1/2}$ in the neighbourhood of 0 (which, in practice, is what happens for a "typical" weight function f). In this case, one can replace $F(\varepsilon)$ in Theorem 3.1 by $\widetilde{F}(\varepsilon) \stackrel{\text{def}}{=} \int_{\varepsilon^2}^1 dt/f^2(t)$.

Example 3.3. Consider two "typical" weight functions f of W for which $\int_0^1 dt/f^2(t) = \infty$. According to Theorem 3.1, for $f_1(t) \stackrel{\text{def}}{=} \sqrt{t\log(1/t)}$ (in the neighbourhood of 0),

$$\limsup_{\varepsilon\to 0} \frac{\varepsilon^2}{\log\log(1/\varepsilon)} \log \mathbb{P}\Big(\|W\|_{f_1} < \varepsilon\Big) \le -\frac{\pi^2}{8}, \tag{3.1}$$

whereas for $f_2(t) \stackrel{\text{def}}{=} \sqrt{t\log\log(1/t)}$ (in the neighbourhood of 0),

$$\limsup_{\varepsilon\to 0} \frac{\varepsilon^2 \log\log(1/\varepsilon)}{\log(1/\varepsilon)} \log \mathbb{P}\Big(\|W\|_{f_2} < \varepsilon\Big) < 0.$$

Remark 3.4. We do not have a lower bound in general for $\log \mathbb{P}(\|W\|_f < \varepsilon)$ when $\int_0^1 dt/f^2(t) = \infty$. Let us consider the particular example $f_1(t) \stackrel{\text{def}}{=} \sqrt{t\log(1/t)}$ treated above. Let $\varepsilon \in (0, 1/2)$ and $\delta = \delta(\varepsilon) \stackrel{\text{def}}{=} \exp(-1/\varepsilon^4)$. By scaling,

$$\mathbb{P}\Big(\sup_{0\le t\le\delta} \frac{|W(t)|}{\sqrt{t\log(1/t)}} < \varepsilon\Big) = \mathbb{P}\Big(\sup_{0\le s\le 1/2} \frac{|W(s)|}{\sqrt{s\log(1/2s\delta)}} < \varepsilon\Big). \tag{3.2}$$

Since for all $\varepsilon \in (0, 1/2)$, $\log(1/2s\delta) \ge 2\sqrt{\log(1/s)}\,\sqrt{\log(1/2\delta)} \ge \sqrt{\log(1/s)}\,/\varepsilon^2$, and since test (1.8) confirms that $\sup_{0\le s\le 1/2} |W(s)|/s^{1/2}(\log(1/s))^{1/4}$ is a well-defined variable, it follows that the probability term on the left hand side of (3.2) is bounded below by a positive absolute constant, uniformly for $\varepsilon \in (0, 1/2)$. This playing the role of (2.9), and applying the same argument as in the proof of Theorem 1.1, we arrive at the following estimate:

$$\liminf_{\varepsilon\to 0} \frac{\varepsilon^2}{\log(1/\varepsilon)} \log \mathbb{P}\Big(\|W\|_{f_1} < \varepsilon\Big) > -\infty. \tag{3.3}$$

There is a clear gap between (3.1) and (3.3).

4. Some applications

4.1. Chung's functional law

The estimate in Theorem 1.1 allows to establish Chung-type functional iterated logarithm laws for the increments of the Wiener process W. Let $\mathbb{S}$ be Strassen's set defined by

$$\mathbb{S} \stackrel{\text{def}}{=} \Big\{g : g \text{ absolutely continuous, with Lebesgue derivative } \dot{g}, \\ \text{such that } g(0)=0 \text{ and } J(g) \stackrel{\text{def}}{=} \int_0^1 \dot{g}^2(t)\,dt \le 1\Big\}.$$

In the language of probability functional analysis, $\mathbb{S}$ is the unit ball of the reproducing kernel Hilbert space pertaining to the Wiener measure on $C[0,1]$, the space of all continuous function on $[0,1]$ endowed with the uniform topology.

Our first application gives the rate of convergence in Strassen's functional law of the iterated logarithm for W.

Theorem 4.1. *If f is a positive function satisfying either* (H1) *or* (H2), *such that* (1.9) *holds, then for all $g \in \mathbb{S}$ with $J(g) < 1$,*

$$\liminf_{T\to\infty} (\log\log T) \Big\| \frac{W(T\,\cdot)}{\sqrt{2T\log\log T}} - g \Big\|_f = \frac{\pi}{4\sqrt{1-J(g)}}\, c_f \qquad \text{a.s.,}$$

where

$$c_f \stackrel{\text{def}}{=} \Big(\int_0^1 \frac{dt}{f^2(t)} \Big)^{1/2}.$$

Remark 4.2. In case $f \stackrel{\text{def}}{=} 1$ and $g \stackrel{\text{def}}{=} 0$, this is Chung's classical law of the iterated logarithm. For $f \stackrel{\text{def}}{=} 1$ and general g, Theorem 4.1 is first discovered by Csáki [7], and ultimately extended by many other mathematicians, cf. for example de Acosta [1], Baldi and Roynette [3], Kuelbs and Li [15] among others.

Proof of Theorem 4.1. From Theorem 1.1 and "standard" arguments,

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\Big(\|W - \frac{g}{\varepsilon}\|_f \le r\varepsilon \Big) = -\frac{\pi^2 (c_f)^2}{8r^2} - \frac{J(g)}{2}, \tag{4.1}$$

for all $r > 0$ and $g \in \mathbb{S}$ (for details, cf. for example de Acosta [1]). On the other hand, thanks to the conditions in Theorem 4.1, there exists a finite constant $C > 0$, depending only on f, such that for all $0 < \delta < 1$,

$$\|g(1 \wedge ((1+\delta)\cdot)) - g\|_f \leq C\sqrt{\delta}. \tag{4.2}$$

The estimates (4.1)–(4.2) together imply Theorem 4.1 as is pointed out by Deheuvels and Mason [11]. □

More generally, from Theorem 1.1, we can get local and global Chung's functional iterated logarithm laws for the increments of W. For more details, cf. Berthet [4].

4.2. Higher dimensions

Let $\{W_d(t);\ 0 \leq t \leq 1\}$ denote standard d–dimensional Brownian motion ($d \geq 1$), and "$\|\cdot\|$" the usual Euclidean modulus in $\mathbb{R}^d$. Here is the analogue of Theorem 1.1 in dimension d.

Theorem 4.3. *If f is a positive function satisfying either* (H1) *or* (H2), *then*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\Big(\sup_{0<t\leq 1} \frac{\|W_d(t)\|}{f(t)} < \varepsilon\Big) = -\frac{j^2_{(d-2)/2}}{2}\int_0^1 \frac{dt}{f^2(t)},$$

where $j_{(d-2)/2}$ is the smallest positive root of the Bessel function $J_{(d-2)/2}$.

Since $j_{-1/2} = \pi/2$, Theorem 4.3 is in agreement with Theorem 1.1 when $d = 1$.

4.3. Brownian bridge

In the theory of empirical processes, the Brownian bridge plays an important role. Let $\{\Xi_d(t);\ 0 \leq t \leq 1\}$ denote a standard $\mathbb{R}^d$-valued Brownian bridge, with $d \geq 1$.

Theorem 4.4. *Assume that $t \mapsto f(t)$ and $t \mapsto f(1-t)$ are both nondecreasing in a neighbourhood of* 0. *Assume that $\inf_{a\leq t\leq b} f(t) > 0$ for all $0 < a \leq b < 1$. Then*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\Big(\sup_{0<t<1} \frac{\|\Xi_d(t)\|}{f(t)} < \varepsilon\Big) = -\frac{j^2_{(d-2)/2}}{2}\int_0^1 \frac{dt}{f^2(t)},$$

where $j_{(d-2)/2}$ is as before the smallest positive root of $J_{(d-2)/2}$.

To see how Theorem 4.4 can be applied to weighted empirical processes, we refer to Csáki [8].

4.4. General moving boundaries

Let h_1 and h_2 be Borel functions on $[0,1]$. Following Mogulskii [22], we consider the probability of the event

$$E_\varepsilon \stackrel{\mathrm{def}}{=} \Big\{ \varepsilon\, h_1(t) \le W(t) \le \varepsilon\, h_2(t), \qquad \text{for all } t \in [0,1] \Big\},$$

when ε goes to 0.

Theorem 4.5. *Assume that $h_2 - h_1$ is a positive function satisfying either* (H1) *or* (H2), *such that $h_1 + h_2$ is absolutely continuous whose Lebesgue derivative is square integrable over* $[0,1]$. *Then*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \, \log \mathbb{P}(E_\varepsilon) = -\frac{\pi^2}{2} \int_0^1 \frac{dt}{(h_2(t) - h_1(t))^2}. \tag{4.3}$$

When $h_2 - h_1$ satisfies (H1) such that h_1 and h_2 are piece-wise continuous, Theorem 4.5 is due to Mogulskii [22].

To prove (4.3), observe that $E_\varepsilon = \{ \|W - \varepsilon\, g\|_f < \varepsilon \}$, with $g \stackrel{\mathrm{def}}{=} (h_1 + h_2)/2$ and $f \stackrel{\mathrm{def}}{=} (h_2 - h_1)/2$. The upper bound in (4.3) clearly follows from the same argument as in Section 2. Its lower bound is a consequence of the Cameron–Martin formula (this is where the condition upon $h_1 + h_2$ comes in) and of our Theorem 1.1.

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