# Evaluating the small deviation probabilities for subordinated Lévy processes

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***Summary.*** We study the small deviation problem for a class of symmetric Lévy processes, namely, subordinated Lévy processes. These processes can be represented as $W \circ A$, where $W$ is a standard Brownian motion, and $A$ is a subordinator independent of $W$. Under some mild general assumption, we give precise estimates (up to a constant multiple in the logarithmic scale) of the small deviation probabilities. These probabilities, also evaluated under the conditional probability given the subordination process $A$, are formulated in terms of the Laplace exponent of $A$. The results are furthermore extended to processes subordinated to the fractional Brownian motion of arbitrary Hurst index.

***Keywords.*** Lévy process, subordination, small deviation, fractional Brownian motion.

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## 1 Introduction

Let $Z := (Z(t),\ t \in T)$ be a mean-zero stochastic process with $\mathbb{P}\{\sup_{t\in T} |Z(t)| < \infty\} > 0$. The so-called small deviation or small ball problem for $Z$ (in the logarithmic level) consists in evaluating the asymptotic behaviour of

$$\log \mathbb{P}\left\{\sup_{t\in T} |Z(t)| < \varepsilon\right\}, \qquad \varepsilon \to 0.$$

When $Z$ is a general Gaussian process, it is known (Kuelbs and Li [4] and Li and Linde [7]) that the small deviation problem for $Z$ is equivalent to the problem of estimating the entropy numbers of an associated (linear, bounded) operator, which, in general, is a challenging problem as well. However, for a large class of Gaussian processes it is possible to obtain quite precise estimates for their small deviation probabilities. We refer to Li and Shao [9] and Lifshits [10] for overviews.

Despite some recent progresses (Samorodnitsky [13], Simon [16]–[17], Ishikawa [3], Li and Linde [8], Lifshits and Simon [11]), far less is known when $Z$ is non Gaussian. In this paper, we are interested in the small deviation problem for a class of symmetric Lévy processes. While earlier paper mainly investigated the question whether or not for a given Lévy process $X$ the probabilities $\mathbb{P}\{\sup_{t\in T}|X(t)| < \varepsilon\}$ are non-zero for all $\varepsilon > 0$ (Simon [16] and Ishikawa [3]), our aim is to evaluate the exact small deviation probabilities for a certain class of Lévy processes.

Let $X := (X(t),\ t \in [0,1])$ be a symmetric Lévy process (i.e., a symmetric process with independent and stationary increments). We assume that

$$X(1) \overset{\text{law}}{=} \mathcal{N}(0, \sigma^2), \tag{1.1}$$

where $\sigma^2$ is a positive infinitely divisible random variable. That is, $X(1)$ is distributed as $G\sigma$, where $G$ is a standard Gaussian $\mathcal{N}(0,1)$ random variable, independent of the infinitely divisible random variable $\sigma^2$. The distribution of $X(1)$ is referred to in the literature as a type G distribution (see Rosinski [12]).

From the Lévy property of $X$ and the representation (1.1), it is clear that, after a possible enlargement of the probability space, the process $X$ can be realized as

$$X(t) = W(A(t)), \qquad t \in [0,1], \tag{1.2}$$

where $W$ is a standard Brownian motion, and $A$ is a subordinator (i.e., a non-decreasing Lévy process) with $A(1)$ distributed as $\sigma^2$ in (1.1), such that the processes $W$ and $A$ are independent. In order to distinguish randomnesses contributed by $W$ and $A$, it is convenient to regard (1.2) as $X(t,\omega,\omega') = W(A(t,\omega),\omega')$, for $(\omega,\omega') \in \Omega \times \Omega'$.

We refer to Bertoin ([2], Section 8.4) and Sato ([15], Chapter 6) for an account of general properties of subordinated Lévy processes. An important and well-known example is the Bochner subordination for the symmetric stable Lévy motion of index $\alpha \in (0,2)$. In this case, $A$ is a stable subordinator of index $\alpha/2$.

Given a stochastic process $X$ indexed by $[0,1]$, we define its $L^q$-norms as follows:

$$(1.3)\qquad \|X\|_q := \left(\int_0^1 |X(t)|^q\,\mathrm{d}t\right)^{1/q}, \quad 1 \le q < \infty, \quad \text{and} \quad \|X\|_\infty := \sup_{t\in[0,1]} |X(t)|.$$

In this paper, we provide accurate estimates for the small ball probabilities of Lévy processes $X$ as in (1.2) under the $L^q$-norm ($1 \le q \le \infty$), with respect to the (product) probability $\mathbb{P}$ (the *annealed* setting) as well as for the conditional probabilities $P_\omega(\cdot) := \mathbb{P}(\cdot \mid A)$ (the *quenched* setting) given the subordination process $A$. Indeed, under some mild general assumption, we will prove that for any $q \in [1,\infty]$, when $\varepsilon \to 0$,

$$(1.4)\qquad \log P_\omega\{\,\|X\|_q < \varepsilon\,\} \;\asymp\; -\Phi(\varepsilon^{-2}), \qquad \text{a.s.,}$$

$$(1.5)\qquad \log \mathbb{P}\{\,\|X\|_q < \varepsilon\,\} \;\asymp\; -\,\Phi(\varepsilon^{-2}),$$

where $f(\varepsilon) \asymp g(\varepsilon)$, means $0 < \liminf_{\varepsilon\to 0} \frac{f(\varepsilon)}{g(\varepsilon)} \le \limsup_{\varepsilon\to 0} \frac{f(\varepsilon)}{g(\varepsilon)} < \infty$, and $\Phi$ is the Laplace exponent of the subordinator $A$, defined by

$$(1.6)\qquad \mathbb{E}\left[\,\mathrm{e}^{-xA(t)}\,\right] = \mathrm{e}^{-t\Phi(x)}, \qquad t \in [0,1],\;\; x \ge 0.$$

It should be mentioned that the constants appearing in the asymptotic of (1.4) may, of course, depend on $\omega$.

In the case that $X$ is a symmetric stable Lévy process of index $\alpha \in (0,2)$ (thus $A$ is a stable subordinator of index $\alpha/2$), $\Phi$ is a constant multiple of $x^{\alpha/2}$, and (1.5) with $q = \infty$ was known to Taylor [19] while, to our knowledge, even in this case (1.4) seems to be new.

The announced two-sided estimates (1.4) and (1.5) describe the small deviation probabilities of $X = W(A)$. Yet it turns out that our methods do not appeal to the Markov property of the Brownian motion $W$ in an essential way. More heavily they depend on the Gaussian property of $W$. Therefore in Section 2 we are going to extend (1.4) and (1.5) for processes $X = W_H(A)$, where $W_H$ is a fractional Brownian motion of arbitrary Hurst index $H \in (0,1)$.

The rest of the paper is as follows. In Section 2 we state the main result (Theorem 2.1), which yields (1.4) and (1.5) as special cases. Section 3 is devoted to some preliminary results about the range of a subordinator. Theorem 2.1 is proved in two distinct parts: the quenched part is treated in Section 4 while the annealed part is proved in Section 5. Finally, we present further remarks and questions in Section 6.

## 2 Main result

Let $W_H := (W_H(t),\ t \ge 0)$ be a fractional Brownian motion of Hurst index $H \in (0,1)$. That is, $W_H$ is a mean-zero Gaussian process with covariance

$$\mathbb{E}\,[\,W_H(s)W_H(t)\,] = \frac{1}{2}\,\big(t^{2H} + s^{2H} - |t-s|^{2H}\big)\,, \qquad s \ge 0,\ t \ge 0.$$

Let $A := (A(t),\ t \in [0,1])$ be as before a subordinator, independent of $W_H$. We are interested in the subordinated process

$$Y_H(t) := W_H(A(t)), \qquad t \in [0,1]. \tag{2.1}$$

Clearly, when $H = 1/2$, $W_H = W_{1/2}$ is a standard Brownian motion, so that $Y_{1/2}$ is the subordinated Lévy process introduced in (1.2).

Let $\Phi$ be the Laplace exponent of $A$, defined by (1.6). Throughout the paper, we assume that

$$\liminf_{x\to\infty} \frac{\Phi(x)}{\log x} > 0. \tag{2.2}$$

Condition (2.2) is to ensure that the subordinator $A$ does not grow too slowly (loosely speaking, it should grow at least like a Gamma process, which increases at a logarithmic rate). Note that in the case that $A$ is a stable subordinator of index $\beta \in (0,1)$, $\Phi(x)$ is a constant multiple of $x^\beta$, which obviously satisfies (2.2).

Here is the main result of the paper.

**Theorem 2.1** *Let $Y_H$ be the process defined in* (2.1)*, and assume* (2.2)*. Let $q \in [1,\infty]$.*
(a) *The quenched case: for $\varepsilon \to 0$,*

$$\log P_\omega\,\{\,\|Y_H\|_q < \varepsilon\,\} \asymp -\Phi(\varepsilon^{-1/H}), \qquad \text{a.s.} \tag{2.3}$$

(b) *The annealed case: we have,*

$$\log \mathbb{P}\,\{\,\|Y_H\|_q < \varepsilon\,\} \asymp -\,\Phi(\varepsilon^{-1/H}), \qquad \varepsilon \to 0. \tag{2.4}$$

Let us say a few words about the proof of Theorem 2.1. The main part, which is treated in Section 4, concerns the quenched case (2.3), whereas the annealed case (see Section 5) follows from the quenched case more or less painlessly. The proof of (2.3) is divided into two parts. The lower bound follows from a general result of Talagrand for the small deviation problem for Gaussian processes, once we have enough information about the range of the subordinator $A$. The upper bound, which involves fine properties of fractional Brownian motion on a fractal-like set, relies on some technical study of the subordinator and the $L^q$ behaviour of the fractional Brownian motion. This latter part gives us some interesting information about the behaviour of a fractional Brownian motion on a fractal set which will be subject of a forthcoming paper.

Throughout the paper, the letter $c$ (with a subscript) denotes some constant which is finite and (strictly) positive. It may depend on $H$, yet not on the investigated processes (besides on the constant appearing in (2.2)).

# 3 Preliminaries on the range of a subordinator

Let $A := (A(t),\ t \geq 0)$ be a subordinator, with Laplace exponent $\Phi$ defined in (1.6). Let $\mathcal{R}$ denote the closure of $\{A(t),\ t \in [0,1]\}$. In other words, $\mathcal{R}$ is the closure of the range of $A$. For any $\delta > 0$, let $N(\mathcal{R}, \delta)$ be the minimal number $N$ necessary to cover $\mathcal{R}$ with $N$ intervals of lengths less than or equal to $\delta$.

**Lemma 3.1** *For all $\delta > 0$ and all $k \geq 20\,\Phi(1/\delta)$,*

$$\mathbb{P}\left\{N(\mathcal{R},\delta) \geq k\right\} \leq \exp\left(-\frac{k}{40}\right), \tag{3.1}$$

$$\mathbb{P}\left\{N(\mathcal{R},\delta) < \frac{1}{4}\,\Phi(1/\delta)\right\} \leq \exp\left[-\frac{1}{4}\,\Phi(1/\delta)\right]. \tag{3.2}$$

*Proof.* Let $T_0 := 0$ and define by induction

$$T_i = T_i(\delta) := \inf\left\{s > T_{i-1} :\ A(s) - A(T_{i-1}) > \delta\right\}, \qquad i \geq 1.$$

Note that for any $k \geq 0$, the following holds:

$$\{N(\mathcal{R},\delta) \geq k\} = \{T_k \leq 1\}.$$

By the strong Markov property, $(\eta_i := T_i - T_{i-1})_{i \geq 1}$ is a sequence of independent and identically distributed random variables. We have,

$$\mathbb{P}\left\{N(\mathcal{R},\delta) \geq k\right\} = \mathbb{P}\left\{\sum_{i=1}^{k} \eta_i \leq 1\right\}. \tag{3.3}$$

We first prove (3.1). By the exponential Chebyshev inequality, for any $\lambda > 0$,

$$\mathbb{P}\left\{N(\mathcal{R},\delta) \geq k\right\} \leq \mathrm{e}^{\lambda}\left[\,\mathbb{E}\left(\mathrm{e}^{-\lambda T_1}\right)\right]^k.$$

Observe that by integration by parts,

$$\mathbb{E}\left(\mathrm{e}^{-\lambda T_1}\right) = 1 - \lambda \int_0^\infty \mathrm{e}^{-\lambda x}\,\mathbb{P}\{T_1 > x\}\,\mathrm{d}x.$$

In view of the monotonicity of $x \mapsto \mathbb{P}\{T_1 > x\}$, this leads to:

$$\begin{aligned}
\mathbb{E}\left(\mathrm{e}^{-\lambda T_1}\right) &\leq 1 - \lambda \int_0^{1/\lambda} \mathrm{e}^{-1}\,\mathbb{P}\{T_1 > 1/\lambda\}\,\mathrm{d}x \\
&= 1 - \mathrm{e}^{-1}\,\mathbb{P}\{T_1 > 1/\lambda\} \\
&= 1 - \mathrm{e}^{-1}\,\mathbb{P}\{A(1/\lambda) < \delta\}.
\end{aligned}$$

It is clear from Chebyshev's inequality that

$$\begin{aligned}
\mathbb{P}\{A(1/\lambda) < \delta\} &= 1 - \mathbb{P}\{A(1/\lambda) \geq \delta\} \\
&\geq 1 - \frac{1 - \mathbb{E}[\,\mathrm{e}^{-A(1/\lambda)/\delta}\,]}{1 - \mathrm{e}^{-1}} \\
&= 1 - \frac{1 - \mathrm{e}^{-(1/\lambda)\Phi(1/\delta)}}{1 - \mathrm{e}^{-1}}.
\end{aligned}$$

Choose now $\lambda := 2\Phi(1/\delta)$, so that

$$\mathbb{P}\{A(1/\lambda) < \delta\} \geq \frac{\mathrm{e}^{-1/2} - \mathrm{e}^{-1}}{1 - \mathrm{e}^{-1}}. \tag{3.4}$$

We have therefore proved that for any $k \geq 0$,

$$\begin{aligned}
\mathbb{P}\left\{N(\mathcal{R},\delta) \geq k\right\} &\leq \mathrm{e}^{2\Phi(1/\delta)}\left(1 - \frac{\mathrm{e}^{-1}(\mathrm{e}^{-1/2} - \mathrm{e}^{-1})}{1 - \mathrm{e}^{-1}}\right)^k \\
&\leq \mathrm{e}^{2\Phi(1/\delta)}\left(1 - \frac{1}{8}\right)^k \\
&\leq \exp\left[2\Phi(1/\delta) - \frac{k}{8}\right],
\end{aligned}$$

where we have used the inequalities $\frac{\mathrm{e}^{-1/2} - \mathrm{e}^{-1}}{1 - \mathrm{e}^{-1}} \geq \frac{1}{8}$ and $1 - u \leq \mathrm{e}^{-u}$ for $u \geq 0$. This readily yields (3.1).

We now turn to the proof of (3.2). We first check that $T_1$ admits finite exponential moments in the neighborhood of 0. By Chebyshev's inequality, for any $y > 0$, $\mathbb{P}\{A(y) < \delta\} \le \mathrm{e}\,\mathbb{E}[\,\mathrm{e}^{-A(y)/\delta}\,] = \mathrm{e}^{1-y\Phi(1/\delta)}$. Accordingly, for any $\lambda > 0$,

$$\begin{aligned}\mathbb{E}\left[\,\mathrm{e}^{\lambda T_1}\,\right] &= \lambda \int_0^\infty \mathrm{e}^{\lambda y}\, \mathbb{P}\left\{T_1 > y\right\}\,\mathrm{d}y \\ &= \lambda \int_0^\infty \mathrm{e}^{\lambda y}\, \mathbb{P}\left\{A(y) < \delta\right\}\,\mathrm{d}y \\ &\le \lambda \mathrm{e} \int_0^\infty \mathrm{e}^{\lambda y - y\Phi(1/\delta)}\,\mathrm{d}y.\end{aligned}$$

We choose now $\lambda := \frac{1}{2}\Phi(1/\delta)$, so that $\mathbb{E}[\,\mathrm{e}^{\lambda T_1}\,] \le \mathrm{e}$.

In view of (3.3) and Chebyshev's inequality, we arrive at the following estimate: for any $k \ge 0$,

$$\mathbb{P}\left\{N(\mathcal{R},\delta) < k\right\} = \mathbb{P}\left\{\sum_{i=1}^k \eta_i > 1\right\} \le \mathrm{e}^{-\lambda}\left\{\mathbb{E}[\,\mathrm{e}^{\lambda T_1}\,]\right\}^k \le \exp\left(-\frac{1}{2}\Phi(1/\delta) + k\right).$$

Taking $k := \frac{1}{4}\Phi(1/\delta)$ yields (3.2). □

**Corollary 3.2** *Assume* (2.2). *Almost surely for all sufficiently small* $\delta$,

$$\frac{1}{8}\Phi(1/\delta) \le N(\mathcal{R},\delta) \le 40\,\Phi(1/\delta). \tag{3.5}$$

*Proof.* By (2.2), there exists $c_1 > 0$ and $\delta_1 > 0$ such that for all $\delta < \delta_1$, $\Phi(1/\delta) \ge c_1 \log(1/\delta)$. In view of (3.1) and (3.2), this implies that for all $\delta < \delta_1$,

$$\mathbb{P}\left\{N(\mathcal{R},\delta) \ge 20\,\Phi(1/\delta)\right\} \le \delta^{c_1/2}, \qquad \mathbb{P}\left\{N(\mathcal{R},\delta) \le \frac{1}{4}\,\Phi(1/\delta)\right\} \le \delta^{c_1/4}.$$

Let $\delta = \delta_j := j^{-\gamma}$, where $\gamma > 4/c_1$. Then

$$\sum_j \mathbb{P}\left\{N(\mathcal{R},\delta_j) \ge 20\,\Phi(1/\delta_j)\right\} < \infty, \qquad \sum_j \mathbb{P}\left\{N(\mathcal{R},\delta_j) \le \frac{1}{4}\,\Phi(1/\delta_j)\right\} < \infty.$$

By the Borel–Cantelli lemma, almost surely for all large $j$, we have

$$\frac{1}{4}\,\Phi(1/\delta_j) \le N(\mathcal{R},\delta_j) \le 20\,\Phi(1/\delta_j).$$

Let $\delta \in [\delta_{j+1}, \delta_j]$. Since

$$\Phi(x) \le \Phi(2x) \le 2\Phi(x), \tag{3.6}$$

(the second inequality being a consequence of the Cauchy–Schwarz inequality), and since $\delta_j/\delta_{j+1} \to 1$ $(j \to \infty)$, we obtain:

$$N(\mathcal{R}, \delta) \le N(\mathcal{R}, \delta_{j+1}) \le 20\, \Phi(1/\delta_{j+1}) \le 20\, \Phi(2/\delta) \le 40\, \Phi(1/\delta),$$

and similarly,

$$N(\mathcal{R}, \delta) \ge N(\mathcal{R}, \delta_j) \ge \frac{1}{4}\, \Phi(1/\delta_j) \ge \frac{1}{4}\, \Phi(1/(2\delta)) \ge \frac{1}{8}\, \Phi(1/\delta),$$

as desired. □

# 4 Proof of Theorem 2.1: the quenched case

In this section, we prove the quenched part of Theorem 2.1. Since $q \mapsto \|Y_H\|_q$ is non-decreasing on $[1, \infty]$, it suffices to prove, under assumption (2.2), the existence of constants $c_2$ and $c_3$ such that almost surely for all sufficiently small $\varepsilon$,

$$\log P_\omega \left\{ \int_0^1 |W_H(A(t))|\, \mathrm{d}t < \varepsilon \right\} \le -c_2\, \Phi(\varepsilon^{-1/H}), \tag{4.1}$$

$$\log P_\omega \left\{ \sup_{t \in [0,1]} |W_H(A(t))| < \varepsilon \right\} \ge -c_3\, \Phi(\varepsilon^{-1/H}). \tag{4.2}$$

For the sake of clarity, these estimates are proved separately.

*Proof of* (4.2). We apply a general result of Talagrand [18], formulated here as in Ledoux [5], p. 257: Let $(Z(t),\ t \in T)$ be a mean-zero Gaussian process, and for $\varepsilon > 0$, let $N(T, d, \varepsilon)$ denote the entropy number under the Dudley metric $d(s,t) := (\,\mathbb{E}|Z(s) - Z(t)|^2\,)^{1/2}$, i.e., the minimal number of $d$-balls of radius less or equal $\varepsilon$ that are necessary to cover $T$. If $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $N(T, d, \varepsilon) \le \psi(\varepsilon)$ for $\varepsilon > 0$, and that for some constants $1 \le c_4 \le c_5 < \infty$,

$$c_4\, \psi(\varepsilon) \le \psi(\varepsilon/2) \le c_5\, \psi(\varepsilon), \qquad \varepsilon > 0,$$

then there exists $c_6 \in \mathbb{R}_+$ such that for all $\varepsilon > 0$,

$$\mathbb{P} \left\{ \sup_{(s,t) \in T^2} |Z(s) - Z(t)| < \varepsilon \right\} \ge \mathrm{e}^{-c_6\, \psi(\varepsilon)}.$$

Since $\sup_{t \in [0,1]} |W_H(A(t))| = \sup_{s \in \mathcal{R}} |W_H(s)|$, we can apply Talagrand's result to $T := \mathcal{R}$, $Z(t) := W_H(t)$ and $d(s,t) := |s - t|^H$, which leads to $N(T, d, \varepsilon) = N(\mathcal{R}, \varepsilon^{1/H})$, with $N(\mathcal{R}, \delta)$

defined as in Section 3. In view of the upper bound in (3.5), Talagrand's result implies that

$$P_\omega \left\{ \sup_{t\in[0,1]} |W_H(A(t))| < \varepsilon \right\} \geq \exp\left[ -c_7\, \Phi(\varepsilon^{-1/H}) \right],$$

as desired. □

The proof of (4.1) is more delicate, and needs a preliminary result. Let $\delta > 0$, and let $(T_i)_{i\geq 0}$ and $N(\mathcal{R}, \delta)$ be as in Section 3.

**Lemma 4.1** *Under assumption* (2.2), *we have, almost surely for all sufficiently small* $\delta$,

$$\#\left\{ i: \;\; 1 \leq i \leq \frac{1}{3} N(\mathcal{R}, \delta), \;\; T_{3i} - T_{3i-1} \geq \frac{1}{2\Phi(1/\delta)} \right\} \geq \frac{\Phi(1/\delta)}{480}.$$

*Proof of Lemma* 4.1. Write $M = M(\delta) := \frac{1}{24}\,\Phi(1/\delta)$, and

$$\Lambda := \sum_{i=1}^{M} \mathbf{1}_{\{T_{3i}-T_{3i-1}\geq \frac{1}{2\Phi(1/\delta)}\}}.$$

By the strong Markov property, $(T_{3i}-T_{3i-1})_{i\geq 1}$ is a sequence of independent random variables having the same distribution as $T_1$. It follows from the exponential Chebyshev inequality that for any $a > 0$,

$$\begin{aligned} \mathbb{P}\{\Lambda < a\} &\leq \mathrm{e}^a\, \mathbb{E}\left[ \mathrm{e}^{-\Lambda} \right] \\ &= \mathrm{e}^a \left[ 1 - (1-\mathrm{e}^{-1})\mathbb{P}\left\{ T_1 \geq \frac{1}{2\Phi(1/\delta)} \right\} \right]^M \\ &\leq \exp\left[ a - M(1-\mathrm{e}^{-1})\mathbb{P}\{T_1 \geq 1/[2\Phi(1/\delta)]\} \right]. \end{aligned}$$

In view of (3.4), this leads to:

$$\mathbb{P}\{\Lambda < a\} \leq \exp\left[ a - M(\mathrm{e}^{-1/2} - \mathrm{e}^{-1}) \right] \leq \exp\left( a - \frac{M}{5} \right) = \exp\left[ a - \frac{1}{120}\,\Phi(1/\delta) \right].$$

We now choose $a := \frac{1}{240}\,\Phi(1/\delta)$, so that

$$\mathbb{P}\left\{ \Lambda < \frac{1}{240}\,\Phi(1/\delta) \right\} \leq \exp\left[ -\frac{1}{240}\,\Phi(1/\delta) \right]. \tag{4.3}$$

From here, a usual Borel–Cantelli argument together with the monotonicity of $\Phi$ gives (for more details, see the proof of Corollary 3.2 in the last section) that almost surely for all small $\delta$,

$$\#\left\{ i:\ \ 1\le i\le M,\ \ T_{3i}-T_{3i-1}\ge \frac{1}{2\Phi(1/\delta)}\right\} \ge \frac{1}{240}\,\Phi(1/(2\delta)) \ge \frac{1}{480}\,\Phi(1/\delta).$$

Since $N(\mathcal{R},\delta)\ge 3M$ almost surely for all small $\delta$ (Corollary 3.2), this implies Lemma 4.1.$\square$

We are now ready to prove (4.1).

*Proof of* (4.1). Instead of working directly with the fractional Brownian motion $W_H$, it turns out to be more convenient to work with an auxiliary process, an idea which already has been used in Li and Linde [6] and in Lifshits and Simon [11]. It is well-known that the fractional Brownian motion $W_H$ admits the following (stochastic) integral representation (possibly in an enlarged space): there exists a two-sided Brownian motion $\beta := (\beta(u),\ u\in\mathbb{R})$, such that with $x_+ := \max\{x,0\}$

$$W_H(t) = \int_{-\infty}^{t} [\,(t-s)^{H-(1/2)} - (-s)_+^{H-(1/2)}\,]\,\mathrm{d}\beta(s) = B_H(t) + \Delta_H(t),$$

where

$$B_H(t) := \int_0^t (t-s)^{H-(1/2)}\,\mathrm{d}\beta(s), \tag{4.4}$$

$$\Delta_H(t) := \int_{-\infty}^0 [\,(t-s)^{H-(1/2)} - (-s)^{H-(1/2)}\,]\,\mathrm{d}\beta(s), \qquad t\in\mathbb{R}_+. \tag{4.5}$$

(By a two-sided Brownian motion, we mean that $(\beta(s),\ s\ge 0)$ and $(\beta(-s),\ s\ge 0)$ are independent Brownian motions with $\beta(0):=0$.) Moreover, by checking the covariances, it is easily seen that the two Gaussian processes $B_H$ and $\Delta_H$, both indexed by $\mathbb{R}_+$, are independent. Therefore, by Anderson's inequality ([1]), for any $\varepsilon>0$,

$$P_\omega\left\{\int_0^1 |W_H(A(t))|\,\mathrm{d}t < \varepsilon\right\} \le P_\omega\left\{\int_0^1 |B_H(A(t))|\,\mathrm{d}t < \varepsilon\right\}. \tag{4.6}$$

Of course, since $W_H$ and $A$ are independent, without loss of generality we can assume that $B_H$ is independent of $A$ as well.

We now fix $\omega$ (thus the only randomness originates from the process $B_H$), so that $\{(T_{3i-3},T_{3i}),\ i\ge 1\}$ is a sequence of disjoint intervals. Observe that

$$\int_0^1 |B_H(A(t))|\,\mathrm{d}t \ge \sum_{i=1}^{N(\mathcal{R},\delta)/3} \int_{T_{3i-3}}^{T_{3i}} |B_H(A(t))|\,\mathrm{d}t. \tag{4.7}$$

Let $n \le N(\mathcal{R}, \delta)/3$. Note that $\int_{T_{3n-3}}^{T_{3n}} |B_H(A(t))|\,\mathrm{d}t$ can be written as

$$\int_{T_{3n-3}}^{T_{3n}} \left| \int_0^{A(T_{3n-3})} (A(t)-s)^{H-(1/2)}\,\mathrm{d}\beta(s) + \int_{A(T_{3n-3})}^{A(t)} (A(t)-u)^{H-(1/2)}\,\mathrm{d}\beta(u) \right| \mathrm{d}t.$$

The $\int_0^{A(T_{3n-3})} \cdots \mathrm{d}\beta(s)$ part (for $t \in [T_{3n-3}, T_{3n}]$) is measurable with respect to the $\sigma$-algebra $\mathcal{F}_{T_{3n-3}} := \sigma\{\beta(s),\ 0 \le s \le T_{3n-3}\}$, whereas the $\int_{A(T_{3n-3})}^{A(t)} \cdots \mathrm{d}\beta(u)$ part is independent of $\mathcal{F}_{T_{3n-3}}$. Therefore, by Anderson's inequality, for any $\varepsilon > 0$,

$$\begin{aligned} & P_\omega\left\{ \sum_{i=1}^{n} \int_{T_{3i-3}}^{T_{3i}} |B_H(A(t))|\,\mathrm{d}t < \varepsilon \right\} \\ \le\ & P_\omega\left\{ \sum_{i=1}^{n-1} \int_{T_{3i-3}}^{T_{3i}} |B_H(A(t))|\,\mathrm{d}t + \int_{T_{3n-3}}^{T_{3n}} \left| \int_{A(T_{3n-3})}^{A(t)} (A(t)-u)^{H-(1/2)}\,\mathrm{d}\widetilde{\beta}(u) \right| \mathrm{d}t < \varepsilon \right\}, \end{aligned}$$

where $\widetilde{\beta}$ denotes a standard Brownian motion independent of $\beta$ (and of $A$). Iterating the procedure, we see that for any $\omega$ and any $n \le N(\mathcal{R}, \delta)/3$,

$$P_\omega\left\{ \sum_{i=1}^{n} \int_{T_{3i-3}}^{T_{3i}} |B_H(A(t))|\,\mathrm{d}t < \varepsilon \right\} \le P_\omega\left\{ \sum_{i=1}^{n} U_i < \varepsilon \right\},$$

where, for any $i$,

$$U_i := \int_{T_{3i-3}}^{T_{3i}} \left| \int_{A(T_{3i-3})}^{A(t)} (A(t)-u)^{H-(1/2)}\,\mathrm{d}\beta_i(u) \right| \mathrm{d}t,$$

and $(\beta_i)_{i\ge1}$ is a sequence of independent standard Brownian motions (which are independent of $A$). In particular, under $P_\omega$, $(U_i)_{i\ge1}$ is a sequence of independent random variables.

Plugging this into (4.7), we obtain:

$$P_\omega\left\{ \int_0^1 |B_H(A(t))|\,\mathrm{d}t < \varepsilon \right\} \le P_\omega\left\{ \sum_{i=1}^{N(\mathcal{R},\delta)/3} U_i < \varepsilon \right\}.$$

For each $i$, we note that $U_i \ge |\int_{T_{3i-3}}^{T_{3i}} \int_{A(T_{3i-3})}^{A(t)} (A(t)-u)^{H-(1/2)}\,\mathrm{d}\beta_i(u)\,\mathrm{d}t|$, and that under $P_\omega$, $\int_{T_{3i-3}}^{T_{3i}} \int_{A(T_{3i-3})}^{A(t)} (A(t)-u)^{H-(1/2)}\,\mathrm{d}\beta_i(u)\,\mathrm{d}t$ is a mean-zero Gaussian random variable with variance

$$\sigma_i^2 := \int_{T_{3i-3}}^{T_{3i}} \int_{T_{3i-3}}^{T_{3i}} \int_{A(T_{3i-3})}^{A(s\wedge t)} (A(s)-u)^{H-(1/2)} (A(t)-u)^{H-(1/2)}\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t.$$

Therefore, writing $(\xi_i)_{i\geq 1}$ for a sequence of independent Gaussian $\mathcal{N}(0,1)$ random variables (which are independent of the subordinator $A$), we arrive at: for any $\varepsilon > 0$ and any $\delta > 0$,

$$P_\omega\left\{\int_0^1 |B_H(A(t))|\,\mathrm{d}t < \varepsilon\right\} \leq P_\omega\left\{\sum_{i=1}^{N(\mathcal{R},\delta)/3} \sigma_i\,|\xi_i| < \varepsilon\right\}. \tag{4.8}$$

We now bound $\sigma_i^2$ from below. The estimate goes slightly differently depending on whether $H \geq 1/2$ or $H < 1/2$. Let us first assume $H \in [\frac{1}{2}, 1)$. In this case, we have

$$\begin{aligned}\sigma_i^2 &\geq \int_{T_{3i-3}}^{T_{3i}}\int_{T_{3i-3}}^{T_{3i}}\int_{A(T_{3i-3})}^{A(s\wedge t)} [\,A(s\wedge t) - u\,]^{2H-1}\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t\\ &= \frac{1}{2H}\int_{T_{3i-3}}^{T_{3i}}\int_{T_{3i-3}}^{T_{3i}} [\,A(s\wedge t) - A(T_{3i-3})\,]^{2H}\,\mathrm{d}s\,\mathrm{d}t,\end{aligned}$$

and since $A(s\wedge t) - A(T_{3i-3}) \geq A(T_{3i-2}) - A(T_{3i-3}) \geq \delta$ for all $s$, $t \geq T_{3i-2}$, this leads to:

$$\begin{aligned}\sigma_i^2 &\geq \frac{1}{2H}\int_{T_{3i-1}}^{T_{3i}}\int_{T_{3i-1}}^{T_{3i}} [\,A(T_{3i-2}) - A(T_{3i-3})\,]^{2H}\,\mathrm{d}s\,\mathrm{d}t\\ &\geq \frac{1}{2H}\,\delta^{2H}\,(T_{3i} - T_{3i-1})^2.\end{aligned}$$

When $H \in (0, \frac{1}{2})$, we argue that by symmetry,

$$\begin{aligned}\sigma_i^2 &= 2\int_{T_{3i-3}}^{T_{3i}}\mathrm{d}t\int_{T_{3i-3}}^{t}\mathrm{d}s\int_{A(T_{3i-3})}^{A(s)} (A(s)-u)^{H-(1/2)}(A(t)-u)^{H-(1/2)}\,\mathrm{d}u\\ &\geq 2\int_{T_{3i-3}}^{T_{3i}}\mathrm{d}t\int_{T_{3i-3}}^{t}\mathrm{d}s\int_{A(T_{3i-3})}^{A(s)} (A(t)-u)^{2H-1}\,\mathrm{d}u\\ &= \int_{T_{3i-3}}^{T_{3i}}\mathrm{d}t\int_{T_{3i-3}}^{t}\mathrm{d}s\,\frac{[\,A(t) - A(T_{3i-3})\,]^{2H} - [\,A(t) - A(s)\,]^{2H}}{H},\end{aligned}$$

and for $T_{3i-1} \leq s \leq t < T_{3i}$, we have $A(t) - A(T_{3i-3}) \geq 2\delta$ whereas $A(t) - A(s) \leq \delta$, so that

$$\sigma_i^2 \geq \int_{T_{3i-1}}^{T_{3i}}\mathrm{d}t\int_{T_{3i-1}}^{t}\mathrm{d}s\,\frac{(2\delta)^{2H} - \delta^{2H}}{H} = \frac{2^{2H}-1}{2H}\,\delta^{2H}\,(T_{3i} - T_{3i-1})^2.$$

Therefore, regardless of the value of $H$, there exists a constant $c_8 = c_8(H) \in \mathbb{R}_+^*$ (actually $c_8 := \frac{2^{(2H)\wedge 1}-1}{2H}$ does the job) such that

$$\sigma_i^2 \geq c_8\,\delta^{2H}\,(T_{3i} - T_{3i-1})^2.$$

from which it follows that

$$\sum_{i=1}^{N(\mathcal{R},\delta)/3} \sigma_i\,|\xi_i| \ge \frac{c_8^{1/2}}{2}\,\frac{\delta^H}{\Phi(1/\delta)} \sum_{i=1}^{N(\mathcal{R},\delta)/3} |\xi_i|\,\mathbf{1}_{\{T_{3i}-T_{3i-1}\ge 1/[2\Phi(1/\delta)]\}}.$$

Plugging this into (4.8) gives that for any $\varepsilon > 0$ and any $\delta > 0$,

$$\begin{aligned} & P_\omega\left\{ \int_0^1 |B_H(A(t))|\,\mathrm{d}t < \varepsilon \right\} \\ \le\ & P_\omega\left\{ \frac{c_8^{1/2}}{2}\,\frac{\delta^H}{\Phi(1/\delta)} \sum_{i=1}^{N(\mathcal{R},\delta)/3} |\xi_i|\,\mathbf{1}_{\{T_{3i}-T_{3i-1}\ge 1/[2\Phi(1/\delta)]\}} < \varepsilon \right\}. \end{aligned} \tag{4.9}$$

In light of Lemma 4.1, we have, almost surely for all small $\delta$,

$$P_\omega\left\{ \int_0^1 |B_H(A(t))|\,\mathrm{d}t < \varepsilon \right\} \le P_\omega\left\{ \frac{c_8^{1/2}}{2}\,\frac{\delta^H}{\Phi(1/\delta)} \sum_{j=1}^{\Phi(1/\delta)/480} |\xi_j| < \varepsilon \right\}.$$

Let $\delta := c_9\,\varepsilon^{1/H}$, where $c_9$ is a constant such that $\frac{c_8^{1/2}}{2}\,c_9^H\,\frac{1}{480}\,\mathbb{E}(|\xi_1|) > 1$. By Chernoff's large deviation theorem,

$$P_\omega\left\{ \frac{c_8^{1/2}}{2}\,\frac{\delta^H}{\Phi(1/\delta)} \sum_{j=1}^{\Phi(1/\delta)/480} |\xi_j| < \varepsilon \right\} \le \exp\left[\,-c_{10}\,\Phi(1/\delta)\,\right], \tag{4.10}$$

where $c_{10} > 0$ is a constant. In view of (3.6), we have $\Phi(1/\delta) \ge c_{11}\,\Phi(\varepsilon^{-1/H})$, so that for all sufficiently small $\varepsilon > 0$,

$$P_\omega\left\{ \int_0^1 |B_H(A(t))|\,\mathrm{d}t < \varepsilon \right\} \le \exp\left[\,-c_{10}c_{11}\,\Phi(\varepsilon^{-1/H})\,\right].$$

This, together with (4.6), completes the proof of (4.1), with $c_2 := c_{10}c_{11}$. □

# 5 Proof of Theorem 2.1: the annealed case

Again, by the monotonicity of $q \mapsto \|Y_H\|_q$, it suffices to prove that

$$\limsup_{\varepsilon\to 0} \frac{1}{\Phi(\varepsilon^{-1/H})} \log \mathbb{P}\left\{ \int_0^1 |W_H(A(t))|\,\mathrm{d}t < \varepsilon \right\} < 0 \quad \text{and} \tag{5.1}$$

$$\liminf_{\varepsilon\to 0} \frac{1}{\Phi(\varepsilon^{-1/H})} \log \mathbb{P}\left\{ \sup_{t\in[0,1]} |W_H(A(t))| < \varepsilon \right\} > -\infty\,. \tag{5.2}$$

We prove these estimates separately.

*Proof of* (5.1). It goes like the proof of (4.1) in Section 4, with some refinement. Let $B_H$ be the Gaussian process defined in (4.4). Let $\delta > 0$, and let $(T_i)_{i\geq 0}$ and $N(\mathcal{R}, \delta)$ be as before. Taking expectations on both sides of (4.9), we obtain the following: for any $\varepsilon > 0$ and any $\delta > 0$,

$$\begin{aligned} & \mathbb{P}\left\{ \int_0^1 |B_H(A(t))| \, \mathrm{d}t < \varepsilon \right\} \\ \leq \; & \mathbb{P}\left\{ \frac{c_8^{1/2}}{2} \frac{\delta^H}{\Phi(1/\delta)} \sum_{i=1}^{N(\mathcal{R},\delta)/3} |\xi_i| \, \mathbf{1}_{\{T_{3i}-T_{3i-1}\geq 1/[2\Phi(1/\delta)]\}} < \varepsilon \right\} \\ \leq \; & I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &:= \mathbb{P}\left\{ N(\mathcal{R}, \delta) < \frac{1}{4}\Phi(1/\delta) \right\}, \\ I_2 &:= \mathbb{P}\left\{ \sum_{i=1}^{\Phi(1/\delta)/12} \mathbf{1}_{\{T_{3i}-T_{3i-1}\geq 1/[2\Phi(1/\delta)]\}} < \frac{1}{240}\,\Phi(1/\delta) \right\}, \\ I_3 &:= \mathbb{P}\left\{ \frac{c_8^{1/2}}{2} \frac{\delta^H}{\Phi(1/\delta)} \sum_{j=1}^{\Phi(1/\delta)/240} |\xi_j| < \varepsilon \right\}. \end{aligned}$$

By (3.2), $I_1 \leq \exp[\,-\frac{1}{4}\Phi(1/\delta)\,]$, whereas (4.3) tells us that $I_2 \leq \exp[\,-\frac{1}{240}\,\Phi(1/\delta)\,]$. Moreover, if we choose $\delta := c_9\, \varepsilon^{1/H}$ as in the previous section, then according to (4.10), we have $I_3 \leq \exp[\,-c_{10}\,\Phi(1/\delta)\,]$ (note that the sequence of random variables $(\xi_i)$ has the same distribution under $P_\omega$ and under $\mathbb{P}$).

Assembling all these pieces, and in light of (3.6), yields

$$\limsup_{\varepsilon\to 0} \frac{1}{\Phi(\varepsilon^{-1/H})} \log \mathbb{P}\left\{ \int_0^1 |B_H(A(t))| \, \mathrm{d}t < \varepsilon \right\} < 0.$$

In view of (4.6), we have $\mathbb{P}\{\,\int_0^1 |W_H(A(t))| \, \mathrm{d}t < \varepsilon\} \leq \mathbb{P}\{\,\int_0^1 |B_H(A(t))| \, \mathrm{d}t < \varepsilon\}$. This implies (5.1). □

*Proof of* (5.2). According to (4.2),

$$\liminf_{\varepsilon\to 0} \exp\left[\, c_3\, \Phi(\varepsilon^{-1/H})\,\right]\; P_\omega\left\{ \sup_{t\in[0,1]} |W_H(A(t))| < \varepsilon \right\} \geq 1, \qquad \text{a.s.}$$

Therefore, by Fatou's lemma,

$$\liminf_{\varepsilon\to 0} \exp\left[\, c_3\, \Phi(\varepsilon^{-1/H})\,\right]\, \mathbb{P}\left\{ \sup_{t\in[0,1]} |W_H(A(t))| < \varepsilon \right\} \geq 1,$$

which, in turn, yields (5.2). □

# 6 Further remarks and questions

In this final section, we present a few related questions.

1. **Existence of limit**

   Theorem 2.1 describes the correct order of the small ball probabilities (in the logarithmic scale), in both quenched and annealed settings. It is interesting to know whether $\lim_{\varepsilon\to 0} \frac{1}{\Phi(\varepsilon^{-1/H})} \log P_\omega\{\, \|Y_H\|_q < \varepsilon \,\}$ and $\lim_{\varepsilon\to 0} \frac{1}{\Phi(\varepsilon^{-1/H})} \log \mathbb{P}\{\, \|Y_H\|_q < \varepsilon \,\}$ exist (in the sense of almost convergence for the first expression). Due to the presence of the subordination, the commonly used sub-additivity argument to prove the existence of such a limit does not seem to be applicable here.

2. **Relating the quenched and the annealed probabilities**

   It is natural to ask whether the quenched and the annealed small deviation probabilities in Theorem 2.1 are closely related. For example, Jensen's inequality (together with Fatou's lemma) yields the following bound: $\mathbb{E}[\liminf_{\varepsilon\to 0} \frac{1}{\Phi(\varepsilon^{-1/H})} \log P_\omega\{\|Y_H\|_q < \varepsilon\}\,] \leq \liminf_{\varepsilon\to 0} \frac{1}{\Phi(\varepsilon^{-1/H})} \log \mathbb{P}\{\|Y_H\|_q < \varepsilon\}$. Does this inequality hold as an equality?

3. **Subordination of Riemann–Liouville processes**

   Let $B_H$ be the Riemann–Liouville process introduced in (4.4). Note that it is a well-defined Gaussian process for all $H > 0$. For example, if $H = 3/2$, then $B_H$ is simply the integrated Brownian motion. Define now the subordinated process $Z_H$ by

   $$Z_H(t) := B_H(A(t))\,, \quad t \in [0,1], \tag{6.1}$$

   where as before $A$ is a subordinator independent of $B_H$.

   A careful inspection of the preceding proofs shows that we have proved the following.

   **Proposition 6.1** *Under assumption* (2.2) *for any* $H > 0$, *we have,*

   $$\limsup_{\varepsilon\to 0} \frac{1}{\Phi(\varepsilon^{-1/H})} \log P_\omega\left\{ \int_0^1 |Z_H(t)|\, \mathrm{d}t < \varepsilon \right\} < 0\,, \qquad \text{a.s.,}$$

*as well as*

$$\limsup_{\varepsilon\to 0} \frac{1}{\Phi(\varepsilon^{-1/H})} \log \mathbb{P}\left\{ \int_0^1 |Z_H(t)|\,\mathrm{d}t < \varepsilon \right\} < 0.$$

*Furthermore, if* $0 < H < 1$*, then for all* $q \in [1, \infty]$*,*

$$\log P_\omega \{ \|Z_H\|_q < \varepsilon \} \asymp -\Phi(\varepsilon^{-1/H}), \qquad \text{a.s.,} \tag{6.2}$$

*as well as*

$$\log \mathbb{P} \{ \|Z_H\|_q < \varepsilon \} \asymp - \Phi(\varepsilon^{-1/H}). \tag{6.3}$$

In the moment we do not know whether or not (6.2) and (6.3) also hold for $H \geq 1$: Talagrand's result as used in the proof of (4.1) does no longer apply to $B_H$ for such $H$.

4. **General symmetric stable processes**

   Let $X$ be a symmetric stable process over $[0, 1]$ in the sense of Samorodnitsky and Taqqu [14]. Then $X$ may be represented as

   $$X(t, \omega, \omega'), \quad t \geq 0, \quad (\omega, \omega') \in \Omega \times \Omega',$$

   such that $X(t, \cdot, \omega')$ is centered Gaussian for $\omega' \in \Omega'$. In Li and Linde [8], this representation was used to derive some small deviation results for $X$ in terms of the conditional Gaussian processes. It is very likely that our methods lead also to some results for general stable (not necessarily Lévy) processes.

5. **General Lévy processes**

   We have studied in this paper the small deviation problem for subordinated Lévy processes. Our method, which involves Gaussian-flavoured inequalities (Anderson's inequality, Talagrand's estimate via metric entropy), can not be extended to attack the problem for general Lévy processes. The latter is believed to be a challenging problem.

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