Corrector Theory for Elliptic Equations with Long-range Correlated Random Potential

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Abstract

We consider an elliptic pseudo-differential equation with a linear potential, whose coefficient contains a highly oscillating part modeled by a stationary ergodic random field, and the random field is constructed as some function of a centered Gaussian process with non-integrable de-correlation rate. We show first that homogenization is simply averaging. We then characterize the corrector: For its mean-zero part, we characterize the magnitude of the fluctuation; further, divided by this magnitude, this mean-zero corrector converges to a Gaussian random process in probability distribution and weakly in the functional space. For the deterministic part of the corrector, we determine its size. As our paper shows, depending on the de-correlation rate of the random field, and the singularity of the Green's function, either the deterministic or the random part of the corrector can dominate.

1 Introduction

We consider elliptic pseudo-differential equations with random potential of the form

$$P(x,D)u_\varepsilon + \tilde{q}_\varepsilon\left(x, \frac{x}{\varepsilon}, \omega\right) u_\varepsilon = f(x), \tag{1.1}$$

for x in an open subset $X \subset \mathbb{R}^d$ with appropriate boundary conditions on ∂X if necessary. Here, $\tilde{q}_\varepsilon\left(x, \frac{x}{\varepsilon}, \omega\right)$ consists of a low frequency part $q_0(x)$ and a high frequency part $q\left(\frac{x}{\varepsilon}, \omega\right)$, which is a re-scaled version of $q(x,\omega)$, a stationary mean zero random field defined on some abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with (possibly multi-dimensional) parameter $x \in \mathbb{R}^d$. The equations are parametrized by $\omega \in \Omega$ denoting a realization, and by $0 < \varepsilon \ll 1$ modeling the correlation length of the random medium. We denote by $\mathbb{E}$ the mathematical expectation with respect to the probability measure $\mathbb{P}$. Equations with coefficients varying at a much smaller scale than the scale at which the phenomenon is observed have many practical applications in the physical modeling of complex media. Two prototypical examples include $P(x,D) = -\nabla \cdot a(x) \cdot \nabla$, in which case (1.1) models steady heat conduction in some heterogeneous absorbing medium, and $P(x,D) = \sqrt{-\Delta + \lambda^2}$, where the equation can be viewed as the boundary equation deduced from a diffusion equation on some domain in $\mathbb{R}^{d+1}$ with Robin boundary conditions on X, $\tilde{q}_\varepsilon$ here being modeling the random impedance; see [4]. In this paper, we adopt the abstract form (1.1) without specifying particular applications.

It is both mathematically and practically interesting to develop asymptotic theories for solutions to (1.1) if only because numerical solutions become prohibitively expensive when $\varepsilon \to 0$. Homogenization theory or averaging theory aims at finding an effective or homogenized equation whose solution u_0 is the limit of u_ε as ε goes to zero. This theory has been well developed since the works of [10, 12, 13], where linear second-order elliptic and parabolic equations with random conductivity tensors were considered; it works quite generally for other types of equations, as for random transport equation [7], and even fully nonlinear equations [6, 11]. The main assumption on the random coefficients is quite mild: stationarity and ergodicity.

Corrector theory aims at further approximating the heterogeneous solution by capturing the leading terms in the corrector $u_\varepsilon - u_0$. Comparing with homogenization, much fewer results are available, whereas more assumptions on the random coefficients are necessary. One folklore is: *de-correlation rate* of the random field matters. Take the second-order elliptic boundary value problem in 1D for instance. When the random coefficient $q(x,\omega)$ there is mixing so that its (auto-)correlation function $\mathbb{E}\{q(y,\omega)q(y+x,\omega)\}$ is integrable, a case we say $q(x,\omega)$ has *short-range* correlation, the corrector converges to a short-range correlated Gaussian process, which can be written as a stochastic integral against Brownian motions [5]. However, as shown in [2], the limit of the corrector can be very different if $q(x,\omega)$ has *long-range* correlation, meaning the correlation function of it does not decay fast enough to be integrable. The other folklore is: *singularity* of Green's function matters as well: For example, for the aforementioned steady diffusion problem, precise characterizations of the limiting correctors in [8, 1] are constrained to low dimensions (less than three), where Green's functions are more than square integrable.

The main objective of this paper is twofolds. First, we consider corrector theory for (1.1) with long-range random potential in possibly multi-dimensional space; so far only the short-range or the long-range but low-dimension cases have been developed [8, 1, 4, 2]. Second, we explore the influence of singularity of the Green's function, as what have been done in the short-range case in [4], on the competition between the deterministic and mean-zero random part of the corrector. We consider the special case when $q(x,\omega)$ is the composition of a function with a stationary centered Gaussian, correlation function of which decays like $|x|^{-\alpha}$ and $\alpha < d$. We find that the mean-zero corrector has magnitude $\varepsilon^{\alpha/2}$, much larger comparing with $\varepsilon^{d/2}$ in the short-range case; though it can be written as a stochastic integral, the integrator is some specially tailored Gaussian process, no longer the standard multi-parameter Wiener process. We assume the Green's function has singularity of the form $G(x,y)\ |x-y|^{-(d-\beta)}$ near the diagonal, and we find that as α increases, 2β is the threshold after which the deterministic part of the corrector starts to dominate the mean-zero part. Details are in the main body of the paper.

The rest of the paper is organized as follows: we state the problem setting and the main results in the next section. In section 3, we prove the homogenization result, and explore how the convergence rate, as well as the size of the deterministic corrector, depends on the de-correlation rate α and the singularity of Green's function β. In section 4, assuming $\alpha < 4\beta$, we characterize the limiting distribution of the random corrector integrated against test functions. Its magnitude is of order ε^α and it admits a stochastic integral presentation. In section 5, for the 1D case under the assumption that Green's function is

Lipschitz continuous and the solution to (1.1) is continuous, we prove that the corrector convergence in distribution in the space of continuous paths. A general discussion on other cases of singularities and relations between α and β is sketched in section 6. Some useful but independent estimates on convolution of potentials are recorded in the appendix.

2 Main results

In this section, we first describe the problem setting with detailed assumptions on the random coefficients, and then state the main results of the paper.

Before proceeding we set some conventions to simplify notations. Very often we ignore the dependence on realization ω of a random field $q(x,\omega)$. For the random fields $q(x)$, $g(x)$, and their correlation functions $R_g(x)$ and $R(x)$ to be defined in (2.3) and (3.1), we denote by $q_\varepsilon(x)$ the rescaled function $q(\frac{x}{\varepsilon})$, etc. For a general function f, the notion $\|f\|_{p,X}$ denotes its L^p on X though X is often omitted; when p is missing, it is understood to be two. We designate $\langle \cdot\,,\,\cdot\rangle$ the usual pairing on L^2. We denote by $a\wedge b$ the smaller one of two positive terms. Throughout the paper we will use C to denote various constants that may vary from line to line. In particular, when C only depends on dimension d and the domain of interest X, we say it is *universal*.

Let us write (1.1) in detail as

$$\begin{cases} P(x,D)u_\varepsilon(x,\omega) + \left(q_0(x) + q_\varepsilon(x,\omega)\right)u_\varepsilon(x,\omega) = f(x), & x \in X, \\ u_\varepsilon(x,\omega) = 0, & x \in \partial X. \end{cases} \tag{2.1}$$

Though we treat $P(x,D)$ as a general elliptic pseudo-differential operator, it helps to keep in mind the two prototypes mentioned earlier, namely the steady diffusion problem with $P(x,D) = -\nabla\cdot a(x)\cdot$, and the Robin boundary equation where $P(x,D) = \sqrt{-\Delta+\lambda^2}$. Pseudo-differential equation is not the objective of this paper. Rather, we assume $P(x,D) + \tilde{q}(x)$ is invertible from $L^2(X)$ to some functional space embedded in $L^2(X)$ as long as $q_0 + q_\varepsilon$ is non-negative; we assume further that the operator norm of $(P(x,D) + q_0 + q_\varepsilon)^{-1}$ can be bounded independent of the non-negative coefficient in the potential term. One can check that this is the case for the two prototypes. In the steady diffusion problem, the solution operator is continuous from $L^2(X)$ to $H^1_0(X)$ whose norm can be bounded by the uniform ellipticity constants of $a(x)$. Similarly, for the Robin boundary equation, the solution operator is continuous from $L^2(X)$ to the Sobolev space $H^{\frac{1}{2}}$ with a bound on its norm only depending on γ.

We assume that $q_0(x)$ is a smooth function bounded from below by some positive number γ. Then the inverse of $P(x,D) + q_0$, denoted by $\mathcal{G}$, is well defined. The operator norm $\|\mathcal{G}\|_{\mathcal{L}(L^2)}$ is bounded by some universal constant C. For simplicity, we assume that $\mathcal{G}$, as a transform on $L^2(X)$, is self-adjoint. Finally, we assume that the Green's function $G(x,y)$ associated to $\mathcal{G}$ satisfies:

$$|G(x,y)| \le \frac{C}{|x-y|^{d-\beta}}, \tag{2.2}$$

for some universal constant C and some real number $\beta \in (0,d)$, which measures how singular the Green's function is near the diagonal $x = y$.

The main assumptions we have on the random process $q(x,\omega)$ are as follows.

(A1) $q(x)$ is constructed by $q(x) = \Phi(g(x))$, where $g(x)$ is a centered stationary Gaussian random field with unit variance. Further, the correlation function of $g(x)$ has heavy tail of the form:

$$R_g(x) := \mathbb{E}\{g(y)g(y+x)\} \sim \kappa_g |x|^{-\alpha} \text{ as } |x| \to \infty, \tag{2.3}$$

for some positive constant κ_g and some real number $\alpha \in (0,d)$.

(A2) The function $\Phi : \mathbb{R} \to \mathbb{R}$ satisfies $|\Phi| \le \gamma \le q_0$, and that

$$\int_{\mathbb{R}} \Phi(s) e^{-\frac{s^2}{2}} ds = 0. \tag{2.4}$$

The upper bound of Φ above ensures that $|q(x)| \le \gamma$. Consequently, $q_0 + q_\varepsilon$ is non-negative, and (2.1) is well-posed almost surely with solution operator bounded uniformly with respect to q. Due to the construction above and (2.4), $q(x)$ is mean-zero and stationary, and has long-range correlation function that decays like $|x|^{-\alpha}$ as we show later.

The first main theorem concerns the homogenization of (2.1). It shows, in particular, how the competition between the de-correlation rate α and the Green's function singularity β affects the convergence rate of homogenization.

Theorem 2.1. *Let u_ε be the solution to (2.1) and u_0 be the solution to the same equation with q_ε replaced by its average zero. Assume that in the equation, $q(x)$ is constructed as in (A1) and (A2), and $f \in L^2(X)$. Then, assuming $2\beta < d$, we have*

$$\mathbb{E}\, \|u_\varepsilon - u_0\|^2 \le \|f\|^2 \times \begin{cases} C\varepsilon^{\alpha}, & \alpha < 2\beta, \\ C\varepsilon^{2\beta} |\log \varepsilon|, & \alpha = 2\beta, \\ C\varepsilon^{2\beta}, & \alpha > 2\beta. \end{cases} \tag{2.5}$$

The constants α and β are defined in (2.3) and (2.2) respectively. When $2\beta \ge d$, only the first case is necessary. The constant C depend on α, β, γ and the uniform bound on the solution operator of (2.1).

This theorem says u_ε and u_0 are close in the energy norm $L^2(\Omega, L^2(X))$. Let ξ_ε denote the corrector $u_\varepsilon - u_0$. We can decompose it into two parts as

$$\xi_\varepsilon = (\mathbb{E}\{u_\varepsilon\} - u_0) + (u_\varepsilon - \mathbb{E}\{u_\varepsilon\}). \tag{2.6}$$

We call the first first part the *deterministic corrector*, and the second mean-zero part the *stochastic corrector*. For the deterministic corrector, we have the following estimates on its size, which depends on the competition of α and β as well. Here and after, $\mathcal{C}(\overline{X})$ denotes the space of functions on X that are continuous up to the boundary.

Theorem 2.2. *Let u_ε, u_0, $q(x)$ and f be as in the previous Theorem. Then for an arbitrary test function $\varphi \in \mathcal{C}(\overline{X})$, we have,*

$$|\langle \mathbb{E}\{u_\varepsilon\} - u_0, \varphi\rangle| \le \|f\|\|\varphi\| \times \begin{cases} C\varepsilon^\alpha, & \alpha < \beta, \\ C\varepsilon^\beta |\log \varepsilon|, & \alpha = \beta, \\ C\varepsilon^\beta, & \alpha > \beta. \end{cases} \tag{2.7}$$

The constant C depends on the same factors as in the previous theorem.

The magnitude of the stochastic corrector is always of order ε^α, as we will see later in this paper. We observe from the theorem above, therefore, the deterministic corrector can be much larger than the stochastic corrector, say when $\alpha > 2\beta$. To describe the stochastic corrector more precisely, we aim to characterize its limiting distribution. To do this, however, we impose the following additional assumption.

(A3) The function Φ satisfies

$$\int_{\mathbb{R}} |\hat{\Phi}(\xi)|\big(1 + |\xi|^3\big) < \infty, \tag{2.8}$$

where $\hat{\Phi}$ denotes the Fourier transform of Φ.

This condition allows one to derive a (non-asymptotic) estimate, Equation (A.4) in the appendix, for the fourth-order moments of $q(x)$, which is a technicality one encounters quite often in corrector theory. With this assumption, we have

Theorem 2.3. *Let u_ε and u_0 solve* (2.1) *and the homogenized solution respectively. Assume $f \in L^2(X)$ and $q(x)$ is constructed by* (A1-A2) *with Φ satisfying* (A3)*. Further, assume $\alpha < 4\beta$. Then:*

$$\frac{u_\varepsilon - \mathbb{E}\{u_\varepsilon\}}{\varepsilon^{\alpha/2}} \xrightarrow[\varepsilon \to 0]{\text{distribution}} -\int_X G(x,y)u_0(y)W^\alpha(dy), \tag{2.9}$$

where $W^\alpha(dy)$ is formally defined to be $\dot{W}^\alpha(y)dy$ and $\dot{W}^\alpha(y)$ is a Gaussian random field with covariance function given by $\mathbb{E}\{\dot{W}^\alpha(x)\dot{W}^\alpha(y)\} = \kappa|x-y|^{-\alpha}$. The convergence is understood as in probability distribution weakly in spatial space.

Remark 2.4. We refer the reader to [9] for theory on multi-parameter random processes. What we really mean by convergence in probability distribution weakly in spatial space is as follows. Fix an arbitrary natural number N, and a set of test functions $\{\varphi_i;\ 1 \le i \le N\}$ in $\mathcal{C}(\overline{X})$. Define $I_i^\varepsilon := \langle \varphi_i, \varepsilon^{-\alpha/2}(u_\varepsilon - \mathbb{E}\{u_\varepsilon\})\rangle$, for $i = 1, \cdots, N$. What (2.9) means is that the N-dimensional random vector $(I_1^\varepsilon, \cdots, I_N^\varepsilon)$ converges in distribution to a centered N-dimensional Gaussian vector $(I_1, \cdots, I_N)$, whose covariance matrix Σ_{ij} is given by

$$\Sigma_{ij} := \int_{X^2} \frac{\kappa}{|y-z|^\alpha}(u_0\varphi_i)(y)(u_0\varphi_j)(z)dydz. \tag{2.10}$$

by the definition of the stochastic integral above, we see I_i is precisely the inner product of φ_i with the right hand side of (2.9).

We observe from Theorem 2.2 that when $\alpha < 2\beta$ we can replace $\mathbb{E}\{u_\varepsilon\}$ in (2.9) by u_0, since the deterministic corrector is much smaller. This is no longer the case for $\alpha \ge 2\beta$.

The condition $\alpha < 4\beta$ in Theorem 2.3 is due to technical reasons which will be clear later. The conclusion of the theorem holds in general, but we need good estimate on high-order (more than four-order) moments of q to carry out the analysis developed in this paper. □

In one dimensional space, with the assumptions that u_ε and u_0 has continuous paths, and that the Green's function is Lipschitz continuous, we can show convergence of the corrector in distribution in the space of continuous paths, as in [5, 2].

Theorem 2.5. *Let X be the unit interval $[0,1]$ in $\mathbb{R}$. Assume that the Green's function $G(x,y)$ is Lipschitz continuous in x with Lipschitz constant $\mathrm{Lip}(G)$ uniform in y. Let u_ε be the solution to (2.1) and u_0 be the homogenized solution. Assume $q(x)$ is constructed as in (A1)-(A3). Then*

$$\frac{u_\varepsilon - u_0}{\varepsilon^{\alpha/2}}(x) \xrightarrow[\varepsilon\to 0]{\text{distribution}} -\sqrt{\frac{\kappa}{H(2H-1)}}\int_0^1 G(x,y)u_0(y)dW_H(y), \tag{2.11}$$

where W_H is the standard fractional Brownian motion with Hurst index $H = 1 - \frac{\alpha}{2}$.

Remark 2.6. We refer the reader to [?] for a review on the definitions of fractional Brownian motions and of the stochastic integral with respect to them. In particular, the random process on the right hand side of (2.11) is a mean-zero Gaussian process which, if designated as $I_H(x)$, has the following covariance function:

$$\mathrm{Cov}[I_H](x,y) = \kappa \int_0^1 \int_0^1 \frac{G(x,t)u_0(t)G(y,s)u_0(s)}{|t-s|^{2(1-H)}} dtds. \tag{2.12}$$

3 Convergence to the homogenized solution

In this section, we prove Theorem 2.1, which says the homogenized equation for (2.1) is obtained by averaging q_ε.

We first verify that the random field $q(x)$ constructed in (A1) and (A2) has the same heavy tail as the underlying Gaussian random field. By stationarity, the correlation function of $q(x)$ is given by

$$R(x) := \mathbb{E}\{q(y)q(y+x)\} = \mathbb{E}\{\Phi(g_0)\Phi(g_x)\}. \tag{3.1}$$

We show that $R(x)$ has the same asymptotic behavior as R_g in (2.3).

Lemma 3.1. *Let $q(x)$ be the random field above. Define $V_1 = \mathbb{E}\{g_0\Phi(g_0)\}$ where g_x is the underlying Gaussian random field. There exist some $T, C > 0$ such that the autocorrelation function $R(x)$ of q satisfies*

$$|R(x) - V_1^2 R_g(x)| \le C R_g^2(x), \quad \textit{for all } |x| \ge T, \tag{3.2}$$

where R_g is the correlation function of g. Further,

$$|\mathbb{E}\{g(y)q(y+x)\} - V_1 R_g(x)| \le C R_g^2(x), \quad \textit{for all } |x| \ge T. \tag{3.3}$$

Proof. A proof of this lemma can be found in [2]; we record it here for the reader's convenience.

$$R(x) = \frac{1}{2\pi\sqrt{1-R_g^2(x)}} \int_{\mathbb{R}^2} \Phi(g_1)\Phi(g_2) \exp\left(-\frac{g_1^2+g_2^2-2R_g(x)g_1g_2}{2(1-R_g^2(x))}\right) dg_1 dg_2.$$

For large $|x|$, the coefficient $R_g(x)$ is small and we can expand the value of the double integral in powers of $R_g(x)$. The zeroth order term is the integration of $\Phi(g_1)\Phi(g_2)$ with respect to $\exp(-|g|^2/2)dg$ where dg is short for dg_1dg_2; this term vanishes due to (2.4). The first order term is integration of $\Phi(g_1)\Phi(g_2)g_1g_2$ with respect to the $\exp(-|g|^2/2)dg$, which gives $V_1^2 R_g(x)$.

Similarly, for the second item in the lemma, we first write

$$\mathbb{E}\{g(y)\Phi(g(y+x))\} = \frac{1}{2\pi\sqrt{1-R_g^2(x)}} \int_{\mathbb{R}^2} g_1\Phi(g_2) \exp\left(-\frac{g_1^2+g_2^2-2R_g(x)g_1g_2}{2(1-R_g^2(x))}\right) dg_1 dg_2.$$

Then we expand the value of the double integral in powers of R_g and characterize the first two orders as before. □

It follows that $R(x)$ behaves like $\kappa|x|^{-\alpha}$, where $\kappa = V_1^2\kappa_g$, for large $|x|$. In particular, there exists some constant C so that $|R(x)| \leq C|x|^{-\alpha}$. Meanwhile, $|R(x)|$ is uniformly bounded, say by $|\Phi|^2 \leq \gamma^2$ according to assumption (A2).

Lemma 3.2. *Let $\mathcal{G}$ be the Green's operator and $q(x)$ be the random field above. Let f be an arbitrary function in $L^2(X)$. Assume $2\beta < d$ and we have*

$$\mathbb{E}\;\|\mathcal{G}q_\varepsilon f\|^2 \leq \|f\|^2 \times \begin{cases} C\varepsilon^\alpha, & \alpha < 2\beta, \\ C\varepsilon^{2\beta}|\log\varepsilon|, & \alpha = 2\beta, \\ C\varepsilon^{2\beta}, & \alpha > 2\beta. \end{cases} \tag{3.4}$$

The constant C depends only on α, β, X, $\|q\|_\infty$ and the bound for $\|\mathcal{G}_\varepsilon\|_{\mathcal{L}}$. If $2\beta \geq d$, then only the first case is necessary.

Proof. The L^2 norm of $\mathcal{G}q_\varepsilon f$ has the following expression:

$$\|\mathcal{G}q_\varepsilon f\|^2 = \int_X \left(\int_X G(x,y)q_\varepsilon(y)f(y)dy\right)^2 dx.$$

After writing the integrand as a double integrals and taking expectation, we have

$$\mathbb{E}\|\mathcal{G}q_\varepsilon f\|^2 = \int_{X^3} G(x,y)G(x,z)R_\varepsilon(y-z)f(y)f(z)dydzdx. \tag{3.5}$$

Use (2.2) to bound the Green's functions. Integrate over x and apply Lemma A.1. We get

$$\mathbb{E}\|\mathcal{G}q_\varepsilon f\|^2 \leq C\int_{X^2} \frac{1}{|y-z|^{d-2\beta}}|R_\varepsilon(y-z)f(y)f(z)|dydz. \tag{3.6}$$

Change variable $(y, y-z) \mapsto (y, z)$. The above integral becomes

$$\int_X \int_{y-X} \frac{1}{|z|^{d-2\beta}} |R_\varepsilon(z) f(y) f(y-z)| dy dz$$

We can further bound the integral from above by enlarging the domain $y - X$ to some finite ball $B(2\rho)$ where $\rho = \sup_{x\in X} |x|$, because the translated region $y - X$ is included in this ball for every y. After this replacement, integrate over y first, and we have:

$$\mathbb{E}\|\mathcal{G} q_\varepsilon f\|^2 \leq C\|f\|^2 \int_{B(2\rho)} \frac{|R_\varepsilon(z)|}{|z|^{d-2\beta}} dz. \tag{3.7}$$

Decompose the integration region into two parts:

$$\begin{cases} D_1 := \{|x\varepsilon^{-1}| \leq T\} \cap B(2\rho), & \text{on which we have } |R_\varepsilon| \leq \gamma^2; \\ D_2 := \{|x\varepsilon^{-1}| > T\} \cap B(2\rho), & \text{on which we have} |R_\varepsilon| \leq C\varepsilon^\alpha |x|^{-\alpha}. \end{cases}$$

The integration on D_1 can be carried out explicitly. The restriction $|x| \leq T\varepsilon$ yields that this term is of order $\varepsilon^{2\beta}$. The integration over D_2 is

$$C \int_{\varepsilon T}^{2\rho} \frac{\varepsilon^\alpha |z|^{d-1}}{|z|^{d-2\beta+\alpha}} d|z|.$$

When $2\beta = \alpha$, the integral equals $C\varepsilon^\alpha(\log(2\rho) - \log(T\varepsilon))$, and is of order $\varepsilon^\alpha |\log \varepsilon|$. When $2\beta \neq \alpha$, the integral equals $C\varepsilon^\alpha((2\rho)^{2\beta-\alpha} - (T\varepsilon)^{2\beta-\alpha})$. This estimate proves the other two cases of the lemma.

The same analysis can be done for $2\beta \geq d$. In this case, the singular term $|y-z|^{-(d-2\beta)}$ in (3.6) should be replaced by either $|\log|y-z||$ or C, which is much smoother. Consequently, $\mathbb{E}\|\mathcal{G} q_\varepsilon f\|^2$ is of order ε^α. □

Proof of Theorem 2.1. The homogenized solution satisfies $(P(x, D) + q_0)u_0 = f$. Define $\chi_\varepsilon = -\mathcal{G} q_\varepsilon u_0$, that is the solution of $(P(x,D)+q_0)\chi_\varepsilon = -q_\varepsilon u_0$. Compare these two equations with the one for u_ε, i.e. (2.1). We get

$$(P(x, D) + q_0 + q_\varepsilon)(\xi_\varepsilon - \chi_\varepsilon) = -q_\varepsilon \chi_\varepsilon,$$

where ξ_ε denotes $u_\varepsilon - u_0$. Since this equation is well-posed a.e. in Ω, we have $\xi_\varepsilon = \chi_\varepsilon - \mathcal{G}_\varepsilon q_\varepsilon \chi_\varepsilon$, which implies

$$\|\xi_\varepsilon\| \leq \|\chi_\varepsilon\| + \|\mathcal{G}_\varepsilon\|_{\mathcal{L}(L^2)} \|q\|_\infty \|\chi_\varepsilon\|. \tag{3.8}$$

Recall that the operator norm $\|\mathcal{G}_\varepsilon\|_{\mathcal{L}(L^2)}$ can be bounded uniformly in Ω; so the right hand side above is further bounded by $C\|\chi_\varepsilon\|$. Since χ_ε is of the form of $\mathcal{G} q_\varepsilon f$, we take expectation and apply the previous lemma to completes the proof. □

We decompose the corrector into the deterministic corrector $\mathbb{E}\{u_\varepsilon\} - u_0$ and the stochastic corrector $u_\varepsilon - \mathbb{E}\{u_\varepsilon\}$. We consider their sizes and limits only in the weak sense, that is after pairing with test functions. We have the following formula for u_ε,

$$u_\varepsilon - u_0 = -\mathcal{G} q_\varepsilon u_0 + \mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon u_0 + \mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon (u_\varepsilon - u_0). \tag{3.9}$$

Pairing this with an arbitrary test function $\varphi \in \mathcal{C}(\overline{X})$, we have

$$\langle u_\varepsilon - u_0, \varphi\rangle = -\langle \mathcal{G}q_\varepsilon u_0, \varphi\rangle + \langle \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon u_0, \varphi\rangle + \langle \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon (u_\varepsilon - u_0), \varphi\rangle. \tag{3.10}$$

Now the deterministic corrector $\langle \mathbb{E}\{u_\varepsilon\} - u_0, \varphi\rangle$ is precisely the expectation of the expression above. In the following, we estimate the size of this corrector using the analysis developed in the proof of Lemma 3.2.

Proof of Theorem 2.2. Take expectation in (3.10). Since the first term on the right is mean zero, we have

$$\langle \mathbb{E}\{u_\varepsilon\} - u_0, \varphi\rangle = \mathbb{E}\langle \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon u_0, \varphi\rangle + \mathbb{E}\langle \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon (u_\varepsilon - u_0), \varphi\rangle. \tag{3.11}$$

Let m denote the L^2 function $\mathcal{G}\varphi$. Rewrite the first term on the right as $\mathbb{E}\langle q_\varepsilon u_0, \mathcal{G}q_\varepsilon m\rangle$, which can be written as

$$\int_X G(x,y) R_\varepsilon(x-y) u_0(x) m(y) dx dy.$$

After controlling the green's function by $C|x-y|^{-d+\beta}$, we have an object similar to (3.6). Following the same procedure there, we can show that $|\mathbb{E}\langle q_\varepsilon u_0, \mathcal{G}q_\varepsilon m\rangle|$ can be bounded as in (2.7). To complete the proof, we only need to control the remainder term in (3.11), which can be written as $\mathbb{E}\langle q_\varepsilon (u_\varepsilon - u_0), \mathcal{G}q_\varepsilon m\rangle$. We have:

$$\mathbb{E}|\langle q_\varepsilon (u_\varepsilon - u_0), \mathcal{G}q_\varepsilon m\rangle| \le \|q_\varepsilon\|_\infty \left(\mathbb{E}\|u_\varepsilon - u_0\|^2\right)^{1/2} \left(\mathbb{E}\|\mathcal{G}q_\varepsilon m\|^2\right)^{1/2}. \tag{3.12}$$

According to Theorem 2.1 and Lemma 3.2, this term can be bounded by the right hand side of (3.4). Therefore, the remainder is smaller than the quadratic term which gives the estimate desired. □

For any fixed test function φ, the random corrector $\langle u_\varepsilon - \mathbb{E}\{u_\varepsilon\}, \varphi\rangle$ is precisely the mean-zero part of the right hand side of (3.10); we are interested its limiting distribution. The size of its variance is presumably given by that of $-\langle \mathcal{G}q_\varepsilon u_0, \varphi\rangle$. We calculate

$$\operatorname{Var}\left(-\langle \mathcal{G}q_\varepsilon u_0, \varphi\rangle\right) = \operatorname{Var}\left(-\langle q_\varepsilon u_0, m\rangle\right) = \int_{X^2} R_\varepsilon(x-y) u_0 m(x) u_0 m(y) dx dy.$$

Estimate this integral by decomposing the domain as in the proof of Lemma 3.2; we verify that this object is of size ε^α, independent of β. Therefore, a more accurate characterization of the stochastic corrector is to find the limiting distribution of $\varepsilon^{-\alpha/2}\langle u_\varepsilon - \mathbb{E}\{u_\varepsilon\}, \varphi\rangle$. This is the main task of Section 4.

4 Corrector theory in higher dimensional space

In this section, we consider the limiting distribution of the stochastic corrector. In the analyses we are going to develop, the following estimate proves very useful. Recall that R is uniformly bounded, and there exists some T so that $|R| \le C|x|^{-\alpha}$ when $|x| > T$.

Lemma 4.1. *Recall that $R(x)$ denotes the correlation function of the random field $q(x)$ constructed in* (A1) *and* (A2)*, and that $R_\varepsilon(x)$ denotes $R(\varepsilon^{-1}x)$. Let $p \geq 1$; we have*

$$\|R_\varepsilon\|_{p,B(\rho)} \leq \begin{cases} C\varepsilon^{\alpha}, & \alpha p < d, \\ C\varepsilon^{\alpha}|\log \varepsilon|^{\frac{1}{p}}, & \alpha p = d, \\ C\varepsilon^{\frac{d}{p}}, & \alpha p > d. \end{cases} \tag{4.1}$$

Here, $B(\rho)$ is the open ball centered at zero with radius ρ. The constant C depends on ρ, dimensionality d, and the constant in the asymptotic behavior of $R(x)$.

Proof. We break the expression for $\|R_\varepsilon\|_p^p$ into two parts as follows:

$$\int_{B(\varepsilon T)} |R_\varepsilon(x)|^p dx + \int_{B(\rho)\setminus B(\varepsilon T)} |R_\varepsilon(x)|^p dx.$$

For the first term, we bound R_ε by its uniform norm and verify this term is of order ε^d. For the second term, which we call I_2, we use the asymptotic behavior of R and have

$$I_2 \leq C \int_{B(\rho)\setminus B(\varepsilon T)} \varepsilon^{\alpha p}|x|^{-\alpha p} dx \leq C\varepsilon^{\alpha p} \int_{T\varepsilon}^{\rho} r^{d-1-\alpha p} dr.$$

Carry out this integral, we find it is of order $\varepsilon^{\alpha p}|\log \varepsilon|$ if $\alpha p = d$, of order $\varepsilon^{\alpha p \wedge d}$ if otherwise.

Now combine the two parts; compare the orders case by case to get the bound for $\|R_\varepsilon\|_p^p$. Then take pth roots to complete the proof. □

Lemma 4.2. *Assume $q(x)$ constructed in* (A1-A2) *satisfies* (A3)*. Let φ be an arbitrary test function in $\mathcal{C}(\overline{X})$. Then we have the following estimate of the variance of the second term in* (3.10)*.*

$$\text{Var } \langle \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon u_0, \varphi\rangle \ll C\|u_0\|^2\|\varphi\|_\infty^2 \varepsilon^{\alpha}. \tag{4.2}$$

Again, the constant C only depend on the factors as stated in Theorem 2.1.

Proof. We observe first that $m := \mathcal{G}\varphi$ is uniformly bounded since φ is, a useful fact in the following. To simplify notations, denote by I the variance of $\langle \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon u_0, \varphi\rangle$. It has the expression:

$$I = \int_{X^4} u_0(x)m(y)u_0(\xi)m(\eta)G(x,y)G(\xi,\eta)\times \\ \times \Big[\mathbb{E}\{q_\varepsilon(x)q_\varepsilon(y)q_\varepsilon(\xi)q_\varepsilon(\eta)\} - \mathbb{E}\{q_\varepsilon(x)q_\varepsilon(y)\}\mathbb{E}\{q_\varepsilon(\xi)q_\varepsilon(\eta)\}\Big] dxdyd\xi d\eta.$$

Apply Lemma A.2 to estimate the variance of the product of q_ε above, and use the bound for the Green's functions. We have

$$I \leq C\int_{X^4} |u_0(x)m(y)u_0(\xi)m(\eta)| \frac{1}{|x-y|^{d-\beta}} \frac{1}{|\xi-\eta|^{d-\beta}} \times \\ \times \sum_{p \neq \{(1,2),(3,4)\}} |R_\varepsilon(x_{p(1)} - x_{p(2)})R_\varepsilon(x_{p(3)} - x_{p(4)})| dxdyd\xi d\eta.$$

Here, $p = \{(p_1, p_2), (p_3, p_4)\}$ denotes possibilities of choosing two different pairs of indices from $\{1, 2, 3, 4\}$ in such a way that, each pair contains different indices though the two pairs may share a same index. There are $C_6^2 = 15$ different choices for p; however, $p = \{(1, 2), (3, 4)\}$ is excluded from the sum above. Identify (x_1, x_2, x_3, x_4) with (x, y, ξ, η); we see that there are 14 terms in the sum, and each of them is a product of two R_ε functions whose arguments are the difference vectors of points in $\{x, y, \xi, \eta\}$; more importantly, at most one of the R_ε functions share the same argument with one of the Green's functions.

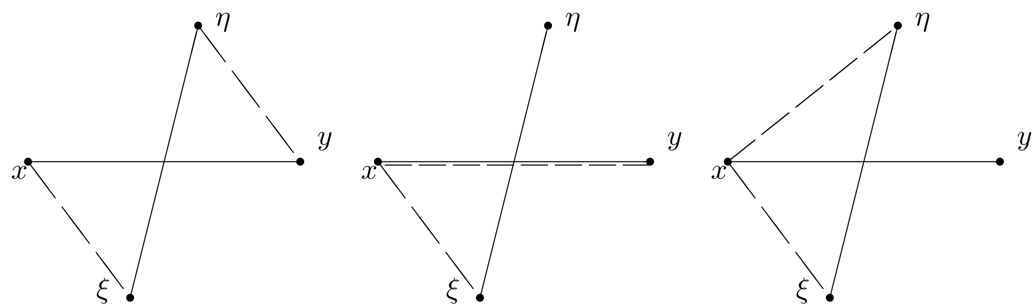

Figure 1: Difference vectors of four points. The *solid* lines represent arguments of the Green's functions, while the *dashed* lines represent those of the correlation functions.

We can divide the fourteen choices of p into three categories as shown in Figure 1. In the first category as illustrated by the first picture, the two vectors in the correlation functions are linear independent with both of the vectors in the Green's functions; in the second category, one of the Green's function shares the same argument with one of the correlation function; finally in the third category, the vector in one of the Green's function is a linear combination of the two vectors of the correlation functions.

For the first category, we consider a typical term of the form:

$$J_1 = \int_{X^4} |u_0(x)m(y)u_0(\xi)m(\eta)| \frac{1}{|x-y|^{d-\beta}} \frac{1}{|\xi-\eta|^{d-\beta}} |R_\varepsilon(x-\xi)R_\varepsilon(y-\eta)|. \tag{4.3}$$

Change variable as follows:

$$(x, x-y, x-\xi, y-\eta) \mapsto (x, y, \xi, \eta).$$

Bound m by its uniform norm. In terms of the new variables, we have

$$J_1 \le \|m\|_\infty^2 \int_X dx \int_{x-X} dy \int_{x-X} d\xi \int_{x-y-X} d\eta \; \frac{|u_0(x)u_0(x-\xi)R_\varepsilon(\xi)R_\varepsilon(\eta)|}{|y|^{d-\beta}|y-(\xi-\eta)|^{d-\beta}}.$$

We can replace the integration region of y and ξ by $B(2\rho)$, and replace that of η by $B(3\rho)$, where ρ as before denotes the maximum distance of a point in X and the origin. After doing this, we integrate over x first to get rid of the u_0 function; then integrate over y and apply Lemma A.1 to get

$$J_1 \le \|m\|_\infty^2 \|u_0\|^2 \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|R_\varepsilon \mathbf{1}_{B(2\rho)}(\xi)| \; |R_\varepsilon \mathbf{1}_{B(3\rho)}(\eta)|}{|\xi-\eta|^{d-2\beta}} d\xi d\eta. \tag{4.4}$$

Here, $\mathbf{1}_A$ is the indicator function of a subset $A \subset \mathbb{R}^d$. We considered the case $2\beta < d$; the other cases can be considered by the same token and is only easier. To estimate the integral above, we apply Hardy-Littlewood-Sobolev inequality [**?**, Theorem 4.3]. With $p = 2d/(d+2\beta) > 1$, we have

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|R_\varepsilon \mathbf{1}_{B(2\rho)}(\xi)|\ |R_\varepsilon \mathbf{1}_{B(3\rho)}(\eta)|}{|\xi - \eta|^{d-2\beta}} d\xi d\eta \le C(d,\beta,p) \|R_\varepsilon\|_{p,B(2\rho)} \|R_\varepsilon\|_{p,B(3\rho)}. \tag{4.5}$$

Now apply Lemma 4.1: If $\alpha p \le d$, we see J_1 is of order $\varepsilon^{2\alpha}$ or $\varepsilon^{2\alpha}|\log \varepsilon|^{2/p}$ which is much smaller than ε^α; if otherwise, J_1 is of order $\varepsilon^{2d/p} \ll \varepsilon^\alpha$ because by our choice of p we have $2d/p - \alpha = d + 2\beta - \alpha > 2\beta > 0$.

In the second category, we consider a typical term of the form:

$$J_2 = \int_{X^4} |u_0(x) m(y) u_0(\xi) m(\eta)| \frac{1}{|x-y|^{d-\beta}} \frac{1}{|\xi - \eta|^{d-\beta}} |R_\varepsilon(x-y) R_\varepsilon(x-\xi)|. \tag{4.6}$$

This time we use the following change of variable,

$$(x, x-y, x-\xi, \xi-\eta) \mapsto (x, y, \xi, \eta).$$

With this change and bounding m, we have

$$J_2 \le \|m\|_\infty^2 \int_X dx \int_{x-X} dy \int_{x-X} d\xi \int_{x-\xi-X} d\eta\ \frac{|u_0(x) u_0(x-\xi) R_\varepsilon(\xi) R_\varepsilon(y)|}{|y|^{d-\beta} |\eta|^{d-\beta}}.$$

Enlarge the integration region of y, ξ, η as before, and then integrate over x and η. We have

$$J_2 \le \|m\|_\infty^2 \|u_0\|^2 \int_{B^2(2\rho)} \frac{1}{|y|^{d-\beta}} |R_\varepsilon(y)| |R_\varepsilon(\xi)| d\xi dy. \tag{4.7}$$

The integration over ξ yields a term of size ε^α; meanwhile, the integration over y can be estimated as what we have done for the integral in (3.7), and is of size given in (2.7). Therefore, $J_2 \ll \varepsilon^\alpha$.

For the third category, we consider a typical term of the form:

$$J_3 = \int_{X^4} |u_0(x) m(y) u_0(\xi) m(\eta)| \frac{1}{|x-y|^{d-\beta}} \frac{1}{|\xi - \eta|^{d-\beta}} |R_\varepsilon(x-\xi) R_\varepsilon(x-\eta)|. \tag{4.8}$$

Change variables according to

$$(x, x-y, x-\xi, x-\eta) \mapsto (x, y, \xi, \eta).$$

After the routine of enlarging integration domains, bounding m, and integrating the non-singular terms, we have

$$J_3 \le \|m\|_\infty^2 \|u_0\|^2 \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|R_\varepsilon \mathbf{1}_{B(2\rho)}(\xi)| |R_\varepsilon \mathbf{1}_{B(2\rho)}(\eta)|}{|\xi - \eta|^{d-\beta}} d\xi d\eta. \tag{4.9}$$

This term can be estimated exactly as what we have done for (4.4). In particular, it is much smaller than ε^α. This completes the proof. □

To prove Theorem 2.3, we are essentially consider the law of random vectors of the form $(J_1^\varepsilon(\omega), \cdots, J_N^\varepsilon(\omega))$, where

$$J_j^\varepsilon(\omega) := -\frac{1}{\varepsilon^{\alpha/2}} \int_X q_\varepsilon(y)\psi_j(y)dy, \tag{4.10}$$

for some collection of $L^2(X)$ functions $\{\psi_k(x);\ 1 \le k \le N\}$. We have the following result characterizing the limiting joint law of them.

Lemma 4.3. *The random vector $(J_1^\varepsilon, J_2^\varepsilon, \cdots, J_N^\varepsilon)$ converges in distribution to the centered Gaussian random vector $(J_1, J_2, \cdots, J_N)$ whose covariance matrix is given by*

$$C_{ik} = \mathbb{E}\{J_i J_k\} = \int_{X^2} \frac{\kappa\ \psi_i(y)\psi_k(z)}{|y-z|^\alpha} dydz. \tag{4.11}$$

Moreover, the random variable J_k admits the following stochastic integral representation.

$$J_k = -\int_X \psi_k(y) W^\alpha(dy). \tag{4.12}$$

Here $W^\alpha(dy)$ is as formally defined in Theorem 2.3.

Proof. We want to show that $\forall t_1, t_2, \cdots, t_N \in \mathbb{R}$, $\sum_{i=1}^N t_i J_i^\varepsilon$ converges in distribution to $\sum_{i=1}^N t_i J_i$. Since

$$\sum_{i=1}^N t_i J_i^\varepsilon = -\frac{1}{\varepsilon^{\alpha/2}} \int_X q_\varepsilon(y) \sum_{i=1}^N t_i \psi_i(y) dy,$$
$$\sum_{i=1}^N t_i J_i = -\int_X (\sum_{i=1}^N t_i\psi_i(y)) W^\alpha(dy),$$

and $\sum_{i=1}^N t_i\psi_i(y) \in L^2(X)$, we only need to show

$$-\frac{1}{\varepsilon^{\alpha/2}} \int_X q_\varepsilon(y) f(y) dy \xrightarrow[\varepsilon\to 0]{\text{distribution}} -\int_X f(y) W^\alpha(dy) \tag{4.13}$$

for any $f \in L^2(X)$.

We prove this convergence in two steps: First, we show it holds when $q(x) = g(x)$, i.e., q is a centered stationary Gaussian field. Second, we generalize the result to the case when $q(x) = \Phi(g(x))$.

The Gaussian case. When $q(x) = g(x)$, the random variable $-\varepsilon^{-\alpha/2}\int_X q_\varepsilon(y)f(y)dy$ is centered, Gaussian, with variance $V_\varepsilon := \varepsilon^{-\alpha}\int_{X^2} R_g(\frac{y-z}{\varepsilon})f(y)f(z)dydz$, so it suffices to show

$$V_\varepsilon \longrightarrow \int_{X^2} \frac{\kappa_g f(y)f(z)}{|y-z|^\alpha} dydz =: \mathrm{Var}\left(-\int_X f(y)W^\alpha(dy)\right) \tag{4.14}$$

as $\varepsilon \to 0$. The equality above holds by the definition of our stochastic integral.

Since $R_g(x) \sim \kappa_g |x|^{-\alpha}$, for any $\delta > 0$, there exists an $M > 0$ so that $|x| > M$ implies $|R_g(x) - \kappa_g|x|^{-\alpha}| < \delta\kappa_g|x|^{-\alpha}$. According to this, we have

$$\Big| V_\varepsilon - \int_{X^2} \frac{\kappa_g f(y)f(z)}{|y-z|^\alpha} dydz \Big| \le \int_{|y-z|>M\varepsilon} \frac{\delta\kappa_g |f(y)f(z)|}{|y-z|^\alpha}\, dydz + \\ + \int_{|y-z|\le M\varepsilon} |f(y)f(z)| \left(\varepsilon^{-\alpha} + \frac{\kappa_g}{|y-z|^\alpha} \right) dydz := (I) + (II) + (III).$$

We've used the fact $\|R\|_\infty = 1$. It's easy to see that $(I) \le C\delta$, $(II) + (III) \le C\varepsilon^{d-\alpha}$. First let $\varepsilon \to 0$, then let $\delta \to 0$, we proved (4.14).

The case of function of Gaussian field. When $q(x) = \Phi(g(x))$, we claim that the random variable $\varepsilon^{-\alpha/2} \int_X q_\varepsilon(y) f(y) dy$ converges to $\varepsilon^{-\alpha/2} \int_X V_1 g_\varepsilon(y) f(y) dy$ in probability. Then (4.13) follows from this, the Gaussian case, and the fact $\kappa = \kappa_g V_1^2$.

To show the convergence in probability, we estimate the second moment as follows:

$$\begin{aligned} & \mathbb{E} \left(\frac{1}{\varepsilon^{\alpha/2}} \int_X (q_\varepsilon(y) - V_1 g_\varepsilon(y)) f(y) dy \right)^2 \\ = \;& \frac{1}{\varepsilon^\alpha} \int_{X^2} \mathbb{E}\{(q_\varepsilon(y) - V_1 g_\varepsilon(y))(q_\varepsilon(z) - V_1 g_\varepsilon(z))\} f(y) f(z) dydz. \end{aligned}$$

The expectation term inside the integral can be written as

$$R_\varepsilon(y-z) - V_1^2 (R_g)_\varepsilon(y-z) + V_1[V_1(R_g)_\varepsilon(y-z) - \mathbb{E}\{g_\varepsilon(y)q_\varepsilon(z)\}] \\ + V_1[(R_g)_\varepsilon(y-z) - \mathbb{E}\{g_\varepsilon(z)q_\varepsilon(y)\}].$$

Recall (3.3) of Lemma 3.1 to estimate these terms. We can bound the second moment above by

$$C\varepsilon^{-\alpha} \int_{|y-z|\le T\varepsilon} |f(y)f(z)| dydz + C\varepsilon^{-\alpha} \int_{|y-z|>T\varepsilon} \frac{\varepsilon^{2\alpha}|f(y)f(z)|}{|y-z|^{2\alpha}} dydz := (I) + (II).$$

Carry out the routine analysis we have developed for this type of integrals; it is easy to verify that $(I) \le C\varepsilon^{d-\alpha}$ and (II) is of order ε^α if $2\alpha < d$, of order $\varepsilon^\alpha |\log\varepsilon|$ if $2\alpha = d$, and of order $\varepsilon^{d-\alpha}$ if $2\alpha > d$. In all cases, we have $(I) + (II)$ converges to zero, which completes the proof. $\square$

According to the interpretation in Remark 2.4, the lemma above implies that $\mathcal{G}q_\varepsilon u_0$ converges to the limit in (2.9). The other terms in the stochastic corrector $u_\varepsilon - \mathbb{E}\{u_\varepsilon\}$ are controlled by Lemmas 3.2 and 4.2. These are sufficient to prove Theorem 2.3 as follows.

Proof of Theorem 2.3. Recall the expressions (3.9) for the corrector; we see its random part, i.e. $u_\varepsilon - \mathbb{E}\{u_\varepsilon\}$, can be decomposed as

$$-\mathcal{G}q_\varepsilon u_0 + (\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u_0 - \mathbb{E}\{\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u_0\}) + (\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon(u_\varepsilon - u_0) - \mathbb{E}\{\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon(u_\varepsilon - u_0)\}).$$

By (4.2), for any test function $\varphi \in \mathcal{C}(\overline{X})$, we have

$$\left\langle \frac{\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u_0 - \mathbb{E}\{\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u_0\}}{\varepsilon^{\alpha/2}}, \varphi \right\rangle \xrightarrow[\varepsilon\to 0]{\text{probability}} 0. \tag{4.15}$$

Recall estimate (3.12); apply (2.5) and (3.4) to it. We find that when $\alpha < 4\beta$, the size of $\mathbb{E}|\langle \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon (u_\varepsilon - u_0), \varphi\rangle|$ is much smaller than $\varepsilon^{\alpha/2}$, which implies

$$\left\langle \frac{\mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon (u_\varepsilon - u_0)}{\varepsilon^{\alpha/2}}, \varphi \right\rangle \xrightarrow[\varepsilon\to 0]{\text{probability}} 0. \tag{4.16}$$

The leading term in the random corrector is therefore $\langle -\mathcal{G}q_\varepsilon u_0, \varphi\rangle$.

Consider an arbitrary set of test functions $\{\varphi_i, 1 \le i \le N\}$. By the same argument above we can verify that the vectors $(Q_1^\varepsilon, \cdots, Q_N^\varepsilon)$, where

$$Q_i^\varepsilon := \varepsilon^{-\alpha/2} \langle \varphi_i, \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon u_0 + \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon (u_\varepsilon - u_0)\rangle,$$

converges in probability to zero vectors. On the other hand, by Lemma 4.3 and the fact that $u_0(y)\mathcal{G}\varphi(y) \in L^2(X)$, we verify that $(I_\varepsilon^i, \cdots, I_\varepsilon^N)$ converges in distribution to $(I_1, \cdots, I_N)$, where

$$I_\varepsilon^i := \varepsilon^{-\alpha/2}\langle \varphi_i, -\mathcal{G}q_\varepsilon u_0\rangle,$$

and $(I_1, \cdots, I_N)$ is the centered Gaussian with covariance matrix given by (2.10). Combining this convergence result with (4.15) and (4.16), we see that $(I_1^\varepsilon, \cdots, I_N^\varepsilon)$, where $I_i^\varepsilon := \varepsilon^{-\alpha/2}\langle u_\varepsilon - \mathbb{E}\{u_\varepsilon\}, \varphi_i\rangle$ as defined in Remark 2.4, converges in distribution to $(I_1, \cdots, I_N)$. This completes the proof. □

5 Corrector in one dimensional space

In this section, we restrict the dimension to be one. With further assumptions that the Green's function is Lipschitz continuous and the solution to (2.1) has continuous path, we derive a stronger convergence result of $u_\varepsilon - u_0$, in probability distribution in the space of continuous paths. The proof resembles and depends on [2] largely.

For simplicity, we assume that the solution to (2.1) has continuous path. This is the case for the first prototype, i.e., the steady diffusion problem, where solution belongs to $H_0^1(X) \subset \mathcal{C}(X)$. We also assume that the Green's function $G(x, y)$ is Lipschitz in x with Lipschitz constant uniform in y. Again, this is the case for the steady diffusion problem. However, it is not the case for the other prototype, i.e., the Robin boundary equation, where even in $1D$, the Green's function has logarithmic singularity. With these assumptions, we are going to characterize the limiting distribution of $(u_\varepsilon - u_0)/\varepsilon^{\alpha/2}$ in the space of continuous paths $\mathcal{C}(X)$.

Recall the decomposition in (3.9); we write

$$\frac{u_\varepsilon - u_0}{\varepsilon^{\alpha/2}}(x) = -\varepsilon^{-\alpha/2}\mathcal{G}q_\varepsilon u_0(x) + \varepsilon^{-\alpha/2}\mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon u_0(x) + \varepsilon^{-\alpha/2}\mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon (u_\varepsilon - u_0)(x). \tag{5.1}$$

We will call the first time on the right hand side $I_\varepsilon(x)$, the second term $Q_\varepsilon(x)$, and the third one $r_\varepsilon(x)$. We verify also that the sum of the last two terms is $\varepsilon^{-\alpha/2}\mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon u_\varepsilon(x)$, which we will call $Q^\varepsilon(x)$.

Our plan is as follows: First, we show that $I_\varepsilon(x)$ has the limiting distribution in $\mathcal{C}(X)$ as desired in (2.11); Second, we show that $Q^\varepsilon(x)$ converges in distribution $\mathcal{C}(X)$ to the zero function. Since the zero process is deterministic, the convergence in fact holds in probability [**?**, p.27]; the conclusion of Theorem 2.5 follows immediately.

To show convergence of $I_\varepsilon(x)$ and $Q^\varepsilon(x)$, we apply the following standard result on weak convergence of probability measures, whose proof can be found for instance in [**?**, p.64].

Proposition 5.1. *Suppose $\{M_\varepsilon\}_{\varepsilon\in(0,1)}$ is a family of random processes parametrized by $\varepsilon \in (0,1)$ with values in the space of continuous functions $\mathcal{C}$ and $M_\varepsilon(0) = 0$. Then M_ε converges in distribution to M_0 as $\varepsilon \to 0$ if the following holds:*

(i) (Finite-dimensional distributions) *for any $0 \le x_1 \le \cdots \le x_k \le 1$, the joint distribution of $(M_\varepsilon(x_1), \cdots, M_\varepsilon(x_k))$ converges to that of $(M_0(x_1), \cdots, M_0(x_k))$ as $\varepsilon \to 0$.*

(ii) (Tightness) *The family $\{M_\varepsilon\}_{\varepsilon\in(0,1)}$ is a tight sequence of random processes in $\mathcal{C}(X)$. A sufficient condition is the Kolmogorov criterion: $\exists \delta, \beta, C > 0$ such that*

$$\mathbb{E}\left\{|M_\varepsilon(s) - M_\varepsilon(t)|^\beta\right\} \le C|t-s|^{1+\delta}, \tag{5.2}$$

uniformly in ε and $t, s \in (0,1)$.

Proof of Theorem 2.5. We carry out the aforementioned two-step plan. Let us denote by $I(x)$ the Gaussian process on the right hand side of (2.11).

Convergence of $I_\varepsilon(x)$ to $I(x)$. We first show convergence of finite dimensional distributions. Fix an arbitrary natural number N, an N-tuple $(x_1, \cdots, x_N)$, we need to show that the joint law of $(I_\varepsilon(x_1), \cdots, I_\varepsilon(x_N))$ converges to that of $(I(x_1), \cdots, I(x_N))$. It suffices to show that for arbitrary N-tuple $(\xi_1, \cdots \xi_N) \in \mathbb{R}^N$, we have

$$\sum_{i=1}^N \xi_i I_\varepsilon(x_i) \xrightarrow[\varepsilon\to 0]{\text{distribution}} \sum_{i=1}^N \xi_i I(x_i),$$

as convergence in distribution of random variables. Recalling the exact form of I_ε and I; our goal is to show, with $\sigma_H := \sqrt{\kappa/(H(2H-1))}$,

$$\frac{1}{\varepsilon^{\alpha/2}} \int_X \sum_{i=1}^N \xi_i G(x_i, y) q_\varepsilon(y) dy \xrightarrow[\varepsilon\to 0]{\text{distribution}} \sigma_H \int_X \sum_{i=1}^N \xi_i G(x_i, y) u_0(y) dW_H(y). \tag{5.3}$$

Set $F_x(y) = \sum_{i=1}^N \xi_i G(x_i, y) u_0(y)$. We verify that $F_x \in L^1 \cap L^\infty(\mathbb{R})$ and apply the following convergence result:

$$\frac{1}{\varepsilon^{\alpha/2}} \int_X F(y) q_\varepsilon(y) dy \xrightarrow[\varepsilon\to 0]{\text{distribution}} \sigma_H \int_X F(y) dW_H(y), \quad \text{for } F \in L^1 \cap L^\infty, \tag{5.4}$$

which is Theorem 3.1 of [2]. This proves the convergence of finite dimensional distributions.

To show tightness of $I_\varepsilon(x)$, we calculate $\mathbb{E}|I_\varepsilon(x) - I_\varepsilon(y)|^2$ which we denote by J_1. Calculation shows:

$$\begin{aligned} J_1 &= \frac{1}{\varepsilon^\alpha} \mathbb{E}\left(\int_X [G(x,z) - G(y,z)] q_\varepsilon(z) u_0(z) dz\right)^2 \\ &= \frac{1}{\varepsilon^\alpha} \int_{X^2} [G(x,z) - G(y,z)][G(x,\xi) - G(y,\xi)] R_\varepsilon(z-\xi) u_0(z) u_0(\xi) dz d\xi. \end{aligned}$$

Use the assumption on the Lipschitz continuity of G to obtain

$$J_1 \le (\mathrm{Lip}G)^2|x-y|^2\frac{1}{\varepsilon^\alpha}\int_X |R_\varepsilon(z-\xi)u_0(z)u_0(\xi)|dzd\xi \le C|x-y|^2. \tag{5.5}$$

We used the fact that the integral above has size ε^α, which can be easily proved as before. This shows tightness and complete the first step.

Convergence of $Q^\varepsilon(x)$ to zero function. For convergence of finite distribution, we can show that $\sum_{i=1}^N \xi_i Q^\varepsilon(x_i)$ converges to zero in $L^2(\Omega,\mathbb{P})$ which is stronger. Since we can group $\sum_{i=1}^N \xi_i G(x_i,y)$ together as in (5.3), it suffices to show $\sup_{x\in X}\mathbb{E}|Q^\varepsilon(x)|\to 0$.

We prove this by showing $\sup_{x\in X}\mathbb{E}|Q_\varepsilon(x)|^2\to 0$ and $\sup_{x\in X}\mathbb{E}|r_\varepsilon(x)|\to 0$. The first term, i.e., $\mathbb{E}|Q_\varepsilon(x)|^2$, has the following expression,

$$\varepsilon^{-\alpha}\int_{X^4} G(x,y)G(y,z)G(x,\xi)G(\xi,\eta)u_0(z)u_0(\eta)\mathbb{E}\{q_\varepsilon(y)q_\varepsilon(z)q_\varepsilon(\xi)q_\varepsilon(\eta)\}d\xi d\eta dz dy. \tag{5.6}$$

Crudely bound the Green's functions and u_0 by their uniform norms. Then apply Lemma A.2 to get

$$\mathbb{E}|Q_\varepsilon(x)|^2 \le C\varepsilon^{-\alpha}\|G\|_\infty^4\|u_0\|_\infty^2\int_{X^4}\sum_p |R_\varepsilon(x_{p(1)}-x_{p(2)})R_\varepsilon(x_{p(3)}-x_{p(4)})|. \tag{5.7}$$

This time p runs over all 15 possible ways to choose two pairs from $\{1,2,3,4\}$. Since R_ε is bounded by $C\varepsilon^\alpha|x|^{-\alpha}$. We verify each item in the sum has a contribution of size $\varepsilon^{2\alpha}$ and so does the sum. Consequently, $\mathbb{E}|Q_\varepsilon(x)|^2\le C\varepsilon^\alpha$ and converges to zero uniformly in x.

For $r_\varepsilon(x)$, we use Cauchy-Schwarz to get

$$|r_\varepsilon(x)| \le \varepsilon^{-\frac{\alpha}{2}}\left(\int_X |q_\varepsilon(z)(u_\varepsilon-u_0)(z)|^2dz\right)^{\frac12}\left(\int_X\left(\int_X G(x,y)q_\varepsilon(y)G(y,z)dy\right)^2dz\right)^{\frac12}.$$

Bound q_ε in the first integral by its uniform norm. Take expectation afterwards; we verify that $\mathbb{E}|r_\varepsilon(x)|$ is bounded by

$$C\varepsilon^{-\frac{\alpha}{2}}\left(\mathbb{E}\|u_\varepsilon-u_0\|^2\right)^{\frac12}\left(\mathbb{E}\int_{X^3}G(x,y)G(y,z)G(x,\xi)G(\xi,z)q_\varepsilon(y)q_\varepsilon(\xi)dyd\xi dz\right)^{\frac12}.$$

The integral above can be estimated as before and is of size ε^α. Expectation of $\|u_\varepsilon-u_0\|^2$ is also of size α as shown before. As a result, $\mathbb{E}|r_\varepsilon(x)|\le C\varepsilon^\alpha$ and converges to zero uniformly with respect to x.

It suffices now to prove tightness of $Q^\varepsilon(x)$. To this end, we calculate $\mathbb{E}|Q^\varepsilon(x)-Q^\varepsilon(y)|^2$ which we denote by J_2.

$$J_2 = \mathbb{E}\left(\varepsilon^{-\frac{\alpha}{2}}\int_{X^2}[G(x,z)-G(y,z)]q_\varepsilon(z)G(z,\xi)q_\varepsilon(\xi)u_\varepsilon(\xi)d\xi dz\right)^2.$$

Use Cauchy-Schwarz and the uniform bound on q_ε; we get

$$J_2 \le \varepsilon^{-\alpha}\mathbb{E}\left\{(\|q\|_\infty\|u_\varepsilon\|)^2\int_X\left(\int_X[G(x,z)-G(y,z)]q_\varepsilon(z)G(z,\xi)dz\right)^2d\xi\right\}.$$

The term $\|u_\varepsilon\|$ can be bounded uniformly with respect to ω because the operator norm of $\mathcal{G}_\varepsilon$ is. Therefore, we have

$$J_2 \le C\mathbb{E}\int_{X^3}[G(x,z)-G(y,z)][G(x,\eta)-G(y,\eta)]q_\varepsilon(z)q_\varepsilon(\eta)G(z,\xi)G(\eta,\xi)dzd\eta d\xi.$$

Use the Lipschitz continuity and the uniform bound of G to get

$$J_2 \le C\varepsilon^{-\alpha}\int_{X^3}(\mathrm{Lip}G)^2|x-y|^2R_\varepsilon(z-\eta)\|G\|_\infty^2 dzd\eta d\xi \le C|x-y|^2. \tag{5.8}$$

The second inequality holds because the integral is of size ε^α as we have seen many times. This completes the proof of Q^ε converging to zero functions. Recall the argument above Proposition 5.1 to complete the proof of the theorem. □

Remark 5.2. We assumed that the random field $q(x)$ satisfies (A3) to take advantage of Lemma A.2. However, this assumption is not necessary for Theorem 2.5 to hold. Indeed, with just (A1) and (A2) we can derive asymptotic behavior of the fourth order moment $\mathbb{E}\{q(x_1)q(x_2)q(x_3)q(x_4)\}$ when the four points are mutually far away from each other. We can use this fact to estimate (5.6) instead. The argument involves routine decomposition of integration domains, which is tedious so we omit it here.

6 Conclusions and further discussions

We considered the deterministic corrector and the stochastic corrector for equation (2.1), where the coefficient in the potential term is constructed as a function of a long-range correlated Gaussian random field. We found that the stochastic corrector has magnitude $\varepsilon^{\alpha/2}$ and its limiting distribution can be characterized by a Gaussian random process in some weak sense. The deterministic corrector, however, may exceed the stochastic corrector in terms of size. We find that the threshold for this to happen is $\alpha = \beta$.

In our analysis, we assumed that the Green's function $G(x,y)$ has singularity of the type $|x-y|^{-(d-\beta)}$ near the origin, i.e., the diagonal $x = y$. However, other types of singularities, such as $G(x,y) \sim \log|x-y|$, can also be analyzed following our technique. In fact, for logarithmic singularity, which indeed is the case for the steady diffusion problem when $d = 2$ and the Robin boundary equation when $d = 1$, our result still holds. Since logarithmic singularity is much smoother (on bounded domain) than the general potentials we considered, the deterministic corrector is of order ε^α while the stochastic corrector has magnitude $\varepsilon^{\alpha/2}$.

When proving convergence in distribution of the stochastic corrector, we assumed $\alpha < 4\beta$, due to a technical reason that only in this case the estimate (3.12) is enough to control the remainder term in (3.10). However, this is not a real problem if we can estimate sufficiently high-order moments of $q(x)$. So far, we only used fourth moments. Suppose we have a good estimate on the six-moments, we could perform one more iteration on (3.10) to get

$$\begin{aligned}\langle u_\varepsilon - u_0, \varphi\rangle = -\langle \mathcal{G}q_\varepsilon u_0, \varphi\rangle + \langle \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u_0, \varphi\rangle - \langle \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u_0, \varphi\rangle \\ - \langle \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon(u_\varepsilon - u_0), \varphi\rangle.\end{aligned}$$

Suppose that the six-moments estimate is sufficiently accurate to control the variance of the third item on the right, and that the estimate on four-moments is sufficient to control the remainder term. Then we expect the same result in Theorem 2.3 to hold for some larger range of α, which eventually include the whole spectrum $(0, d)$ that α can belong.

A Two useful lemmas

A.1 Estimates of convolution of potentials

Here, we record a lemma which estimates the convolution of two potential functions, or the convolution of a potential function with a logarithmic function. Its proof can be found in the appendix of [3].

Lemma A.1. *Let X be an open and bounded subset in $\mathbb{R}^d$, and $x \neq y$ two points in X. Let α, β be positive numbers in $(0, d)$. We have the following convolution results.*

$$\int_X \frac{1}{|z-x|^\alpha} \cdot \frac{1}{|z-y|^\beta} dz \le \begin{cases} C|x-y|^{d-(\alpha+\beta)}, & \alpha+\beta > d, \\ C(\log|x-y|+1), & \alpha+\beta = d, \\ C, & \alpha+\beta < d. \end{cases} \tag{A.1}$$

The convolution of logarithms with a weak singular potential turns out to be finite as follows:

$$\int_X |\log|z-x|| \frac{1}{|z-y|^\alpha} dz \le C. \tag{A.2}$$

A.2 Fourth-order moments of $q(x, \omega)$

The following lemma provides a non-asymptotic estimate of the four-moments of $q(x)$ constructed in (A1-A2), with the additional assumption (A3). In the following, we set $F = \{1, 2, 3, 4\}$ and denote by $\mathcal{U}$ the collections of two pairs of unordered numbers in F, i.e.,

$$\mathcal{U} = \big\{p = \{\big(p(1), p(2)\big), \big(p(3), p(4)\big)\} \mid p(i) \in F, p(1) \neq p(2), p(3) \neq p(4)\big\}. \tag{A.3}$$

As members in a set, the pairs $(p(1), p(2))$ and $(p(3), p(4))$ are required to be distinct; however, they can have one common index. There are three elements in $\mathcal{U}$ whose indices $p(i)$ are all different. They are precisely $\{(1, 2), (3, 4)\}$, $\{(1, 3), (2, 4)\}$ and $\{(1, 4), (2, 3)\}$. Let us denote by $\mathcal{U}_*$ the subset formed by these three elements, and its complement by $\mathcal{U}^*$.

Lemma A.2. *Let $q(x, \omega)$ be the random field constructed in* (A1)-(A3)*. Fix four arbitrary points $\{x_i \in \mathbb{R}^d;\ 1 \le i \le 4\}$. Then we have the following.*

$$\begin{aligned} &\left| \mathbb{E} \prod_{i=1}^4 q(x_i) - \sum_{p \in \mathcal{U}_*} R(x_{p(1)} - x_{p(2)}) R(x_{p(3)} - x_{p(4)}) \right| \\ \le& C \sum_{p \in \mathcal{U}^*} R(x_{p(1)} - x_{p(2)}) R(x_{p(3)} - x_{p(4)}). \end{aligned} \tag{A.4}$$

The constant C is the one in (2.8) raised to the fourth power.

For a proof of this lemma, we refer the reader to [4, Proposition 4.1].

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