Probability Theory and Related Fields. Manuscript-Nr.
(will be inserted by hand later)

Stochastic invariant imbedding

Application to stochastic differential equations with boundary conditions

Josselin Garnier

Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau Cedex, France.
e-mail : garnier@paris.polytechnique.fr

September 22, 1997

Summary. We study stochastic differential equations of the type :

$$dx_t = f(t, x_t)dt + \sum_{k=1}^{d} \sigma^k(t, x_t) \circ dw_t^k, \quad x \in \mathbb{R}^d, \quad t \in [0, T_0].$$

Instead of the customary initial value problem, where the initial value x_0 is fixed, we impose an affine boundary condition :

$$h_0 x_0 + h_1 x_{T_0} = v_0,$$

where h_0, h_1 are deterministic matrices and v_0 is a fixed vector. Our main aim is to prove existence and uniqueness results for such anticipating stochastic differential equations.

Mathematics Subject Classification (1991): 34K10, 60H10.

1. Introduction

Stochastic integrals with anticipating integrands have been studied for several years. As a result, usual stochastic calculus on Itô's integrals has been extended to some class of integrals where the integrated term is not adapted to the driving Brownian motion (see in particular [11]). Therefore it has been possible to deal with some stochastic differential equations containing parameters which anticipate the Brownian motion.

The most investigated problem is the study of the solution of a stochastic differential equation of the form :

$$dX(t) = f(t, X(t))dt + \sum_{k=1}^{d} \sigma^k(t, X(t)) \circ dw_t^k, \qquad X \in \mathbb{R}^d, \tag{1}$$

with a boundary condition, which means that we impose that the solution X should satisfy a condition which involves both $X(0)$ and $X(1)$:

$$F(X(0), X(1)) = 0, \tag{2}$$

instead of the classical initial condition where only the value of X_0 is fixed.

Literature contains some results about such equations : Ocone and Pardoux [13] have succeeded in developing a theory for the linear case (i.e. the case where $f(t,x) = Ax + a$, $\sigma(t,x) = Bx + b$ and $F(x_0, x_1) = H_0 x_0 + H_1 x_1 - V_0$). Nualart and Pardoux [12] have presented existence and uniqueness theorems for the case $\sigma \equiv B$. Donati-Martin [5] has dealt with the one-dimensional case. Zeitouni and Dembo [4] have proved existence results for a class of equations by using optimal control techniques. Finally Alabert et al. [1] have discussed the Markov field property for solutions of one-dimensional stochastic differential equations with a boundary condition of the form $x_0 = \psi(x_1)$, $\psi' \leq 0$.

It appears that the problem is quite difficult in the deterministic case (i.e. when $\sigma \equiv 0$). Indeed, it seems hardly to found a general theory which could unify the results. We only find some particular results in literature, which apply to some classes of problems. Our aim is to develop a stochastic version of one of these methods, called invariant imbedding. This method can be applied to differential equations with affine boundary conditions :

$$H_0 X(0) + H_1 X(1) = V_0, \tag{3}$$

where H_0 and H_1 are fixed matrices and V_0 is a fixed vector. It produces interesting results concerning some deterministic problems, so we translate this technique to a class of stochastic equations of the form (1) and (3). We shall see that this extension requires some smoothness in the Malliavin sense of the integrands.

The paper is organized as follows. First we present in preliminaries the deterministic invariant imbedding method, and we review some results of Nualart and Pardoux [11] about the definition and the stochastic calculus associated with the generalized Stratonovich integral. In Section 3 we develop a stochastic invariant imbedding method and give general conditions for existence and uniqueness of solutions. Section 4 is devoted to some examples of application of our existence and uniqueness result. Finally we study in Section 5 a diffusion approximation problem with boundary conditions, whose limiting process is identified as the unique solution of a stochastic boundary value problem.

2. Preliminary results

2.1. Deterministic invariant imbedding method

Here we shall outline briefly the deterministic version of the invariant imbedding method. Fore more detail we refer to [2]. Let us consider the dynamical system, that will be denoted by (SBC) henceforth :

$$\frac{dX}{dt}(t) = F(t, X(t)),$$

with the boundary condition $H_0 X(0) + H_1 X(1) = V_0$, where H_0 and H_1 are fixed matrices, V_0 is a fixed vector, and F is a smooth vector field.

The invariant imbedding method consists in regarding the size of the interval τ defined by the two endpoints of the boundary condition and the fixed value v of the boundary condition as variables instead of parameters. Therefore we consider X as a function which depends on three variables t, τ and v and we write $X = X(t, \tau, v)$ from now on. Moreover we introduce a new unknown variable, which is the value at the final point of the solution $R(\tau, v) = X(\tau, \tau, v)$.

The result of the invariant imbedding theory which is of a great interest for us is the transformation of the boundary value problem into a family of problems with initial conditions. If we resume Chapter 2 in [2], then we have under nice regularity assumptions :

Proposition 1. *If the couple $(R(\tau, v), X(t, \tau, v))$ satisfies the following system, denoted by (SIC) :*

$$\frac{\partial R_i}{\partial \tau} + \sum_{k,l=1}^{d} \frac{\partial R_i}{\partial v_k} H_{1\,kl} F_l(\tau, R(\tau, v)) = F_i(\tau, R(\tau, v)), \tag{4}$$

$$\textit{starting from } \tau = 0 : R(0, v) = (H_0 + H_1)^{-1} v, \tag{5}$$

$$\frac{\partial X_i}{\partial \tau} + \sum_{k,l=1}^{d} \frac{\partial X_i}{\partial v_k} H_{1\,kl} F_l(\tau, R(\tau, v)) = 0, \tag{6}$$

$$\textit{starting from } \tau = t : X(t, t, v) = R(t, v), \tag{7}$$

then $X(t, 1, V_0)$ is a solution of system (SBC).

The system (SIC) consists of two coupled problems with initial conditions. An important fact to be noticed is that (6) is a linear partial differential equation. As a consequence it possesses a solution as soon as (4) admits a solution. To sum up, if there exists a function $R(\tau, v)$ solution of the differential equation (4) with the initial condition (5), then there exists a solution $X(t)$ of the boundary value problem (SBC).

2.2. Characteristics and invariant imbedding

The problem of integrating (4) can be reduced to the following characteristic system of ordinary differential equations :

$$\begin{cases} \dfrac{d\tilde{R}}{d\tau}(\tau, v) = F(\tau, \tilde{R}(\tau, v)), & \tilde{R}(0, v) = (H_0 + H_1)^{-1} v, \\ \dfrac{d\tilde{V}}{d\tau}(\tau, v) = H_1 F(\tau, \tilde{R}(\tau, v)), & \tilde{V}(0, v) = v. \end{cases}$$

If $\tilde{V}(\tau, v)$ is invertible with respect to v for every $\tau \in [0, 1]$, then the function $R(\tau, v)$ defined by $\tilde{R}(\tau, \tilde{V}^{-1}(\tau, v))$ is the unique solution of (4). Further, if we set $X(t, \tau, v) = \tilde{R}(t, \tilde{V}^{-1}(\tau, v))$, then the couple $(R(\tau, v), X(t, \tau, v))$ is the unique solution of the system (SIC). Therefore $X(t, 1, V_0)$ is a solution of the system (SBC).

2.3. Generalized Stratonovich integral

Let us return to the stochastic differential equation (1). First of all we have to define the stochastic integral $\int_0^t \sigma^k(s,X(s)) \circ dw_s^k$ which appears in the S.D.E.. Indeed we cannot expect to obtain a solution X which is adapted to the driving Brownian motion because of the boundary condition. The stochastic integral $\int_0^t \sigma^k(s,X(s)) \circ dw_s^k$ cannot be understood using the usual definitions of stochastic integrals according to Itô or Stratonovich. We shall use a new approach of the stochastic integral which has been developed by Skorohod first. What is interesting for us is that no restriction about the adaptness of the integrand is presupposed. However that generality has a price, which is the substitution of some smoothness requirement (in the sense of the derivation on Wiener space) for the adaptedness requirement. Therefore we will have to prove some smoothness properties for the processes that will be supplied by the invariant imbedding method, so that we will be able to give sense to the stochastic integrals.

In this section we present the generalized Stratonovich integral and its associated stochastic calculus. For more detail we refer to Nualart-Pardoux [11]. Throughout this subsection and the remainder of the paper we work on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω denotes the space of all continuous functions in $\mathcal{C}^0([0,1], \mathbb{R}^d)$ equipped with the topology generated by the sup norm, $\mathcal{F}$ is the Borel σ-algebra of Ω and $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$.

Definition 1. *A real valued process $\{u_t; t \in [0,1]\}$ is said to be Stratonovich integrable if for any $i = 1, \ldots, d$ and for any $t \in [0,1]$, the sequence $\{\xi_n^i(t); n \in \mathbb{N}\}$ defined by :*

$$\xi_n^i(t) = \sum_{l=0}^{2^n-1} \frac{w^i_{t_n^{l+1}\wedge t} - w^i_{t_n^l \wedge t}}{t_n^{l+1} - t_n^l} \int_{t_n^l}^{t_n^{l+1}} u_s ds, \quad t_n^l = l2^{-n},$$

converges in probability as n tends to infinity.
In that case, the limit will be denoted by :

$$\int_0^t u_s \circ dw_s^i.$$

We review the notion of derivation on Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$, which will be needed below.
$\mathbf{S}$ denotes the set of all random variables of the type : $F = f(w_{t_1}^{i_1}, \ldots, w_{t_n}^{i_n})$, where $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, $0 \leq t_1 < \ldots < t_n \leq 1$ and $i_1, \ldots, i_n \in \{1, \ldots, d\}$.
If $F \in \mathbf{S}$, we define its derivative in the i-th direction as the process $(D_t^i(F))_{t\in[0,1]}$ given by : $D_t^i(F) = \sum_{l; i_l = i} \frac{\partial f}{\partial x_l}(w_{t_1}^{i_1}, \ldots, w_{t_n}^{i_n}) \mathbb{1}_{[0,t_l]}(t)$.

Spaces $\mathbb{D}^{1,p}$, $p \geq 1$.
$\mathbb{D}^{1,p}$ denotes the completion of $\mathbf{S}$ with respect to the norm :

$$\|F\|_{1,p} = \|F\|_p + \| \|DF\|_{HS} \|_p,$$

where $\|.\|_p$ denotes the $L^p(\Omega)$ norm and $\|DF\|_{HS}^2 = \sum_{j=1}^{d} \int_0^1 (D_t^j F)^2 dt$.

Spaces $\mathbb{L}^{1,p}$ *and* $\mathbb{L}_C^{1,p}$, $p \geq 1$.
$\mathbb{L}^{1,p} = L^p([0,1], \mathbb{D}^{1,p})$. $\mathbb{L}_C^{1,p}$ denotes the set of all random processes $(u_t)_{t\in[0,1]}$ which satisfy :
1. $u \in \mathbb{L}^{1,p}$,
2. $s \mapsto D_t u_s$ is continuous with values in $L^p(\Omega)$, both on $(0,t)$ and on $(t,1)$, uniformly with respect to t.
3. $\operatorname{ess\,sup}_{(s,t)\in(0,1)^2} \mathbb{E}[|D_s u_t|^p] < \infty$.

Localization.
$\mathbb{D}_{loc}^{1,p}$ denotes the set of all random variables F which are such that there exists a sequence $\{(\Omega_n, F_n); n \in \mathbb{N}\} \subset \mathcal{F} \times \mathbb{D}^{1,p}$ with the two following properties :
1. $\Omega_n \nearrow \Omega$ when $n \to \infty$,
2. $F = F_n$ almost surely on Ω_n.
We say that (Ω_n, F_n) localizes F, and $D_t F$ is clearly defined by : $D_t F = D_t F_n$ on Ω_n, $n \in \mathbb{N}$. Moreover we introduce the notion of localizer : A sequence of random variables β_n is said to localize a process $u \in \mathbb{L}_{loc}^{1,p}$ if
1. β_n belongs to $\mathbb{D}^{1,q}$ for any $q \geq 2$,
2. $\{\beta_n = 1\} \nearrow \Omega$ as n tends to infinity almost surely,
3. $\beta_n u$ belongs to $\mathbb{L}^{1,p}$.
An important result is the following : If $u \in \mathbb{L}_{C,loc}^{1,2}$, then u is Stratonovich integrable.

3. Stochastic invariant imbedding method

3.1. Formulation and notations

Throughout the paper we shall use the convention of summation upon repeated indices. A process $(X(t))_{t\in[0,1]}$ is said to be a solution of the stochastic differential equation

$$dX(t) = f(t, X(t))dt + \sigma^k(t, X(t)) \circ dw_t^k \tag{8}$$

if $\sigma^k(., X(.))$ is Stratonovich integrable and :

$$X(t) - X(0) = \int_0^t f(s, X(s))ds + \int_0^t \sigma^k(s, X(s)) \circ dw_s^k \quad \text{a.s..}$$

If in addition $(X(0), X(1))$ satisfies almost surely

$$H_0 X(0) + H_1 X(1) = V_0, \tag{9}$$

where H_0, H_1 are fixed matrices and V_0 is a fixed vector, then we say that the process X is solution of the stochastic differential equation (8) with the boundary condition (9).

We aim at applying the invariant imbedding method which has just been presented to the stochastic case. Let us introduce the characteristic processes $\tilde{R}$ and $\tilde{V}$ defined by :

$$\begin{cases} d\tilde{R} = f(\tau,\tilde{R})d\tau + \sigma^k(\tau,\tilde{R}) \circ dw_\tau^k, \\ \tilde{R}(0,v) = (H_0+H_1)^{-1}v, \end{cases} \tag{10}$$

$$\begin{cases} d\tilde{V} = H_1 f(\tau,\tilde{R})d\tau + H_1\sigma^k(\tau,\tilde{R}) \circ dw_\tau^k, \\ \tilde{V}(0,v) = v. \end{cases} \tag{11}$$

We can rewrite these S.D.E.'s with Itô's integrals instead of Stratonovich's integrals if we replace f by $\hat{f} = f + \frac{1}{2}\frac{\partial \sigma^k}{\partial x_j}\sigma_j^k$. It comes immediately that :

$$\tilde{V}(\tau,v) = H_1\tilde{R}(\tau,v) + H_0(H_0+H_1)^{-1}v. \tag{12}$$

We introduce the random matrices $\{Y_{i,j} = \frac{\partial \tilde{R}_i}{\partial v_j}; i,j = 1,..,d\}$ and $\{Z_{i,j} = \frac{\partial \tilde{V}_i}{\partial v_j}; i,j = 1,..,d\}$. They are related by :

$$Z(\tau,v) = H_1 Y(\tau,v) + H_0(H_0+H_1)^{-1}. \tag{13}$$

We denote by $\mathcal{H}1$ and $\mathcal{H}2$ the following conditions :

$\mathcal{H}1$ $\hat{f}$ and σ^k are functions from $[0,1]\times \mathbb{R}^d$ into $\mathbb{R}^d$ which are
- continuous with respect to (t,x),
- five times differentiable with respect to x, with uniformly bounded and continuous partial derivatives.

$\mathcal{H}2$ There exists some $\alpha > 0$ such that
$|\det Z(\tau,v)| \geq \alpha$ for every $(\tau,v) \in [0,1]\times \mathbb{R}^d$ almost surely.

Proposition 2. *Under $\mathcal{H}1$ and $\mathcal{H}2$, there exists a unique adapted process $\bar{V}$ solution of :*

$$\begin{cases} d\bar{V} = a(\tau,\bar{V}(\tau))d\tau + b^k(\tau,\bar{V}(\tau)) \circ dw_\tau^k, \\ \bar{V}(0) = V_0, \end{cases} \tag{14}$$

where $a(\tau,v) = -Z^{-1}H_1 f(\tau,\tilde{R}(\tau,v))$, $b^k(\tau,v) = -Z^{-1}H_1\sigma^k(\tau,\tilde{R}(\tau,v))$.
Moreover
1. $\bar{V}$ is well-defined on $[0,1]$ almost surely,
2. $\bar{V}$ is the inverse of $\tilde{V}$, that means that almost surely, for every τ, $\bar{V}(\tau)$ is the unique element of $\mathbb{R}^d$ such that :

$$\tilde{V}(\tau,\bar{V}(\tau)) = V_0. \tag{15}$$

3. $\bar{V} \in \mathbb{L}^{1,p}_{loc}$ for every $p \geq 2$.

Proof . The proof of the first two points relies strongly on Lemma 6-1-1 [9] whose key argument is the implicit function theorem. It claims that (14) possesses a unique local solution, which is the inverse $\tilde{V}^{-1}(.,V_0)$ until time $T(V_0)$, defined by $T(V_0) = \inf\{\tau \text{ such that } V_0 \in \tilde{V}(\tau,A_\tau)\}$ ($=\infty$ if $\{\ldots\} = \emptyset$), where A_τ is the random subset of $\mathbb{R}^d$ given by : $A_\tau = \{y \text{ s.t. } \exists s < \tau,\ \det Z(s,y) = 0\}$. However Hypothesis $\mathcal{H}2$ yields that $\det Z(\tau,v)$ is never 0 for every (τ,v) almost surely, which implies that $A_\tau = \emptyset$. As a consequence, $T(V_0) = \infty$ almost surely.

The proof of the third point is rather technical and is deferred to the appendix. □

3.2. Existence

In this subsection we assume that $\mathcal{H}1$ and $\mathcal{H}2$ are fulfilled.

Definition 2. *We introduce the new processes ($t \leq \tau$) :*

$$R(\tau) = \tilde{R}(\tau, \bar{V}(\tau)), \qquad X(t,\tau) = \tilde{R}(t, \bar{V}(\tau))$$

Lemma 1. *R and X can be expanded according to :*

$$\begin{aligned} R(\tau) = \quad & (H_0+H_1)^{-1}V_0 + \int_0^\tau \left(I_d - YZ^{-1}\right)(s,\bar{V}(s))H_1 f(s,R(s))ds \\ & + \int_0^\tau \left(I_d - YZ^{-1}\right)(s,\bar{V}(s))H_1\sigma^k(s,R(s)) \circ dw_s^k, \end{aligned}$$

$$\begin{aligned} X(0,\tau) = \quad & (H_0+H_1)^{-1}V_0 - \int_0^\tau (H_0+H_1)^{-1}Z^{-1}(s,\bar{V}(s))H_1 f(s,R(s))ds \\ & - \int_0^\tau (H_0+H_1)^{-1}Z^{-1}(s,\bar{V}(s))H_1\sigma^k(s,R(s)) \circ dw_s^k. \end{aligned}$$

As a consequence $H_0X(0,\tau) + H_1R(\tau) = V_0$ for every τ almost surely.

Proof . We can get the expansion of R from the generalized Itô's formula (see Theorem 2-3 [3]) if we apply it to $\tilde{R}(\tau, v)$ and $\bar{V}(\tau)$:

$$dR(\tau) = d\tilde{R}(\tau,v)|_{v=\bar{V}(\tau)} + Y(\tau,\bar{V}(\tau)) \circ d\bar{V}(\tau).$$

The expansion of X is obvious since $X(0,\tau) = (H_0+H_1)^{-1}\bar{V}(\tau)$.
Finally we add the first equation multiplied by H_1 to the second one multiplied by H_0 and we use (13), so that we obtain the last relation. □

Theorem 1. *The following assertions hold :*
- *$X(.,1) \in \mathbb{L}^{1,p}_{C,loc}$ for every $p \geq 2$,*
- *almost surely, for every $t \leq 1$:*

$$X(t,1) - X(0,1) = \int_0^t f(s,X(s,1))ds + \sigma^k(s,X(s,1)) \circ dw_s^k. \qquad (16)$$

- *almost surely, $H_0X(0,1) + H_1X(1,1) = V_0$.*

Proof . The first statement follows readily from Lemma 2-2-3 [14]. We have in particular :

$$D_\nu^i \tilde{R}(t,\bar{V}(1)) = (D_\nu^i \tilde{R})(t,\bar{V}(1)) + Y(t,\bar{V}(1))D_\nu^i\bar{V}(1).$$

By Proposition 7-8 [11], $\sigma^k(.,\tilde{R}(.,\bar{V}(1)))$ is Stratonovich integrable and :

$$\int_0^t \sigma^k(s,\tilde{R}(s,v)) \circ dw_s^k \,|_{v=\bar{V}(1)} = \int_0^t \sigma^k(s,\tilde{R}(s,\bar{V}(1))) \circ dw_s^k.$$

Combining with (10) yields (16). Finally the boundary condition is satisfied almost surely according to the third equality of Lemma 1. □

3.3. Uniqueness

In this subsection, we still assume that $\mathcal{H}1$ and $\mathcal{H}2$ are fulfilled.
By Lemmas 2-2-1 and 2-2-2 [14], $v \mapsto \tilde{R}(\tau, v)$ admits an inverse $v \mapsto \tilde{R}^{-1}(\tau, v)$ which satisfies :

$$\begin{aligned}\tilde{R}^{-1}(\tau, v) = \quad & (H_0 + H_1)v \\ & - \int_0^\tau U_s(\tilde{R}^{-1}(s,v)) f(s,v) ds - \int_0^\tau U_s(\tilde{R}^{-1}(s,v)) \sigma^k(s,v) \circ dw_s^k,\end{aligned}$$

where $U_s(v) = Y^{-1}(s,v)$. The proof of the following proposition relies strongly on this Itô's formula.

Theorem 2. *If X belongs to $\mathbb{L}^{1,8}_{C,loc}$ and is a solution with continuous sample paths of the problem (8) with the boundary condition (9), then, almost surely,*

$$X_. = \tilde{R}(., \tilde{V}^{-1}(1, V_0)).$$

Proof . The key argument is the generalized Itô-Ventzell formula proved by Ocone and Pardoux in [14]. We begin by checking that we can apply it to $\tilde{R}^{-1}(\tau, v)$ and $X(\tau)$. Under the above assumptions on X, we have
1. $s \mapsto f(s, X(s)) \in L^2([0,1], \mathbb{R}^d)$ almost surely, since it belongs to $\mathcal{C}^0([0,1], \mathbb{R}^d)$,
2. $s \mapsto \sigma^k(s, X(s)) \in \mathbb{L}^{1,8}_{C,loc}$,
3. $\tilde{R}^{-1}(\tau, v)$ fulfils all the required conditions of Theorem 1-4-1 [14] by Lemmas 2-2-1 and 2-2-2 [14].
As a consequence we can apply a localized version of the generalized Itô-Ventzell formula (we do not explicit the straightforward localization procedure) :

$$\begin{aligned}\tilde{R}^{-1}(\tau, X(\tau)) &= (H_0 + H_1) X(0) \\ - \int_0^\tau U_s(\tilde{R}^{-1}(s, X(s))) f(s, X(s)) ds &- \int_0^\tau U_s(\tilde{R}^{-1}(s, X(s))) \sigma^k(s, X(s)) \circ dw_s^k \\ + \int_0^\tau \frac{\partial \tilde{R}^{-1}}{\partial v}(s, X(s)) f(s, X(s)) ds &+ \int_0^\tau \frac{\partial \tilde{R}^{-1}}{\partial v}(s, X(s)) \sigma^k(s, X(s)) \circ dw_s^k.\end{aligned}$$

From $\frac{\partial \tilde{R}^{-1}}{\partial v}(s,v) = U_s(\tilde{R}(s,v))$ we obtain

$$\tilde{R}^{-1}(\tau, X(\tau)) = (H_0 + H_1) X(0) \text{ for every } \tau,$$

and in particular $X(1) = \tilde{R}(1, (H_0 + H_1)X(0))$. Thus we can rewrite the boundary condition as $\tilde{V}(1, (H_0 + H_1)X(0)) = V_0$. Since $v \mapsto \tilde{V}(1, v)$ is invertible almost surely by Proposition 2, we get that $X(.) = \tilde{R}(., \tilde{V}^{-1}(1, V_0))$. □

Remark 1. The smoothness requirement on $\bar{V}(1)$ is essential for our purpose. Indeed, a very stringent point is that we can only state uniqueness amongst the processes of class $\mathbb{L}^{1,8}_{C,loc}$, because we need to apply the Itô-Ventzell formula. As a consequence we have to prove the existence of a solution in $\mathbb{L}^{1,8}_{C,loc}$.
The stochastic invariant imbedding method is twice powerful for solving stochastic differential equations with affine boundary conditions. Firstly it provides us

with a way to prove existence. Secondly it is helpful to prove the smoothness of the random variable $\bar{V}(1)$ very efficiently, since we actually deal with the stochastic process $\bar{V}(\tau)$ defined by a S.D.E.. It is much easier to prove that this process belongs to $\mathbb{L}^{1,p}_{loc}$ than to prove that the inverse of $v \mapsto \tilde{V}(1,v)$ at the point V_0 belongs to $\mathbb{D}^{1,p}_{loc}$.

4. Applications of the existence and uniqueness theorem

4.1. One-dimensional problem

Let f and σ be functions from $[0,+\infty) \times \mathbb{R}$ into $\mathbb{R}$, h_0, h_1 and V_0 be real constants such that $h_0 + h_1 \neq 0$.

Theorem 3. *If f and σ fulfil $\mathcal{H}1$ and moreover $h_0 h_1 \geq 0$, then $\mathcal{H}2$ is satisfied. As a consequence, there exists a unique process in $\mathbb{L}^{1,8}_{C,loc}$ solution of :*

$$\begin{cases} dX(t) = f(t, X(t))dt + \sigma(t, X(t)) \circ dw_t, \\ h_0 X(0) + h_1 X(1) = V_0. \end{cases}$$

Note that in the case where $h_0 = 0$ or $h_1 = 0$, we deal with a S.D.E. with an initial condition and the result is well known.

Proof . Consider for instance the case $h_0 > 0$ and $h_1 > 0$. The proof relies on the fact that $v \to \tilde{R}(\tau, v)$ is monotone increasing for every τ, in other words that $Y(\tau, v) > 0$ almost surely.

Thus $\det Z(\tau, v) = h_1 Y(\tau, v) + h_0(h_0 + h_1)^{-1} \geq h_0(h_0 + h_1)^{-1} > 0$. □

Remark 2. This theorem has already been proved by Donati-Martin in [5] by means of a different way. The stochastic invariant imbedding method provides us with a very simple way to get back this result. Moreover, we shall see in the further subsections that the stochastic invariant imbedding method is efficient for solving some multi-dimensional cases.

4.2. Some results in multi-dimensional cases

We first deal with a simple situation which has been studied by Nualart and Pardoux in [12]. We are able to exhibit new existence and uniqueness results.

We consider the space of $d \times d$ matrices $\mathcal{M}_d(\mathbb{R})$ equipped with the norm : $\|A\| = \sup_{i=1,\ldots,d} \sum_{j=1}^{d} |A_{ij}|$. We begin by recalling the elementary following lemma, which will be needed below.

Lemma 2. *If $\|A\| < 1$, then $I_d + A$ is invertible and $\|(I_d + A)^{-1}\| \leq \dfrac{1}{1 - \|A\|}$.*

We assume that :

- H_0 and H_1 belong to $\mathcal{M}_d(\mathbb{R})$ and V_0 belongs to $\mathbb{R}^d$,
- $t \mapsto B_t$ is a continuous mapping from $[0,1]$ into $\mathcal{M}_d(\mathbb{R})$,
- $(t,x) \mapsto f(t,x)$ fulfils the conditions of $\mathcal{H}1$.

We denote by ∇f the matrix $\dfrac{\partial f_i}{\partial x_j}$ and by K its sup norm $\sup_{t \in [0,1], x \in \mathbb{R}^d} \|\nabla f(t,x)\|$.

Theorem 4. *If $H_0 + H_1$ is invertible and :*

$$K < \ln\left(1 + \frac{1}{\|H_1\|\|(H_0+H_1)^{-1}\|}\right), \tag{17}$$

then $\mathcal{H}2$ is fulfilled and there exists a unique process in $\mathbb{L}^{1,8}_{C,loc}$ solution of :

$$\begin{cases} X(t) - X(0) = \int_0^t f(s, X(s))ds + \int_0^t B_s \circ dw_s, \\ H_0 X(0) + H_1 X(1) = V_0. \end{cases} \tag{18}$$

Before we turn to the proof, we state two straightforward corollaries of this theorem.

Corollary 1. *Let n be an integer between 1 and d.*
Let $V_0^n \in \mathbb{R}^n$ and $V_0^{d-n} \in \mathbb{R}^{d-n}$.
If $K < \ln 2$ then there exists a unique process in $\mathbb{L}^{1,8}_{C,loc}$ solution of :

$$\begin{cases} X(t) - X(0) = \int_0^t f(s, X(s))ds + \int_0^t B_s \circ dw_s, \\ \begin{pmatrix} X_1(0) \\ \vdots \\ X_n(0) \end{pmatrix} = V_0^n \quad \text{and} \quad \begin{pmatrix} X_{n+1}(1) \\ \vdots \\ X_d(1) \end{pmatrix} = V_0^{d-n}. \end{cases}$$

Corollary 2. *If H_0 is invertible and $\|H_0^{-1}H_1\| < e^{-K}$, then there exists a unique process in $\mathbb{L}^{1,8}_{C,loc}$ solution of (18).*

Proof . This corollary follows easily from Theorem 4. Indeed, if H_0 is invertible then we can introduce the new boundary condition

$$\tilde{H}_0 X(0) + \tilde{H}_1 X(1) = \tilde{V}_0,$$

where $\tilde{H}_0 = I_d$, $\tilde{H}_1 = H_0^{-1}H_1$ and $\tilde{V}_0 = H_0^{-1}V_0$. Now we can readily check that (17) is satisfied :

$$\begin{aligned} \ln\left(1 + \frac{1}{\|\tilde{H}_1\|\|(\tilde{H}_0+\tilde{H}_1)^{-1}\|}\right) &= \ln\left(1 + \frac{1}{\|H_0^{-1}H_1\|\|I_d + H_0^{-1}H_1\|}\right) \\ &\geq \ln\left(1 + \frac{1}{\|H_0^{-1}H_1\|\left(\frac{1}{1-\|H_0^{-1}H_1\|}\right)}\right) \\ &= -\ln(\|H_0^{-1}H_1\|) > -\ln e^{-K} = K. \end{aligned}$$

□

Proof of Theorem 4 . We need to check that $\mathcal{H}2$ is fulfilled. It is sufficient to show that there exists $\beta < 1$ such that :

$$\sup_{\tau\in[0,1]} \sup_{v\in\mathbb{R}^d} \|H_1(Y(\tau,v) - (H_0+H_1)^{-1})\| \leq \beta \text{ almost surely.} \tag{19}$$

Indeed, we can rewrite (13) as $Z(\tau,v) = I_d + H_1(Y(\tau,v) - (H_0+H_1)^{-1})$. As a consequence, if (19) holds, then by Lemma 2 we obtain that $Z(\tau,v)$ is invertible and $\|Z^{-1}(\tau,v)\| \leq \frac{1}{1-\beta}$ for any τ, v almost surely. Hence $|\det Z| \geq \frac{(1-\beta)^d}{d!}$, and $\mathcal{H}2$ is fulfilled. Finally it remains to prove the following lemma.

Lemma 3. *(19) is satisfied with* $\beta = \|H_1\| \|(H_0+H_1)^{-1}\|(e^K - 1)$.

Proof . Let $M(\tau, v) = Y(\tau, v) - (H_0+H_1)^{-1}$. M is solution of :

$$\left\{ \begin{array}{l} dM = \nabla f(\tau, \tilde{R}) M d\tau + \nabla f(\tau, \tilde{R})(H_0+H_1)^{-1} d\tau, \\ M(0,v) = 0. \end{array} \right.$$

Therefore

$$\|M(\tau, v)\| \leq \int_0^\tau \|\nabla f(s, \tilde{R}(s,v))\| \left(\|M(s,v)\| + \|(H_0+H_1)^{-1}\| \right) ds,$$

We can apply Gronwall's inequality and the definition of K provides us with the estimate :

$$\sup_{\tau \in [0,1]} \|M(\tau, v)\| \leq \|(H_0+H_1)^{-1}\|(e^K - 1),$$

from which we can deduce the result. □

4.3. Triangular systems

To simplify the presentation, we shall assume that the dimension is equal to 2. However we can easily extend the following result to higher dimensions. We assume that :

- $(t,x) \mapsto f(t,x)$ and $(t,x) \mapsto \sigma^k(t,x)$, $k=1,2$ fulfil $\mathcal{H}1$ and have the following forms : $f(t,x_1,x_2) = \begin{pmatrix} f_1(t,x_1) \\ f_2(t,x_1,x_2) \end{pmatrix}$, $\sigma^k(t,x_1,x_2) = \begin{pmatrix} \sigma_1^k(t,x_1) \\ \sigma_2^k(t,x_1,x_2) \end{pmatrix}$, $k=1,2$.
- H_0 and H_1 belong to $\mathcal{M}_2(\mathbb{R})$ and are inferiorly triangular. Both the coefficients H_{011} and H_{111} (resp. H_{022} and H_{122}) have the same signs.
- The diagonal coefficients of the matrix H_0, i.e. H_{011} and H_{022}, are different from 0.

Theorem 5. *$\mathcal{H}2$ is fulfilled and there exists a unique process in $\mathbb{L}^{1,8}_{C,loc}$ solution of :*

$$\left\{ \begin{array}{l} X_1(t) - X_1(0) = \displaystyle\int_0^t f_1(s, X_1(s)) ds + \int_0^t \sigma_1^k(s, X_1(s)) \circ dw_s^k, \\ X_2(t) - X_2(0) = \displaystyle\int_0^t f_2(s, X_1(s), X_2(s)) ds + \int_0^t \sigma_2^k(s, X_1(s), X_2(s)) \circ dw_s^k, \end{array} \right.$$

with the boundary conditions

$$\left\{ \begin{array}{l} H_{011} X_1(0) + H_{111} X_1(1) = V_1^0, \\ H_{021} X_1(0) + H_{022} X_2(0) + H_{121} X_1(1) + H_{122} X_2(1) = V_2^0. \end{array} \right.$$

Proof . At the expense of having to modify V_0, we may take H_0 and H_1 to have the form :

$$H_0 = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \qquad H_1 = \begin{pmatrix} c & 0 \\ b & e \end{pmatrix},$$

with $c \geq 0$ and $e \geq 0$. $H_0 + H_1$ is obviously invertible.

Some straightforward calculations yield that the characteristics $\tilde{R}_1(\tau, v)$ and $\tilde{V}_1(\tau, v)$ do not depend on v_2. As a consequence $Z_{12}(\tau, v) = 0$. Besides

$$Z_{11}(\tau, v) = \frac{1 + c \exp N_1(\tau, v)}{1 + c}, \quad Z_{22}(\tau, v) = \frac{1 + e \exp N_2(\tau, v)}{1 + e},$$

where $N_j(\tau, v) = \int_0^\tau \frac{\partial f_j}{\partial x_j}(\tau, \tilde{R}(\tau, v))ds + \int_0^\tau \frac{\partial \sigma_j^k}{\partial x_j}(\tau, \tilde{R}(\tau, v)) \circ dw_\tau^k$.

This implies $\det Z(\tau, v) = Z_{11}(\tau, v) Z_{22}(\tau, v) \geq \frac{1}{(1+c)(1+e)} > 0$. □

4.4. Second-order equations

In this subsection we are looking for real processes $(x(t), y(t))$ which solve the system :

$$\begin{cases} dx(t) = y(t)dt, \\ dy(t) = f(t, x(t), y(t))dt + \sigma(t, x(t), y(t)) \circ dw_t, \end{cases} \tag{20}$$

with the affine boundary conditions :

$$H_0 \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} + H_1 \begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = V_0,$$

where H_0 et H_1 are matrices in $\mathcal{M}_2(\mathbb{R})$ and V_0 is a vector in $\mathbb{R}^2$.

Theorem 6. *Suppose that $\mathcal{H}^1$ holds, $H_0 = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ and $H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Assume moreover that one of the following conditions is satisfied :*

$$\mathcal{H}2' : \begin{cases} \sigma(t,x,y) \text{ does not depend on } (x,y), \\ K = \sup_{t \in [0,1]} \sup_{x,y \in \mathbb{R}} \left\{ |\frac{\partial f}{\partial x}(t,x,y)| + |\frac{\partial f}{\partial y}(t,x,y)| \right\} \text{ satisfies } K < \ln(2+a), \\ a \text{ is a positive real such that } a > e - 2. \end{cases}$$

$$\mathcal{H}2'' : \begin{cases} \sigma(t,x,y) \text{ does not depend on } x, \\ \frac{\partial f}{\partial x}(t,x,y) \geq 0 \text{ for all } (t,x,y), \\ a \text{ is a non-negative real.} \end{cases}$$

$$\mathcal{H}2''' : \begin{cases} \sigma(t,x,y) \text{ does not depend on } (x,y), \\ f(t,x,y) \text{ does not depend on } y \text{ and } \frac{\partial f}{\partial x}(t,x) \geq -\frac{\pi^2}{4}, \\ a \text{ is a positive real.} \end{cases}$$

Then $\mathcal{H}2$ is fulfilled and there exists a unique process in $\mathbb{L}^{1,8}_{C,loc}$ solution of (20) with the boundary conditions : $\begin{cases} x(1) + ax(0) = V_0^1, \\ y(0) = V_0^2. \end{cases}$

Proof . We shall only sketch the proof.
If we assume $\mathcal{H}2'$ then the conditions of Theorem 4 are clearly satisfied, which produces the expected result.
If we assume $\mathcal{H}2''$ or $\mathcal{H}2'''$, then it can be checked that the matrix Z is triangular and $\det Z(\tau, v) = Y_{11}(\tau, v) + \frac{a}{1+a}$. On the one hand, the assumption $\mathcal{H}2''$ implies that $\tau \mapsto Y_{11}(\tau, v)$ is nondecreasing for every v almost surely, which leads to the conclusion. On the other hand the lower bound $\frac{\partial f}{\partial x}(t,x) \geq -\frac{\pi^2}{4}$ is a sufficient condition which insures that $\mathcal{H}2$ is fulfilled. Indeed, applying the technical Lemma 4 about the behaviour of the solution of a second-order ordinary differential equation, we get that $Y_{11}(\tau, v)$ is non-negative for every $\tau \in [0,1]$ almost surely. This implies that $\det Z(\tau, v) \geq \frac{a}{a+1}$, which completes the proof of the theorem. Hence it remains to prove the lemma :

Lemma 4. *Let g be a continuous function from $[0, \infty)$ into $\mathbb{R}$ and u be the solution of the real second-order ordinary differential equation :*

$$\begin{cases} \dfrac{d^2u}{dt^2}(t) = g(t)u(t), \\ u(0) = A > 0, \quad u'(0) = 0. \end{cases}$$

If $\inf_{t\geq 0} g(t)$ is finite and if we set $K_0 = \dfrac{1}{(-\inf_{t\geq 0} g(t)) \vee 0}$ ($K_0 = +\infty$ if $\inf_{t\geq 0} g(t) \geq 0$), then $u > 0$ on $[0, \dfrac{\pi\sqrt{K_0}}{2})$.

Proof of Lemma 4 . If we study the Wronskian associated with u and u_K, $K < K_0$, solution of the ordinary differential equation :

$$\begin{cases} \dfrac{d^2u_K}{dt^2}(t) = -\dfrac{1}{K}u_K(t), \\ u_K(0) = A, \quad u'_K(0) = 0, \end{cases}$$

then we notice that u cannot take the value zero while u_K is positive. Since $u_K(t) = A\cos(t/\sqrt{K})$, we easily deduce the result of the lemma. □

Remark 3. The regularity hypotheses on the function f can be weakened in the particular case where σ^k depends only on time t.

Remark 4. Of course I have not exhibited all the cases where the stochastic invariant imbedding method could be applied (in fact, I have only presented the most simple cases). I have aimed at showing that the invariant imbedding method could be successfully applied to stochastic boundary value problems.

5. Limit theorem

5.1. Problem and result

We shall study in this section the asymptotic behaviour of the solution of a boundary value problem with random coefficients. The tools used here are on the one hand diffusion approximation results according to Kunita [9] and on the other hand our existence and uniqueness results concerning stochastic differential equations with boundary conditions.

Let $\eta^k(t)$ be independent Markov processes defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and w_t be a d-dimensional Brownian motion defined on a stochastic basis $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$, where $\Omega_0 = \mathcal{C}^0([0,1], \mathbb{R}^d)$.

We assume that η^k take values in a compact subset of $\mathbb{R}$, are centered, have unique invariant probability measures under which they are ergodic, and fulfil Doeblin's condition (as a consequence their generators fulfil the Fredholm alternative). Moreover we denote $\alpha_k = \int_0^\infty \mathbb{E}[\eta^k(0)\eta^k(t)]dt$ (we assume $\alpha_k \neq 0$ and $\alpha_k < \infty$ and, consequently, $0 < \alpha_k < \infty$).

Let f and σ^k be functions, H_0 and H_1 be two matrices and V_0 be some vector, such that $\mathcal{H}1$ and $\mathcal{H}2$ are fulfilled with H_0, H_1, f and $\sqrt{\alpha_k}\sigma^k$. Thus there exists a unique process X in $\mathbb{L}^{1,8}_{C,loc}$ solution of :

$$\left\{ \begin{array}{l} dX(t) = f(t, X(t))dt + \sqrt{\alpha^k}\sigma^k(t, X(t)) \circ dw_t^k, \\ H_0 X(0) + H_1 X(1) = V_0. \end{array} \right. \tag{21}$$

Let us consider the boundary value problem :

$$\left\{ \begin{array}{l} \frac{dX^\varepsilon}{dt}(t) = f(t, X^\varepsilon(t)) + \sigma^k(t, X^\varepsilon(t))\frac{1}{\varepsilon}\eta^k(\frac{t}{\varepsilon^2}), \\ H_0 X^\varepsilon(0) + H_1 X^\varepsilon(1) = V_0. \end{array} \right. \tag{22}$$

We denote by $\mathbb{D}^\varepsilon$ the subset of Ω such that the problem (22) admits a solution. Our main result is the following.

Theorem 7. *$\mathbb{D}^\varepsilon$ is a measurable subset of Ω and $\mathbb{P}(\mathbb{D}^\varepsilon) \stackrel{\varepsilon \to 0}{\longrightarrow} 1$.*
The smallest solution $(X^\varepsilon(t))_{t\in[0,1]}$ of the problem (22) (in the sense of the sup norm) defined on $\mathbb{D}^\varepsilon$ converges weakly to $(X(t))_{t\in[0,1]}$ in $\mathcal{C}^0([0,1], \mathbb{R}^d)$.

One difficulty is due to the fact that X^ε is not defined on the whole set Ω. In fact we shall prove on the one hand that $\mathbb{D}^\varepsilon$ is measurable and $\mathbb{P}(\mathbb{D}^\varepsilon) \to 1$, and on the other hand that the process $\tilde{X}^\varepsilon$ which is equal to X^ε on $\mathbb{D}^\varepsilon$ and 0 otherwise converges weakly to X.

Before we turn to the proof, some comments are called for. We have had some trouble to state uniqueness of the solution of the boundary value problem (22) by means of a rigorous way. Indeed we cannot deduce from the almost sure hypothesis $\mathcal{H}2$ that a similar condition is fulfilled by the problem (22), because the processes $t \mapsto \int_0^t \frac{1}{\varepsilon}\eta^k(\frac{s}{\varepsilon^2})ds$ take values in the Cameron Martin space $\mathbf{H}^1$, which is a negligeable subset of Ω_0 with respect to the Wiener measure. On the other hand we are not allowed to use a density argument to extend the claim of $\mathcal{H}2$ to the space $\mathbf{H}^1$, since the map defined by the process Z on Ω_0 (see Section 3) need not to be continuous. Besides we cannot even state uniqueness for the problem (22) when ε goes to zero, because only weak convergence results for the flow Z^ε associated with (22) are available to us. As a consequence we lack control over the flow Z^ε for large values of v and do not manage to prove an asymptotic lower bound for $|\det Z^\varepsilon|$ over the whole space $\mathbb{R}^d$. Hence we do not succeed in deducing from $\mathcal{H}2$ and the convergence results that (22) possesses no more than one solution when ε goes to zero. Namely we shall prove that a second solution, if any, has no option but to take larger and larger values as ε goes to zero.

Nonetheless the hypothesis $\mathcal{H}2$ is very close to insure existence and uniqueness for the problem (22), so we may have assumed without loss of generality that the boundary value problem admits a unique solution $\mathbb{P}$-almost surely for every $\varepsilon > 0$. Indeed, it is natural to expect that explicit assumptions on f and σ^k under which $\mathcal{H}2$ holds will also insure that $\mathcal{H}2$ holds true with the stochastic integral "$\circ dw_t^k$" replaced by "$\dot{g}^k(t)dt$", $g^k \in \mathbf{H}^1$. This is satisfied in practice. In particular, every situation we have studied in Section 4 does actually take place in this simple case, so that the claim of the theorem may be reduced to : $\mathbb{P}(\mathbb{D}^\varepsilon) = 1$ *for every* $\varepsilon > 0$. *The unique solution of the problem (22) defined on* $\mathbb{D}^\varepsilon$ *converges weakly to* $(X(t))_{t\in[0,1]}$ *in* $\mathcal{C}^0([0,1], \mathbb{R}^d)$.

Remark 5.
a. This theorem has already been proved in the particular case where f and σ^k are linear in [7].
b. In this model we have studied an equation driven by a Markov process η. However we can extend the limit theorem to much more situations, namely every situation where usual diffusion approximation results can be applied (see [9] or [6]).

5.2. Proof of the theorem

We introduce the characteristics $\tilde{R}^\varepsilon$ and $\tilde{V}^\varepsilon$ associated with the boundary value problem (22) as in Section 3. Let $W^1 = \mathcal{C}^0([0,1], \mathcal{C}^1)$ be the set of all the continuous maps $\tilde{r}$ from $[0,1]$ into $\mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$ such that $\tilde{r}(0, v) = (H_0 + H_1)^{-1}v$. W^1 is equipped with the topology generated by the sup norm over compact subsets.

Lemma 5. $\tilde{R}^\varepsilon$ *converges weakly to* $\tilde{R}$ *in* W^1.

Proof . This is a straightforward corollary of Kunita's results [9]. It is sufficient to notice that $\tilde{R}^\varepsilon(\tau, v) = \phi^\varepsilon_\tau((H_0 + H_1)^{-1}v)$ and $\tilde{R}(\tau, v) = \phi_\tau((H_0 + H_1)^{-1}v)$, where ϕ^ε and ϕ denote the standard flows. □

Let $\tilde{r}$ be an element of W^1. Using the now classical notations, we set $\tilde{v}(\tau, v) = H_1\tilde{r}(\tau, v) + H_0(H_0+H_1)^{-1}v$ and $Z(\tau, v) = H_1\frac{\partial \tilde{r}}{\partial v}(\tau, v) + H_0(H_0+H_1)^{-1}$. By the implicit function theorem (see also Lemma 6-1-1 [9]),

- the map $\tilde{v}(\tau, .)$ is a diffeomorphism from ${}^cA_\tau$ into $\mathbb{R}^d$ for every τ almost surely,
- the inverse $\tilde{v}^{-1}(\tau, V)$, $\tau < T(\tilde{r}, V) \wedge 1$ is continuous,

where we have set $T(\tilde{r}, V) = \inf\{t > 0, V \in \tilde{v}(t, A_t)\}$ ($= \infty$ if $A_t = \emptyset$ for every t) and $A_t = \{v \text{ such that } \exists s < t, \det Z(s, v) = 0\}$. We denote by $\bar{v}$ the inverse of $\tilde{v}$ at the point V_0, i.e. the continuous function defined on $[0, T(\tilde{r}, V_0))$ by $\tilde{v}^{-1}(., V_0)$.

As a consequence we can introduce the measurable mappings F from W^1 into $\mathbb{R}^d$ and G from W^1 into $\mathcal{C}^0([0,1], \mathbb{R}^d)$ given by :

$$F(\tilde{r}) = \begin{cases} \bar{v}(1) \text{ if } T(\tilde{r}, V_0) > 1, \\ 0 \text{ otherwise,} \end{cases} \qquad G(\tilde{r})(.) = \begin{cases} \tilde{r}(., \bar{v}(1)) \text{ if } T(\tilde{r}, V_0) > 1, \\ 0 \text{ otherwise.} \end{cases}$$

Finally we denote by B_n the ball of radius n in $\mathbb{R}^d$ and we introduce some subsets of W^1 : $D_n = \{\tilde{r} \text{ such that } \inf_{|v|\le n, \tau\in[0,1]} |\det Z[\tau, v)| > 0\}$ and $D = \bigcap_{n\in\mathbf{N}} D_n$.

Lemma 6. *If $\tilde{r}^p$ is a sequence of W^1 which converges to some $\tilde{r} \in D$, then for p large enough*

$$\bar{v}^p(.) \text{ is defined on } [0,1] \text{ and } \bar{v}^p(1) \stackrel{p\to\infty}{\longrightarrow} \bar{v}(1). \tag{23}$$

If $\phi(p) \to \infty$ and $\hat{v}^{\phi(p)}$ is a solution of $\tilde{v}^{\phi(p)}(1,v) = V_0$ distinct from $\bar{v}^{\phi(p)}(1)$, then $\liminf_{p\to\infty} |\hat{v}^{\phi(p)}| = \infty$.

Proof . Since $\tilde{r} \in D$, by the implicit function theorem $\tilde{v}(\tau,.)$ is a diffeomorphism for every τ whose inverse is continuous in (τ, v). Thus there exists an integer n such that

$$\sup_{s\in[0,1]} |\bar{v}(s)| \leq n-2. \tag{24}$$

Since D_n is an open subset of W^1, $\tilde{r}^p$ belongs to D_n for p large enough. As a consequence, again by the implicit function theorem, $\bar{v}^p(.)$ is well defined until it goes out of B_n. Let us denote by $\sigma_n^p = \inf\{\tau \in [0,1], |\bar{v}^p(\tau)| \geq n\}$ ($=1$ if $\{\ldots\} = \emptyset$), i.e. the first time when $\bar{v}^p$ goes out of B_n. Since

$$\tilde{v}(s\wedge\sigma_n^p, \bar{v}(s\wedge\sigma_n^p)) = V_0 = \tilde{v}^p(s\wedge\sigma_n^p, \bar{v}^p(s\wedge\sigma_n^p))$$

for every $s \in [0,1]$, we have

$$|\tilde{v}(s\wedge\sigma_n^p, \bar{v}^p(s\wedge\sigma_n^p)) - \tilde{v}(s\wedge\sigma_n^p, \bar{v}(s\wedge\sigma_n^p))| \leq \sup_{\tau\in[0,1]} \sup_{|v|\leq n} |\tilde{v}^p(\tau,v) - \tilde{v}(\tau,v)|,$$

leading at once to the estimate

$$\sup_{s\in[0,\sigma_n^p]} |\tilde{v}(s,\bar{v}^p(s)) - \tilde{v}(s,\bar{v}(s))| \stackrel{p\to\infty}{\longrightarrow} 0. \tag{25}$$

Let us consider $(s_p)_{p\in\mathbb{N}}$ a sequence of real numbers which belong to $[0,\sigma_n^p]$. By (25) we have $|\tilde{v}(s_p,\bar{v}^p(s_p)) - \tilde{v}(s_p,\bar{v}(s_p))| \stackrel{p\to\infty}{\longrightarrow} 0$. Besides $(s_p)_{p\in\mathbb{N}}$ is a bounded sequence, so there exists a subsequence $(s_{\phi(p)})_{p\in\mathbb{N}}$ which converges to some $s \in [0,1]$. The continuity of $\tilde{v}$ then implies that

$$|\tilde{v}(s,\bar{v}^{\phi(p)}(s_{\phi(p)})) - \tilde{v}(s,\bar{v}(s_{\phi(p)}))| \stackrel{p\to\infty}{\longrightarrow} 0.$$

Since $v \mapsto \tilde{v}(s,v)$ has a continuous inverse, $|\bar{v}^{\phi(p)}(s_{\phi(p)}) - \bar{v}(s_{\phi(p)})|$ converges to 0 as p goes to infinity. Thus we have just proved that we can extract from any sequence $(s_p)_{p\in\mathbb{N}}$ of real numbers which belongs to $[0,\sigma_n^p]$ a subsequence $(s_{\phi(p)})_{p\in\mathbb{N}}$ which satisfies $|\bar{v}^{\phi(p)}(s_{\phi(p)}) - \bar{v}(s_{\phi(p)})| \stackrel{p\to\infty}{\longrightarrow} 0$. This yields

$$\sup_{s\in[0,1]} |\bar{v}^p(s\wedge\sigma_n^p) - \bar{v}(s\wedge\sigma_n^p))| \stackrel{p\to\infty}{\longrightarrow} 0. \tag{26}$$

Combining (24) and (26) we get that there exists some p_0 such that

$$(p\geq p_0) \Rightarrow \left(\sup_{s\in[0,\sigma_n^p]} |\bar{v}^p(s)| \leq n-1\right).$$

Thus σ_n^p is equal to 1 for $p \geq p_0$ and (23) follows from (26).

Let us prove the second claim of the lemma. We begin by fixing an integer m. There exists some $p(m)$ such that $\tilde{r}^p$ belongs to D_m if $p \geq p(m)$ and $\tilde{v}^p$ is a diffeomorphism from B_m into $\mathbb{R}^d$. In particular $\tilde{v}^p$ is one-to-one from B_m into $\mathbb{R}^d$. Now, if $p \geq \max(p_0, p(m))$, then $\bar{v}^p(1)$ is the unique solution of $\tilde{v}^p(1,v) = V_0$ in B_m. This completes the proof of the second statement. □

Proof of Theorem 7 . $\mathbb{D}^\varepsilon$ is a measurable subset of Ω. Indeed $\omega \in \mathbb{D}^\varepsilon$ if and only if $\tilde{V}^\varepsilon(1,.)^{-1}(\{V_0\})$ is non-empty, i.e. $\tilde{R}^\varepsilon \in E$, where

$$E = \bigcup_{k\in\mathbb{N}} \bigcap_{p\in\mathbb{N}} \bigcup_{v\in B_k\cap\mathbb{Q}^d} \{\tilde{r} \in W^1 \text{ such that } |\tilde{v}(1,v) - V_0| < \frac{1}{p}\}$$

is a measurable subset of W^1.

By Lemma 6 the mappings F and G are continuous at every point of D. Since $\mathbb{P}_0(\tilde{R} \in D) = 1$, Corollary II-1-9 [6] yields that $G(\tilde{R}^\varepsilon)$ converges weakly to $G(\tilde{R})$ in $\mathcal{C}^0([0,1],\mathbb{R}^d)$. Remarking on the one hand that $G(\tilde{R}^\varepsilon)$ is a solution of (22), namely the smallest solution by the second statement of Lemma 6, and on the other hand that $G(\tilde{R})$ is equal to the solution X of (21) by Theorem 2, the proof of the theorem is complete. □

6. Appendix

The aim of this appendix is to prove the third claim of Proposition 2, which states that $\bar{V} \in \mathbb{L}^{1,p}_{loc}$. Henceforth $\dfrac{\partial^j \tilde{R}}{\partial v_j}$ is a shorthand for the tensor of the j-order derivatives of the components of $\tilde{R}$ and $p > 2$ is a fixed real number. We begin by stating some uniform L^r-estimates of $\tilde{R}$ and its derivatives.

Lemma 7. *We can find a modification of the flow $\tilde{R}$ such that $\tilde{R}$ is continuous with respect to (τ, v), $\tilde{R}(\tau,.)$ is a $\mathcal{C}^4$ diffeomorphism for any τ almost surely and $\dfrac{\partial^j \tilde{R}}{\partial v^j}(.,v)$ belongs to $\bigcap_{r\geq 2} \mathbb{L}^{1,r}_C$ for every $v \in \mathbb{R}^d$ and $0 \leq j \leq 4$. Moreover for any real $r \geq 2$ and any integer n there exists constants c_r and $c_{r,n}$ such that*

$$\mathbb{E}[\sup_{\tau\in[0,1]} |\tilde{R}(\tau,v)|^r] \leq c_r(1+|v|^r), \quad v \in \mathbb{R}^d,$$

$$\mathbb{E}[\sup_{\tau\in[0,1]} |\frac{\partial^j \tilde{R}}{\partial v^j}(\tau,v)|^r] \leq c_r, \quad 1 \leq j \leq 4, \quad v \in \mathbb{R}^d,$$

$$\sup_{\nu\in[0,1]} \mathbb{E}[\sup_{\tau\in[0,1]} \sup_{|v|\leq n} |D_\nu \frac{\partial^j \tilde{R}}{\partial v^j}(\tau,v)|^r] \leq c_{r,n}, \quad 0 \leq j \leq 2, \quad n \in \mathbb{N}.$$

Proof . These statements may be found in [9] and in Lemmas 2-2-1 and 2-2-2 [14]. □

We now define the random variables that will be localizers. Here and below we denote by $\phi_{n,d}$ a sequence of functions in $\mathcal{C}^\infty_c(\mathbb{R}^d,\mathbb{R})$ such that $\phi_{n,d}(v) = 0$ for every $|v| \geq n$ and $\phi_{n,d}(v) = 1$ for every $|v| \leq n-1$.

Definition 3. *$\beta_{n,p}$, $n \geq 2$, is the random variable defined by :*

$$\beta_{n,p} = \phi_{r(n),1}\left(S_{n,p}\right), \quad S_{n,p} = \int_0^1 \int_{B_n} 1 + \sum_{i=0}^{4} |\frac{\partial^i \tilde{R}}{\partial v^i}(\tau, v)|^{4d(d+1)q} dv d\tau,$$

where B_n is the ball of radius n, $q(p)$ is an integer such that $q > \dfrac{2p}{p-2}$ and $r(n)$ is equal to $n^{4d(d+1)q+d+1}$.

Proposition 3. *$\{\beta_{n,p}, n \geq 2\}$ satisfies the required conditions to be a localizer :*
1. $\{\beta_{n,p} = 1\} \nearrow \Omega$ almost surely when n goes to infinity,
2. $\beta_{n,p} \in \bigcap_{r \geq 2} \mathbb{D}^{1,r}$ for every $n \geq 2$.

Proof . Applying the estimates of Lemma 7 we get $\mathbb{E}\left[S_{n,p}\right] \leq C(1+n^{4d(d+1)q+d})$. Hence

$$\mathbb{P}\left(\beta_{n,p} = 0\right) \leq \mathbb{P}\left(S_{n,p} \geq r(n) - 1\right) \leq \frac{\mathbb{E}\left[S_{n,p}\right]}{r(n) - 1} \underset{n \to \infty}{\longrightarrow} 0,$$

which yields the first statement. Besides $S_{n,p} \in \bigcap_{r \geq 2} \mathbb{D}^{1,r}$ and $\phi_{r(n),1} \in \mathcal{C}_c^\infty$ imply the second claim by Lemma 3-5 [13]. □

We are going to consider a truncated version of the process $\bar{V}$. We begin by introducing the truncated random functions :

$$a^n(\tau, v) = a(\tau, v)\phi_{n,d}(v), \quad b^{n,k}(\tau, v) = b^k(\tau, v)\phi_{n,d}(v), \tag{27}$$

$$\hat{a}^n(\tau, v) = a^n(\tau, v) + \frac{1}{2}\frac{d < b^{n,k}(.,v), w^k_. >_\tau}{d\tau} + \frac{1}{2}\frac{\partial b^{n,k}}{\partial v_j} b_j^{n,k}(\tau, v), \tag{28}$$

where a and b^k have been defined in Proposition 2. To express the corrective terms of $\hat{a}^n$ it is convenient to introduce the notation :

$$Z^{-1} = \frac{1}{\det Z} F(Y), \text{ where } F(Y) = {}^t\mathrm{com}\left(H_1 Y + H_0(H_0 + H_1)^{-1}\right).$$

$F(y)$ is a matrix whose coefficients are polynomial functions of degree $d-1$ of the coefficients of the matrix y.

$$\begin{aligned}
\frac{d < b^{n,k}(.,v), w^k_. >_\tau}{d\tau} &= -\frac{1}{\det Z} F(Y) H_1 \frac{\partial \sigma^k}{\partial x_l}(\tau, \tilde{R}) \sigma_l^k(\tau, \tilde{R}) \phi_{n,d}(v) \\
&- \frac{1}{\det Z} \frac{\partial F}{\partial y_{l,m}}(Y) \left(H_1 \nabla \sigma^k(\tau, \tilde{R}) Y\right)_{l,m} H_1 \sigma^k(\tau, \tilde{R}) \phi_{n,d}(v) \\
&+ \frac{1}{(\det Z)^2} \mathrm{Tr}\left(F(Y) H_1 \nabla \sigma^k(\tau, \tilde{R}) Y\right) F(Y) H_1 \sigma^k(\tau, \tilde{R}) \phi_{n,d}(v),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial b^{n,k}}{\partial v_j} &= -\frac{1}{\det Z} F(Y) H_1 \left(\frac{\partial \sigma^k}{\partial x_l}(\tau, \tilde{R}) Y_{l,j} \phi_{n,d}(v) + \sigma^k(\tau, \tilde{R}) \frac{\partial \phi_{n,d}}{\partial v_j}(v)\right) \\
&- \frac{1}{\det Z} \frac{\partial F}{\partial y_{l,m}}(Y) \frac{\partial Y_{l,m}}{\partial v_j} H_1 \sigma^k(\tau, \tilde{R}) \phi_{n,d}(v) \\
&+ \frac{1}{(\det Z)^2} \mathrm{Tr}\left(F(Y) H_1 \frac{\partial Y}{\partial v_j}\right) F(Y) H_1 \sigma^k(\tau, \tilde{R}) \phi_{n,d}(v).
\end{aligned}$$

Before stating the next proposition we set out some straightforward estimates.

Lemma 8. *The following statements hold for $\hat{a}^n$ and for $b^{n,k}$.*
1. For $i=0,1$ there exists a constant K such that $\hat{a}^n$ satisfies almost surely :

$$|\frac{\partial^i \hat{a}^n}{\partial v^i}(\tau,v)| \le K\left(\sum_{j=0}^{3+i}\frac{1}{\alpha^j}\right)\left(1+\sum_{j=0}^{i+2}|\frac{\partial^j \tilde{R}}{\partial v^j}(\tau,v)|^{(i+3)d}\right),$$

where α is the lower bound of $|\det Z|$ introduced in Hypothesis $\mathcal{H}2$.
2. For every $v\in\mathbb{R}^d$, $\hat{a}^n(.,v)$ belongs to $\mathbb{L}_C^{1,r}$ for any $r\ge 2$.
Moreover, for any $i=1,..,d$ and $\nu\in[0,1]$,

$$|D^i_\nu \hat{a}^n(\tau,v)| \le K\left(\sum_{j=0}^{4}\frac{1}{\alpha^j}\right)\left(1+\sum_{j=0}^{2}|\frac{\partial^j \tilde{R}}{\partial v^j}(\tau,v)|^{4d}+\sum_{j=0}^{2}|D^i_\nu\frac{\partial^j \tilde{R}}{\partial v^j}(\tau,v)|^{4d}\right).$$

We now show that a truncated version of the process $\bar{V}$ is smooth in the sense of the Malliavin calculus.

Proposition 4. *There exists a unique process $\bar{V}_n$ solution of :*

$$\begin{cases} d\bar{V}_n = \hat{a}^n(\tau,\bar{V}_n(\tau))d\tau + b^{n,k}(\tau,\bar{V}_n(\tau))dw^k_\tau, \\ \bar{V}_n(0)=V_0. \end{cases} \tag{29}$$

Moreover, $\beta_{n,p}\bar{V}_n\in\mathbb{L}^{1,p}$.

Proof . Let us define $\bar{V}_{n,l}$ as the unique process solution of :

$$\begin{cases} d\bar{V}_{n,l} = \hat{a}^n(\tau,\bar{V}_{n,l}(\phi_l(\tau)))d\tau + b^{n,k}(\tau,\bar{V}_n(\phi_l(\tau)))dw^k_\tau, \\ \bar{V}_{n,l}(0)=V_0, \end{cases}$$

where $\phi_l(\tau)=m/2^l$ if $\tau\in[m/2^l,(m+1)/2^l)$.

Step 1 . $\beta_{n,p}\bar{V}_{n,l}\in\mathbb{L}^{1,p}$.

$\bar{V}_{n,l}(0)$ is equal to V_0 and obviously belongs to $\mathbb{D}^{1,p}$. By recursion with respect to m, if we assume that $\bar{V}_{n,l}(\tau)\in\mathbb{D}^{1,p}$ for any $\tau\le m/2^l$, then by Proposition 3-4 [11] we find that $\bar{V}_{n,l}(\tau)\in\mathbb{D}^{1,p}$ for any $\tau\in[m/2^l,(m+1)/2^l]$ and

$$\begin{aligned} D^i_\nu\bar{V}_{n,l}(\tau) &= D^i_\nu\bar{V}_{n,l}(\frac{m}{2^l}) + b^{n,k}_i(\nu,\bar{V}_{n,l}(\frac{m}{2^l}))\mathbb{I}_{\nu\in(\frac{m}{2^l},\tau]} \\ &+\int_{\frac{m}{2^l}}^{\tau} D^i_\nu\hat{a}^n(s,\bar{V}_{n,l}(\frac{m}{2^l}))ds + \int_{\frac{m}{2^l}}^{\tau} D^i_\nu b^{n,k}(s,\bar{V}_{n,l}(\frac{m}{2^l}))dw^k_s \\ &+\int_{\frac{m}{2^l}}^{\tau}\frac{\partial\hat{a}^n}{\partial v}(s,\bar{V}_{n,l}(\frac{m}{2^l}))D^i_\nu\bar{V}_{n,l}(\frac{m}{2^l})ds + \int_{\frac{m}{2^l}}^{\tau}\frac{\partial b^{n,k}}{\partial v}(s,\bar{V}_{n,l}(\frac{m}{2^l}))D^i_\nu\bar{V}_{n,l}(\frac{m}{2^l})dw^k_s. \end{aligned}$$

Finally, $\bar{V}_{n,l}\in\mathbb{L}^{1,p}$ and $\xi^i_{n,l,\tau}(\nu)=D^i_\nu(\bar{V}_{n,l}(\tau))$ satisfies, for $\tau\ge\nu\ge 0$:

$$\begin{aligned} \xi^i_{n,l,\tau}(\nu) &= b^{n,k}_i\left(\nu,\bar{V}_{n,l}(\phi_l(\nu))\right) \\ &+\int_{\psi_l(\nu)\wedge\tau}^{\tau} D^i_\nu\hat{a}^n\left(s,\bar{V}_{n,l}(\phi_l(s))\right)ds + \int_{\psi_l(\nu)\wedge\tau}^{\tau} D^i_\nu b^{n,k}\left(s,\bar{V}_{n,l}(\phi_l(s))\right)dw^k_s \\ &+\int_{\psi_l(\nu)\wedge\tau}^{\tau}\frac{\partial\hat{a}^n}{\partial v}\left(s,\bar{V}_{n,l}(\phi_l(s))\right)\xi^i_{n,l,s}ds + \int_{\psi_l(\nu)\wedge\tau}^{\tau}\frac{\partial b^{n,k}}{\partial v}\left(s,\bar{V}_{n,l}(\phi_l(s))\right)\xi^i_{n,l,s}dw^k_s, \end{aligned} \tag{30}$$

where $\psi_l(\nu) = m/2^l$ if $\nu \in ((m-1)/2^l, m/2^l]$.
In order to simplify the notations it is convenient to set $A_s = \dfrac{\partial \hat{a}^n}{\partial v}(s, \bar{V}_{n,l}(\phi_l(s)))$, $B_s^k = \dfrac{\partial b^{n,k}}{\partial v}(s, \bar{V}_{n,l}(\phi_l(s)))$ and $\eta_\tau = \sup_{s\in[0,\tau]} |\xi^i_{n,l,s}(\nu)|$.
We also introduce the stopping times T_M and $T_{n,p}$ defined by :

$$T_M = \inf\{\tau \text{ such that } |\xi^i_{n,l,\tau}(\nu)| \geq M\},$$

$$T_{n,p} = \inf\{\tau \text{ such that } \int_0^\tau \int_{B_n} 1 + \sum_{i=0}^4 |\frac{\partial^i R}{\partial v^i}(s,v)|^{4d(d+1)q} dv ds \geq r(n)\},$$

where $q(p)$ and $r(n)$ have been defined in Definition 3.
Since A_s and B_s^k are bounded by $K \sup_{|v|\leq n} \left(1 + \sum_{i=0}^3 |\frac{\partial^i R}{\partial v^i}|^{4d}\right)$, applying Sobolev's inequality we obtain that there exists a constant C such that

$$\int_0^{T_{n,p}\wedge 1} |A_s|^{p/p-1} ds \leq Cr(n) \text{ and } \int_0^{T_{n,p}\wedge 1} |B_s^k|^{2p/p-2} ds \leq Cr(n) \tag{31}$$

Now, applying Burkholder-Gandy's inequality and combining with the estimates of Lemmas 7 and 8, we deduce from (30) that :

$$\begin{aligned}\mathbb{E}[\eta^p_{\tau\wedge T_M\wedge T_{n,p}}] \leq C_{p,n}\times & \left(1 + \mathbb{E}\left[\left|\int_{\psi_l(\nu)\wedge\tau\wedge T_M\wedge T_{n,p}}^{\tau\wedge T_M\wedge T_{n,p}} |A_s \xi^i_{n,l,s}(\nu)| ds\right|^p\right]\right. \\ & \left. + \sum_{k=1}^d \mathbb{E}\left[\left|\int_{\psi_l(\nu)\wedge\tau\wedge T_M\wedge T_{n,p}}^{\tau\wedge T_M\wedge T_{n,p}} |B_s^k \xi^i_{n,l,s}(\nu)|^2 ds\right|^{p/2}\right]\right).\end{aligned}$$

Using Hölder's inequality and (31), we get :

$$\mathbb{E}[\eta^p_{\tau\wedge T_{n,p}\wedge T_M}] \leq K_{p,n}\left(1 + \int_{\psi_l(\nu)\wedge\tau}^\tau \mathbb{E}[|\xi^i_{n,l,s\wedge T_{n,p}\wedge T_M}(\nu)|^p] ds\right).$$

Since $|\xi^i_{n,l,s}(\nu)| \leq \eta_s$ by the definition of η, we finally obtain from Gronwall's inequality that

$$\mathbb{E}[\eta^p_{\tau\wedge T_{n,p}\wedge T_M}] \leq K_{p,n} e^{K_{p,n}\tau}.$$

As a consequence, letting $M \to \infty$ and applying the monotone convergence theorem yields :

$$\mathbb{E}[\eta^p_{1\wedge T_{n,p}}] < \infty.$$

However $\beta_{n,p}\eta_1 \leq \eta_{1\wedge T_{n,p}}$ almost surely, moreover the L^p-estimate of η is uniform with respect to ν and l. Therefore it follows that :

$$\sup_{l\in\mathbb{N}^*} \sup_{\nu\in[0,1]} \mathbb{E}[\sup_{\tau\in[0,1]} |\xi^i_{n,l,\tau}(\nu)|^p |\beta_{n,p}|^p] < \infty. \tag{32}$$

The proof of the first step is now complete.

Step 2 . $\beta_{n,p}\bar{V}_n \in \mathbb{L}^{1,p}$.
Let us introduce the stopping time τ_M defined by :

$$\tau_M = \inf\{\tau \text{ such that } \sup_{|v|\leq n} \sum_{i=0}^{3} |\frac{\partial^i R}{\partial v^i}(\tau, v)| \geq M\}.$$

This stopping time satisfies $\limsup_{M\to\infty} \tau_M \geq 1$ almost surely by the uniform L^r-estimates of Lemma 7 and Sobolev's inequality. Now, if we take care to remain on $[0, \tau_M \wedge 1]$ in order to deal with almost sure bounds for $\frac{\partial \hat{a}^n}{\partial v}$, $\frac{\partial b^{n,k}}{\partial v}$, $\hat{a}^n$ and $b^{n,k}$, we can apply a slight modification of Lemma 2-1 [8], so that we get :

$$\mathbb{E}[\sup_{\tau\in[0,\tau_M\wedge 1]} |\bar{V}_{n,l}(\tau) - \bar{V}_n(\tau)|^p] \stackrel{l\to\infty}{\longrightarrow} 0, \tag{33}$$

$$\sup_{\nu\in[0,1]} \mathbb{E}[\sup_{\tau\in[0,\tau_M\wedge 1]} |\beta_{n,p}|^p |\xi^i_{n,l,\tau}(\nu) - \xi^i_{n,\tau}(\nu)|^p] \stackrel{l\to\infty}{\longrightarrow} 0, \tag{34}$$

where $\bar{V}_n$ is the solution of (29) and $\xi^i_{n,\tau}(\nu)$ is the solution of :

$$\begin{aligned}\xi^i_{n,\tau}(\nu) = \;& b^{n,k}_i(\nu, \bar{V}_n(\nu)) + \int_\nu^\tau D^i_\nu \hat{a}^n(s, \bar{V}_n(s))ds + \int_\nu^\tau D^i_\nu b^{n,k}(s, \bar{V}_n(s))dw^k_s \\ &+ \int_\nu^\tau \frac{\partial \hat{a}^n}{\partial v}(s, \bar{V}_n(s))\xi^i_{n,s}(\nu)ds + \int_\nu^\tau \frac{\partial b^{n,k}}{\partial v}(s, \bar{V}_n(s))\xi^i_{n,s}(\nu)dw^k_s.\end{aligned}$$

From (32) and (34) it follows that, for every M :

$$\sup_{\nu\in[0,1]} \mathbb{E}[\sup_{\tau\in[0,\tau_M\wedge 1]} |\beta_{n,p}|^p |\xi^i_{n,\tau}(\nu)|^p] \leq C.$$

Letting $M \to \infty$ and applying the monotone convergence theorem, we get that :

$$\sup_{\nu\in[0,1]} \mathbb{E}[\sup_{\tau\in[0,1]} |\beta_{n,p}|^p |\xi^i_{n,\tau}(\nu)|^p] \leq C,$$

which produces the desired result : $\beta_{n,p}\bar{V}_n \in \mathbb{L}^{1,p}$. □

We are now able to state the smoothness of the process $\bar{V}$.

Proposition 5. *The process $\bar{V}_n$ defined by (29) is the unique solution of :*

$$\begin{cases} d\bar{V}_n = a^n(\tau, \bar{V}_n(\tau))d\tau + b^{n,k}(\tau, \bar{V}_n(\tau)) \circ dw^k_\tau, \\ \bar{V}_n(0) = V_0. \end{cases} \tag{35}$$

Obviously $\bar{V}_n = \bar{V}$ on $\Omega^n = \{\omega$ such that $\sup_{\tau\in[0,1]} |\bar{V}(\tau)| \leq n - 1\}$.
Thus $\big(\beta_{n,p}\bar{V}_n, \Omega^n \cap \{\beta_{n,p} = 1\}\big)$ localizes $\bar{V}$ in $\mathbb{L}^{1,p}$.

Proof . (35) is the Stratonovich's form of (29). Indeed Itô's and Stratonovich's integrals are related by :

$$b^{n,k}(\tau,\bar{V}_n(\tau))\circ dw^k_\tau = b^{n,k}(\tau,\bar{V}_n(\tau))dw^k_\tau + \frac{1}{2}d < (b^{n,k}(.,\bar{V}_n(.))), w^k_. >_\tau,$$

and $\hat{a}^n$ has been chosen so that the relation

$$\frac{1}{2}d < b^{n,k}(.,\bar{V}_n(.)), w^k_. >_\tau = \hat{a}^n(\tau,\bar{V}_n(\tau))d\tau - a^n(\tau,\bar{V}_n(\tau))d\tau$$

holds by the definition (28) and by the generalized Itô-Stratonovich formula proved by Bismut in [3] :

$$d(b^{n,k}(\tau,\bar{V}_n(\tau))) = db^{n,k}(\tau,v)\mid_{v=\bar{V}_n(\tau)} + \frac{\partial b^{n,k}}{\partial v_j}(\tau,\bar{V}_n(\tau))\circ d\bar{V}_n(\tau)_j.$$

□

References

1. Alabert, A., Ferrante, M., Nualart, D.: Markov field property of stochastic differential equations (to appear)
2. Bellman, R., Wing, G.M.: An introduction to invariant imbedding, Wiley, New York, 1976
3. Bismut, J.M.: A generalized formula of Itô and some other properties of stochastic flows, Z. Wahrscheinlichkeitstheorie verw. Gebiete **55**, 331-350 (1981)
4. Dembo, A., Zeitouni, O.: A change of variables formula for Stratonovich integrals and existence of solutions for two-point stochastic boundary value problems, Probab. Theory Rel. Fields. **84**, 411-425 (1990)
5. Donati-Martin, C.: Equations différentielles stochastiques dans $\mathbb{R}$ avec conditions aux bords, Stoch. and Stoch. Reports **35**, 143-173 (1991)
6. Ethier, S.N., Kurtz, T.G.: Markov processes, Wiley, 1986
7. Fouque, J.P., Merzbach, E.: A limit theorem for linear boundary value problems in random media, Annals of Applied Prob., **4**, No. 2, 549-569 (1994)
8. Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes, North Holland, 1988
9. Kunita, H.: Stochastic flows and stochastic differential equations, Cambridge studies in advanced mathematics **24**, 1990
10. Kushner, H.J.: Approximation and weak convergence methods for random processes, MIT Press, 1984
11. Nualart, D., Pardoux, E.: Stochastic calculus with anticipating integrands, Probab. Theory Rel. Fields. **78**, 535-581 (1988)
12. Nualart, D., Pardoux, E.: Boundary value problems for stochastic differential equations, Ann. Probab. **19**, 1118-1144 (1991)
13. Ocone, D., Pardoux, E.: Linear stochastic differential equations with boundary conditions, Probab. Theory Rel. Fields. **82**, 489-526 (1989)
14. Ocone, D., Pardoux, E.: A generalized Itô-Ventzell formula. Application to a class of anticipating stochastic differential equations, Annales de l'I.H.P. **25**, No. 1, 39-71 (1989)

This article was processed by the author using the LaTeX style file from Springer-Verlag.