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Modulational instability induced by randomly varying coefficients for the nonlinear Schrödinger equation

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Abstract

We introduce the theory of modulational instability (MI) of electromagnetic waves in optical fibers. The model at hand is the one-dimensional nonlinear Schrödinger equation with random group velocity dispersion and random nonlinear coefficient. We compute the MI gain which reads as the Lyapunov exponent of a random linear system. We show that the distribution of the MI gain can be expressed in terms of the log-normal statistics. The heavy tail of this probability distribution function involves very different behaviors for the sample and moment MI gains. In the anomalous dispersion regime, random fluctuations of the nonlinear coefficient reduces the sample MI gain peak, although the moment MI peak is enhanced, and the unstable bandwidth is widened. Still in the anomalous dispersion regime, random fluctuations of the group velocity dispersion reduces both the sample MI gain peak and the moment MI peak. Finally, in the normal dispersion regime, randomness extends the MI domain to the whole spectrum of modulations, and increases the MI gain peak. The linear stability analysis is confirmed by numerical simulations of the full stochastic nonlinear Schrödinger equation.

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1. Introduction

As is well known, the interplay between optical Kerr effect and chromatic dispersion leads to the phenomenon of modulational instability (MI) of light waves [1]. This instability was discovered almost simultaneously in several contexts in the mid-1960s. In deep-water gravity waves, it is called the Benjamin–Feir instability [2,3] after Benjamin and Feir found that they were experimentally unable to sustain a nonlinear constant-amplitude wavetrain of gravity waves in deep water. In plasma physics it was called for the first time modulational instability and leads to the collapse of Langmuir waves [4]. It actually occurs in many different physical environments: plasmas, fluids, anharmonic lattices, electrical circuits and nonlinear optics. MI leads to the breakup of a continuum or quasi-continuum wave into a train of ultrashort pulses and it can be used to generate a train of soliton-like pulses [5,6]. MI also sets a fundamental

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nonlinear limiting factor in the transmission of dense wavelength-division multiplexed signals in long-distance fiber links. The MI gain in homogeneous medium has been extensively studied for the scalar nonlinear Schrödinger equation [1], and also for the vector nonlinear Schrödinger equation both in the normal [7] and anomalous [8] regimes. It has been shown that MI depends on the frequencies of initial modulations and the powers of waves.

All these results were obtained in media where the characteristic parameters are constant. In realistic fiber transmission links, the chromatic dispersion and nonlinearity are not constant but can fluctuate stochastically around their mean values. The inhomogeneity of the medium may be inherent to the medium [9] or induced by other propagating waves [10]. Recently it was pointed out that random inhomogeneities in nonlinear dispersive media may extend the domain of the homogeneous MI of nonlinear plane waves over the whole spectrum of modulations. This effect is completely analogous to the stochastic parametric resonance which occurs for a harmonic oscillator with randomly modulated frequency. Indeed, previous work analyzed the MI of electromagnetic waves in nonlinear Kerr media with random nonlinearity [11] or random group-velocity dispersion (GVD) [12,13]. In these references, the authors adopt the following methodology. They consider white-noise models for the coefficients and compute the mean of the intensity of the growing instability (i.e. the Lyapunov exponent of the mean, the so-called mean Lyapunov exponent) in the limit of small deviations. We shall revisit this approach and extend it to a very general class of fluctuating coefficients. We shall obtain the same qualitative picture as in [11–13] for the mean Lyapunov exponent, but we shall show that the statistical distribution of the MI gain is more complicated than expected, and that the mean Lyapunov exponent is larger than the sample Lyapunov exponent that governs the typical growth of the modulation. Indeed the intensity of the modulation has rare but very large fluctuations, and these rare configurations impose the mean value. This behavior is predicted by the random matrix products theory [14], which claims that the sample and moment stability of random dynamical systems may be very different.

This situation is relevant for telecommunication fiber cables where all characteristic parameters slightly fluctuate around their mean values. Note that MI is an important nonlinear limiting factor in long-distance fiber-based telecommunication systems, but it is also a way to generate a train of soliton-like pulses. Indeed when a small modulation is applied to an input signal, MI can be induced if the side band frequency Ω falls within the MI gain spectrum. If we make use of this induced MI, it is possible to generate a train of soliton-like pulses with a repetition frequency determined by the inverse of the input modulation frequency Ω [6]. As pointed out in [5], in order to produce the desired pulse train, it is necessary to remove the pulse train at an appropriate distance which is very sensitive to the MI gain at frequency Ω. It is therefore relevant to obtain precise expressions of the MI gain in realistic fibers, which take into account the unavoidable random fluctuations of the parameters of the fiber. The underlying physical purpose is to prove whether or not randomness leads to a substantial extension of the MI domain with respect to the homogeneous case, and whether it involves a decay or an enhancement of the MI gain peak. Precise expressions of the different Lyapunov exponents will provide the answers.

The paper is organized as follows. In Section 2, we state the problem at hand, and we remember the reader with the standard homogeneous MI gain in Section 3. In Section 4, we consider white-noise coefficients, and study and growth of the mean intensity of the modulation. Sections 5–7 are devoted to the complete statistical descriptions of the sample and mean MI gain in a general framework. The theoretical results are discussed in Section 8 and compared with numerical simulations in Section 9.

2. Formulation

The evolution of the field in random fibers is governed by the nonlinear Schrödinger equation with random coefficients [1]:

$$\mathrm{i}u_z + \beta u_{tt} + \gamma |u|^2 u = 0. \tag{1}$$

Here the standard dimensionless variables are used. γ is the nonlinear coefficient. The GVD coefficient is $\beta > (<)\, 0$ for anomalous (normal) dispersion. Both coefficients are fluctuating around their respective mean values γ_0 and β_0 so that they can be described as

$$\gamma(z) = \gamma_0(1 + \sigma m_\gamma(z)), \tag{2}$$

$$\beta(z) = \beta_0(1 + \sigma m_\beta(z)), \tag{3}$$

where σ is a dimensionless parameter which characterizes the amplitudes of the fluctuations. The fluctuations of the nonlinear coefficient are characterized by the random process m_γ and the fluctuations of the GVD coefficient by m_β. We shall assume that $\mathbf{m} := (m_\beta, m_\gamma)$ is a stationary, zero mean and random process. Note that a usual model for random fiber is the random concatenation of different fiber sections whose coefficients have constant values [15,16]. Typically the lengths of these segments are of the order of 10–100 km, much less than the dispersion distance $L_\mathrm{d} = t_0^2/|\beta_0|$ (t_0 is the pulse duration of the order of 10 ps, while the mean GVD coefficient is of the order of 0.1 $\mathrm{ps}^2\,\mathrm{m}^{-1}$ for most glass fibers so that $L_\mathrm{d} \sim 1000$ km). We shall see that such a configuration may be described by a white-noise model. Eq. (1) has plane wave solutions

$$u(z,t) = \sqrt{P_0} \exp\left(\mathrm{i}\int_0^z \gamma(z')\,\mathrm{d}z' P_0\right). \tag{4}$$

Their linear stability is determined by considering a perturbed solution of the form

$$u(z,t) = (\sqrt{P_0} + u_1(z,t)) \exp\left(\mathrm{i}\int_0^z \gamma(z')\,\mathrm{d}z' P_0\right). \tag{5}$$

By substituting Eq. (5) into Eq. (1), and retaining only the first order terms, one obtains a linear equation for $u_1(z,t)$:

$$\mathrm{i}u_{1z} + \beta u_{1tt} + 2\gamma P_0 \,\mathrm{Re}(u_1) = 0. \tag{6}$$

Performing the Fourier transform $\hat{u}_1 = \int u_1\, \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t$, and using the complex representation: $\hat{u}_1 = \hat{u}_{1\mathrm{r}} + \mathrm{i}\hat{u}_{1\mathrm{i}}$, one obtains a system for the various Fourier components of the perturbation term

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}\hat{u}_{1\mathrm{r}}\\ \hat{u}_{1\mathrm{i}}\end{pmatrix} = Y(z)\begin{pmatrix}\hat{u}_{1\mathrm{r}}\\ \hat{u}_{1\mathrm{i}}\end{pmatrix}, \quad Y(z) = \begin{pmatrix}0 & \beta(z)\omega^2\\ 2\gamma(z)P_0 - \beta(z)\omega^2 & 0\end{pmatrix}. \tag{7}$$

The sample MI gain is defined as the Lyapunov exponent which governs the exponential growth of the modulation:

$$G(\omega) := \lim_{z\to\infty} \frac{1}{z} \ln |\hat{u}_1(z,\omega)|^2. \tag{8}$$

Note that $G(\omega)$ could be random since β and γ are. So it should be relevant to study the mean and fluctuations of the MI gain. For this purpose, we shall analyze the $2n$th mean MI gain defined as the normalized Lyapunov exponent which governs the exponential growth of the nth moment of the intensity of the modulation:

$$G_{2n}(\omega) := \lim_{z\to\infty} \frac{1}{nz} \ln \mathbb{E}[|\hat{u}_1(z,\omega)|^{2n}], \tag{9}$$

where $\mathbb{E}$ stands for the expectation with respect to the distribution of the process (m_γ, m_β). Note that this is a generalization of the standard mean MI gain G_2, which characterizes the exponential growth of the mean intensity of the modulation. If the MI process were deterministic, then we would have $G_{2n}(\omega) = G_2(\omega)$ for every n since

$$\frac{1}{n} \ln |\hat{u}_1(z,\omega)|^{2n} = \ln |\hat{u}_1(z,\omega)|^2.$$

But due to randomness, this does not hold true since we cannot invert the nonlinear power function "$|\cdot|^n$" and the linear statistical averaging "$\mathbb{E}[\cdot]$". Actually Jensen's inequality establishes that if $n \geq 1$, then

$$\mathbb{E}[|\hat{u}_1(z,\omega)|^{2n}] \geq \mathbb{E}[|\hat{u}_1(z,\omega)|^2]^n,$$

and consequently $G_{2n}(\omega) \geq G_2(\omega)$. In case of equality $G_{2n} = G_2 = G$, we can claim that the exponential growth is deterministic, but if G_{2n} increases with n, it means that the MI process is fluctuating.

The random matrix products theory applies to the problem (7). For instance, assume that the random processes m_γ and m_β are piecewise constant over intervals $[n, n+1)$ and take random values on the successive intervals. Under appropriate assumptions on the laws of the values taken by m_γ and m_β, [17, Theorem 4] proves that there exists an analytic function $g(p)$ such that

$$\lim_{z\to\infty} \frac{1}{z} \ln \mathbb{E}[|\hat{u}_1(z,\omega)|^p] = g(p), \tag{10}$$

$$\lim_{z\to\infty} \frac{1}{z} \ln |\hat{u}_1(z,\omega)| = g'(0) \text{ almost surely}, \tag{11}$$

$$\frac{\ln|\hat{u}_1(z,\omega)| - zg'(0)}{\sqrt{z}} \overset{\text{dist.}}{\to} \mathcal{N}(0, g''(0)), \tag{12}$$

where $\mathcal{N}(0,s)$ denotes the Gaussian distribution with mean 0 and variance s. Moreover, the convergence is uniform for $\hat{u}_1(z=0)$ with unit modulus, and the function $p \mapsto g(p)/p$ is monotone increasing. This proves in particular that $G(\omega) = 2g'(0)$ is non-random. In case of non-piecewise constant processes m_β and m_γ, various versions of the above theorem exist which yield the same conclusion [14,18–20]. Unfortunately, the expression of $g(p)$ is very intricate even for very simple random processes m_β and m_γ. We aim in this paper at deriving closed form expressions for the Lyapunov exponents $G(\omega)$ and $G_{2n}(\omega)$ in the natural framework corresponding to the telecommunication applications, where the noise level is low. These expressions could be of practical interest for predicting the modifications of the MI gain peak and spectrum induced by random fluctuations of the coefficients β and γ.

3. The homogeneous modulational instability

We remember the reader with well-known results which can be found in the literature (see for instance [1, Section 5.1]). The matrix Y is constant here ($\beta(z) \equiv \beta_0$, $\gamma(z) \equiv \gamma_0$), so that the MI gain is given by twice the largest value of the real parts of the eigenvalues of Y. In the anomalous dispersion regime $\beta_0 > 0$, there exists a characteristic frequency ω_c above which there is no MI:

$$\omega_c^2 = \frac{1}{\beta_0 \lambda_c}, \quad \lambda_c = \frac{1}{2\gamma_0 P_0}. \tag{13}$$

The frequencies below ω_c are unstable and the corresponding MI gain is

$$G_{\text{det}}(\omega) = 2\beta_0|\omega|\sqrt{\omega_c^2 - \omega^2}.$$

The MI peak $G_{\text{det,opt}}$ and optimal frequency $\omega_{\text{det,opt}}$ are consequently

$$G_{\text{det,opt}} = \beta_0\omega_c^2, \qquad \omega_{\text{det,opt}}^2 = \tfrac{1}{2}\omega_c^2.$$

In the normal dispersion regime $\beta_0 < 0$, there is no MI: $G_{\text{det}} \equiv 0$.

4. The moment equations for the white-noise case

Unlike the homogeneous case, the propagation matrix is no longer constant but varies randomly with distance. So a stochastic approach is required to exhibit the statistical properties of the random MI gain and spectrum. In this section, we study the influence of white random modulations of GVD and nonlinear coefficient on MI in a fiber. The fluctuations are assumed to obey independent Gaussian white noises with delta-correlated functions:

$$\sigma^2\mathbb{E}[m_\beta(x)m_\beta(y)] = \sigma_\beta^2\delta(x-y), \qquad \sigma^2\mathbb{E}[m_\gamma(x)m_\gamma(y)] = \sigma_\gamma^2\delta(x-y), \tag{14}$$

so that (7) reads as a system of stochastic differential equations:

$$\mathrm{d}\hat{u}_{1\mathrm{r}} = \beta_0\omega^2\hat{u}_{1\mathrm{i}}\,\mathrm{d}z + \sigma_\beta\beta_0\omega^2\hat{u}_{1\mathrm{i}}\circ\mathrm{d}W_z^1,$$
$$\mathrm{d}\hat{u}_{1\mathrm{i}} = (2\gamma_0P_0 - \beta_0\omega^2)\hat{u}_{1\mathrm{r}}\,\mathrm{d}z + 2\sigma_\gamma\gamma_0P_0\hat{u}_{1\mathrm{r}}\circ\mathrm{d}W_z^2 - \sigma_\beta\beta_0\omega^2\hat{u}_{1r}\circ\mathrm{d}W_z^1,$$

where W^1 and W^2 are independent and identically distributed standard Brownian motions and $\circ$ stands for the Stratonovitch integral [21, p. 136]. This particular choice of stochastic integral will be made clear in Section 8.

4.1. The second moment MI gain

The analysis of the system of equations for the first moments $\mathbb{E}[\hat{u}_{1\mathrm{r}}], \mathbb{E}[\hat{u}_{1\mathrm{i}}]$ shows that the dynamics is the exponential decreasing of first moments and that the resonant phenomena are absent. This is an expected result since $\hat{u}_{1\mathrm{r}}$ and $\hat{u}_{1\mathrm{i}}$ are strongly oscillating and these oscillations make the first moments average to 0. So we shall analyze the behaviors of the second moments $\mathbb{E}[\hat{u}_{1\mathrm{r}}^2], \mathbb{E}[\hat{u}_{1\mathrm{i}}^2]$ and $\mathbb{E}[\hat{u}_{1\mathrm{r}}\hat{u}_{1\mathrm{i}}]$ (we add $\mathbb{E}[\hat{u}_{1\mathrm{r}}\hat{u}_{1\mathrm{i}}]$ so as to close the equations for the second moments). Applying Itô's formula [21, p. 145], we obtain that the evolution of the row vector of the second moments $X^{(2)} :=^t (\mathbb{E}[\hat{u}_{1\mathrm{r}}^2], \mathbb{E}[\hat{u}_{1\mathrm{r}}\hat{u}_{1\mathrm{i}}], \mathbb{E}[\hat{u}_{1\mathrm{i}}^2])$ reads as a closed form linear system:

$$\frac{\mathrm{d}X^{(2)}}{\mathrm{d}z} = M^{(2)}X^{(2)}, \quad M^{(2)} = \begin{pmatrix} -\beta_0^2\omega^4\sigma_\beta^2 & 2\beta_0\omega^2 & \beta_0^2\omega^4\sigma_\beta^2 \\ \lambda_\mathrm{c}^{-1} - \beta_0\omega^2 & -2\beta_0^2\omega^4\sigma_\beta^2 & \beta_0\omega^2 \\ \beta_0^2\omega^4\sigma_\beta^2 + \lambda_\mathrm{c}^{-2}\sigma_\gamma^2 & 2(\lambda_\mathrm{c}^{-1} - \beta_0\omega^2) & -\beta_0^2\omega^4\sigma_\beta^2 \end{pmatrix},$$

where λ_c is the characteristic wavelength defined by (13). Instability emerges if an eigenvalue of $M^{(2)}$ has a positive real part, and the mean MI gain G_2 defined by (9) is the largest value of the real parts of the eigenvalues of $M^{(2)}$. We have found that the matrix $M^{(2)}$ has three eigenvalues denoted by (p_1, p_2, p_3), which can be expanded as powers of σ_β^2 and σ_γ^2 uniformly with respect to ω:

$$p_1 = 2\omega\sqrt{\beta_0(\lambda_\mathrm{c}^{-1} - \beta_0\omega^2)} + \frac{\beta_0\omega^2}{4}\frac{(8\beta_0^2\omega^4 - 8\beta_0\omega^2\lambda_\mathrm{c}^{-1} + \lambda_\mathrm{c}^{-2})\sigma_\beta^2 + \lambda_\mathrm{c}^{-2}\sigma_\gamma^2}{\lambda_\mathrm{c}^{-1} - \beta_0\omega^2}, \tag{15}$$

$$p_2 = -2\omega\sqrt{\beta_0(\lambda_\mathrm{c}^{-1} - \beta_0\omega^2)} + \frac{\beta_0\omega^2}{4}\frac{(8\beta_0^2\omega^4 - 8\beta_0\omega^2\lambda_\mathrm{c}^{-1} + \lambda_\mathrm{c}^{-2})\sigma_\beta^2 + \lambda_\mathrm{c}^{-2}\sigma_\gamma^2}{\lambda_\mathrm{c}^{-1} - \beta_0\omega^2}, \tag{16}$$

$$p_3 = -\frac{\beta_0\omega^2\lambda_\mathrm{c}^{-2}}{\lambda_\mathrm{c}^{-1} - \beta_0\omega^2 - \beta_0^3\omega^6\sigma_\beta^4}\frac{\sigma_\beta^2 + \sigma_\gamma^2}{2}. \tag{17}$$

These general formulas coincide with the particular cases studied in [11–13].

4.1.1. Normal dispersion $\beta_0 < 0$

In this case, the real parts of p_1 and p_2 are negative, and p_3 is positive real-valued for every frequency ω, which proves that there is instability for all frequencies. The gain is equal to p_3:

$$G_2(\omega) = \frac{|\beta_0|\omega^2\lambda_c^{-2}}{\lambda_c^{-1} + |\beta_0|\omega^2 + |\beta_0|^3\omega^6\sigma_\beta^4}\frac{\sigma_\beta^2 + \sigma_\gamma^2}{2}.$$

The optimal frequency is high in the sense that it is of the order of $\sigma_\beta^{-2/3}$:

$$\omega_{2,\mathrm{opt}}^2 = (2\beta_0^3\lambda_c\sigma_\beta^4)^{-1/3}. \tag{18}$$

The corresponding MI peak is given by

$$G_{2,\mathrm{opt}} = \frac{\lambda_c^{-2}}{1 + 3(2^{-1}\lambda_c^{-1}\sigma_\beta^2)^{2/3}}\frac{\sigma_\beta^2 + \sigma_\gamma^2}{2}. \tag{19}$$

If $\sigma_\beta > 0$, then $G_2(\omega) \to 0$ as $\omega \to \infty$, otherwise $G(\omega)$ converges to $G_{2,\mathrm{opt}} = \frac{1}{2}\lambda_c^{-2}\sigma_\gamma^2$.

4.1.2. Anomalous dispersion $\beta_0 > 0$

In the standard stable region $\omega > \omega_c$, the gain is now positive for every frequency. Comparing carefully the real parts of p_1, p_2 and p_3 establishes that the gain is imposed by p_3:

$$G_2(\omega) = \beta_0^2\frac{\omega^2\omega_c^4}{\omega^2 - \omega_c^2}\frac{\sigma_\beta^2 + \sigma_\gamma^2}{2}.$$

For the study of the standard unstable region $\omega < \omega_c$, we introduce the dimensionless parameter $\delta := \sigma_\gamma^2/\sigma_\beta^2$. The following two cases can be distinguished:

Case 1. If $\delta \le 1$, which means that randomness essentially originates from GVD fluctuations, then the gain is imposed by p_1:

$$G_2(\omega) = 2\beta_0\omega\sqrt{\omega_c^2 - \omega^2} + 2\frac{\omega^2(\omega^2 - \omega_{-,\delta}^2)(\omega^2 - \omega_{+,\delta}^2)}{\omega_c^2 - \omega^2}\beta_0^2\sigma_\beta^2,$$

where the two characteristic frequencies $\omega_{-,\delta}$ and $\omega_{+,\delta}$ are

$$\omega_{-,\delta}^2 := \tfrac{1}{4}(2 - \sqrt{2(1-\delta)})\omega_c^2, \qquad \omega_{+,\delta}^2 := \tfrac{1}{4}(2 + \sqrt{2(1-\delta)})\omega_c^2. \tag{20}$$

This shows that the random dispersion makes the instability increase for $\omega \in (0, \omega_{-,\delta})$ and $\omega \in (\omega_{+,\delta}, \omega_c)$, and decrease for $\omega \in (\omega_{-,\delta}, \omega_{+,\delta})$. In particular, the optimal frequency $\omega_{2,\mathrm{opt}}$ is slightly below $\omega_{\mathrm{det,opt}}$:

$$\omega_{2,\mathrm{opt}}^2 = \tfrac{1}{2}\omega_c^2 - \tfrac{1}{4}\beta_0\omega_c^4(\sigma_\beta^2 - \sigma_\gamma^2), \tag{21}$$

and the corresponding MI peak is reduced:

$$G_{2,\mathrm{opt}} = \beta_0\omega_c^2 - \tfrac{1}{4}\beta_0^2\omega_c^4(\sigma_\beta^2 - \sigma_\gamma^2). \tag{22}$$

Case 2. If $\delta > 1$, which means that randomness essentially originates from fluctuations of the nonlinear coefficient, then the gain is

$$G_2(\omega) = 2\beta_0\omega\sqrt{\omega_c^2 - \omega^2} + \frac{\omega^2(2\omega^2 - \omega_c^2)^2 + (\delta - 1)\omega_c^4/2}{2(\omega_c^2 - \omega^2)}\beta_0^2\sigma_\beta^2,$$

which shows that the random nonlinearity makes instability increase for all frequencies $\omega \in (0, \omega_c)$. In particular, the MI peak obtained at $\omega_{2,\mathrm{opt}}$ defined by (21) is enhanced and given by (22).

4.2. Higher order moments

The results derived in the previous section give accurate expressions for the moment MI gain G_2, but the relationship with the sample MI gain G is not obvious. Further, we have chosen to analyze the second moment of the modulation $\hat{u}_1$ to characterize its stability, but this choice may seem arbitrary and should be discussed. In this section, we study the exponential growth of higher order moments of $\hat{u}_1$ so as to get a picture of the fluctuations of the exponential growth of $\hat{u}_1$. Let n be an integer and $X^{(2n)}$ the $(2n+1)$-dimensional row vector of the $2n$th moments:

$$X_j^{(2n)} := \mathbb{E}[\hat{u}_{1\mathrm{r}}^{2n-j}\hat{u}_{1\mathrm{i}}^{j}], \quad j = 0, \dots, 2n.$$

Applying Itô's formula establishes that the vector $X^{(2n)}$ satisfies

$$\frac{\mathrm{d}X^{(2n)}}{\mathrm{d}z} = M^{(2n)} X^{(2n)},$$

where $M^{(2n)}$ is the $(2n+1) \times (2n+1)$ matrix:

$$M^{(2n)} = (\beta_0\omega^2 - 2\gamma_0 P_0)A^{(2n)} - \beta_0\omega^2 B^{(2n)} + \tfrac{1}{2}\beta_0^4\omega^4\sigma_\beta^2 C^{(2n)} + 2\gamma_0^2 P_0^2\sigma_\gamma^2 D^{(2n)}.$$

The matrices $A^{(2n)}$, $B^{(2n)}$, $C^{(2n)}$ and $D^{(2n)}$ are null but for the following components: $A^{(2n)}_{j,j+1} = 2n - j$, $B^{(2n)}_{j,j-1} = j$, $C^{(2n)}_{j,j} = -2n - 4nj + 2j^2$, $C^{(2n)}_{j,j-2} = j(j-1)$, $C^{(2n)}_{j,j+2} = (2n-j)(2n-j-1)$, and $D^{(2n)}_{j,j-2} = j(j-1)$. In this framework, $nG_{2n}(\omega)$ is equal to the maximum of the real parts of the roots of the characteristic polynomial of the matrix $M^{(2n)}$ whose degree is $2n+1$. This implies that the derivation of a closed-form expression of $G_{2n}(\omega)$ for any $n \geq 2$ is quite intricate, but it can be carried out numerically. Figs. 1–4 display the MI gains G_{2n} for different values of n, σ_β, and σ_γ.

Let us first consider the influence of GVD fluctuations and take $\sigma_\beta > 0$ and $\sigma_\gamma = 0$ (Figs. 1 and 2). In the anomalous regime ($\beta_0 = 1$, Fig. 1), the optimal frequency is found below $\omega_{\mathrm{det,opt}}$ and the MI peak is reduced in agreement with the formulae (21) and (22) for $2n = 2$. But the striking point is that these quantities do not depend on

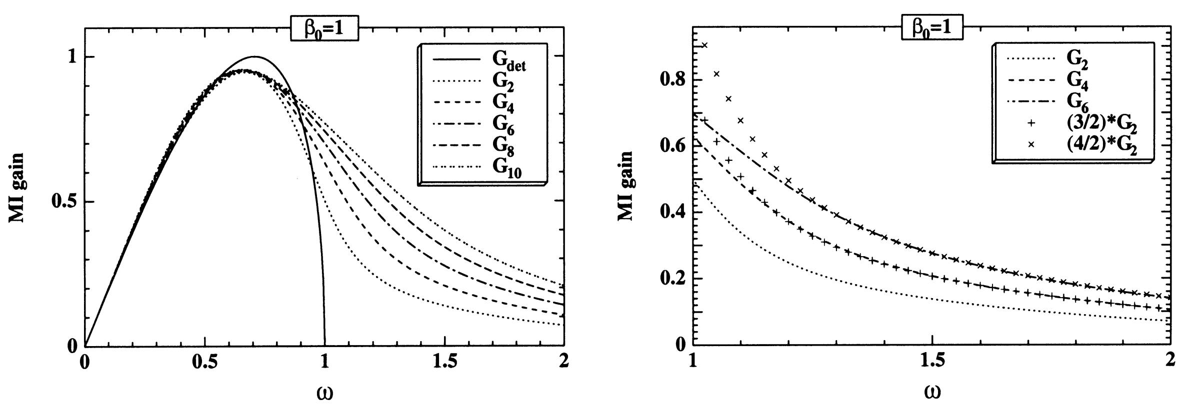

Fig. 1. MI gain versus frequency for an anomalous dispersion regime $\beta_0 = 1$, random GVD $\sigma_\beta = \frac{1}{2}$, and constant nonlinear coefficient $\sigma_\gamma = 0$, $2\gamma_0 P_0 = 1$. The lines correspond to G_{2n} for $2n = 2, 4, 6, 8, 10$.

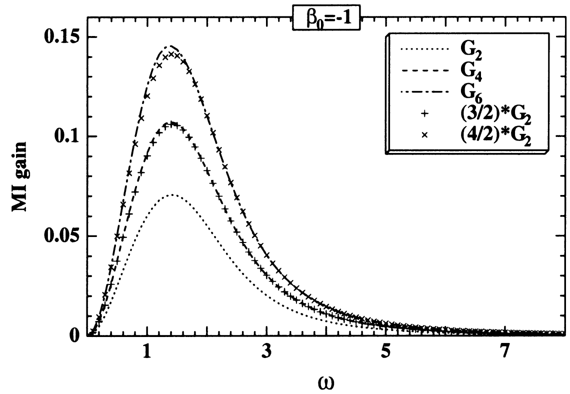

Fig. 2. MI gain versus frequency for a normal dispersion regime $\beta_0 = -1$, random GVD $\sigma_\beta = \frac{1}{2}$, and constant nonlinear coefficient $\sigma_\gamma = 0$, $2\gamma_0 P_0 = 1$. The lines correspond to G_{2n} for $2n = 2, 4, 6, 8, 10$.

n, since the figure shows that the maxima of all curves merge into one point. This indicates that the optimal frequency and MI peak should be nonrandom quantities equal to the values given by (21) and (22): $\omega_{2,\mathrm{opt}} = \sqrt{7}/4 \simeq 0.66$ and $G_{2,\mathrm{opt}} = \frac{15}{16} \simeq 0.94$. Another relevant feature is that the GVD fluctuations extend the MI spectral width with respect to the $\sigma_\beta = 0$ case. Indeed the MI gain for frequencies above $\omega_\mathrm{c} = 1$ is now positive. Furthermore, the curves are now distinct for different n. An empirical relationship can be exhibited, as shown by the right picture of Fig. 1. It seems that $G_{2n}(\omega) \sim \frac{1}{2}(n+1)G_2(\omega)$ for $\omega > \omega_\mathrm{c}$. This relationship will be proved in Section 8. In the normal regime ($\beta_0 = -1$, Fig. 2), there is no MI in the homogeneous case. Taking into account GVD fluctuations, the MI gain is positive for all frequencies and reaches a maximum at some optimal frequency which seems independent of n, but the MI peak depends on n. For $2n = 2$, the MI peak $G_{2,\mathrm{opt}}$ is reached at $\omega_{2,\mathrm{opt}}$ defined by (19) and (18), respectively: $G_{2,\mathrm{opt}} = \frac{1}{14} \simeq 0.071$ and $\omega_{2,\mathrm{opt}} = \sqrt{2} \simeq 1.41$. Moreover, the empirical law $G_{2n}(\omega) \sim \frac{1}{2}(n+1)G_2(\omega)$ is still observed for every frequency ω.

Let us now consider the influence of fluctuations of the nonlinear coefficient and take $\sigma_\beta = 0$ and $\sigma_\gamma > 0$ (Figs. 3 and 4). In the anomalous regime, the optimal frequency is above $\omega_{\mathrm{det,opt}}$ and the MI peak is enhanced as shown by Eqs. (21) and (22). The values found for the optimal frequencies and MI peaks are in good agreement with (21)

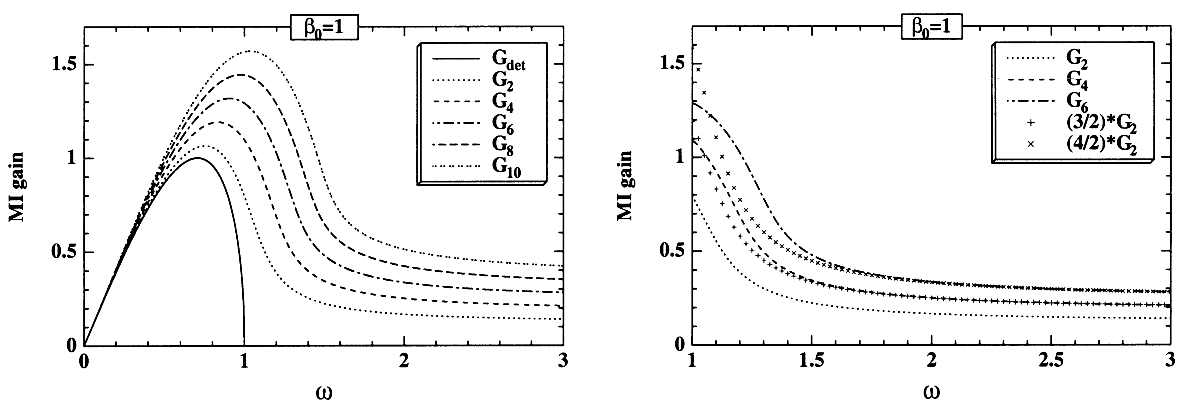

Fig. 3. MI gain versus frequency for an anomalous dispersion regime $\beta_0 = 1$, constant GVD $\sigma_\beta = 0$, and random nonlinear coefficient $\sigma_\gamma = \frac{1}{2}$, $2\gamma_0 P_0 = 1$. The lines correspond to G_{2n} for $2n = 2, 4, 6, 8, 10$.

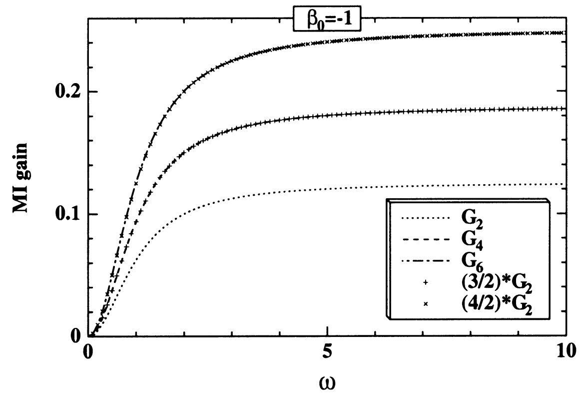

Fig. 4. MI gain versus frequency for a normal dispersion regime $\beta_0 = -1$, constant GVD $\sigma_\beta = 0$, and random nonlinear coefficient $\sigma_\gamma = \frac{1}{2}$, $2\gamma_0 P_0 = 1$. The lines correspond to G_{2n} for $2n = 2, 4, 6$.

and (22) which claim that $\omega_{2,\text{opt}} = \frac{3}{4} = 0.75$ and $G_{2,\text{opt}} = \frac{17}{16} \simeq 1.04$. Further, these quantities depend on n, while they do not in case of GVD fluctuations. This shows that the optimal frequency and MI peak should be random quantities. Furthermore, as for GVD fluctuations, the MI spectrum is widened with respect to the $\sigma_\gamma = 0$ case, and the same empirical law is found between the G_{2n}'s for $\omega > \omega_c$. In the normal dispersion regime, all frequencies are unstable as soon as $\sigma_\gamma > 0$, and the MI gain is an increasing function of ω. For $2n = 2$, the MI peak corresponding to $\omega \to \infty$ is given by (19): $G_{2,\text{opt}} = \frac{1}{8} = 0.125$.

It thus appears that the moments of the modulation $\hat{u}_1$ grow with different exponential rates, which proves that the behavior of the modulation is not deterministic but exhibits strong random fluctuations. It should thus be relevant to study the complete distribution of the MI gain, and not only the first moments. We would also like to underline that the result would have been quantitatively different if we had considered the classical Itô stochastic integral instead of the Stratonovitch integral. It is therefore important to justify why this particular stochastic integral is the suitable one in our framework. Finally, we would like to underline that we have addressed in this section the white-noise case, but it should be also relevant to consider a general colored noise and study the influence of the power spectrum of the noise. All these points will be discussed in the following sections.

5. Sample exponential growth

5.1. The modulation in polar coordinates

We denote by $\mathbf{m}(z)$ the process $(m_\beta(z), m_\gamma(z))$ and we introduce the terms r and s:

$$r(\omega) = \frac{\gamma_0 P_0 - \beta_0 \omega^2}{2\gamma_0 P_0 - \beta_0 \omega^2}, \tag{23}$$

$$s(\omega) = \frac{\gamma_0 P_0}{2\gamma_0 P_0 - \beta_0 \omega^2}. \tag{24}$$

We first assume that $\beta_0 < 0$. Introducing the polar coordinates $(R(z), \psi(z))$ as $\hat{u}_{1\text{r}}(z) = \sqrt{-\beta_0 \omega^2} R(z) \cos(\psi(z))$ and $\hat{u}_{1\text{i}}(z) = \sqrt{2\gamma_0 P_0 - \beta_0 \omega^2} R(z) \sin(\psi(z))$, the system (7) reads as

$$R(z) = R_0 \exp\left(\int_0^z q(\psi(z'), \mathbf{m}(z'))\, \mathrm{d}z' \right), \tag{25}$$

$$\frac{d\psi(z)}{dz} = h(\psi(z), \mathbf{m}(z)), \tag{26}$$

with $q(\psi, \mathbf{m}) = q_0(\psi) + \sigma q_1(\psi, \mathbf{m})$ and $h(\psi, \mathbf{m}) = h_0(\psi) + \sigma h_1(\psi, \mathbf{m})$:

$$q_0(\psi) = 0, \qquad q_1(\psi, \mathbf{m}) = k(\omega)s(\omega)\sin(2\psi)(m_\gamma - m_\beta),$$
$$h_0(\psi) = k(\omega), \qquad h_1(\psi, \mathbf{m}) = k(\omega)s(\omega)(m_\gamma + (m_\gamma - m_\beta)\cos(2\psi)),$$

where $k(\omega)$ is the wave number defined by

$$k(\omega) = \sqrt{\beta_0\omega^2(\beta_0\omega^2 - 2\gamma_0 P_0)}. \tag{27}$$

If $\beta_0 > 0$ and $\beta_0\omega^2 \geq 2\gamma_0 P_0$, then denoting $\hat{u}_{1\mathrm{r}}(z) = \sqrt{\beta_0\omega^2}R(z)\cos(\psi(z))$ and $\hat{u}_{1\mathrm{i}}(z) = -\sqrt{\beta_0\omega^2 - 2\gamma_0 P_0}R(z)\sin(\psi(z))$, the system (7) also reads as (25) and (26). If $\beta_0 > 0$ and $\beta_0\omega^2 < 2\gamma_0 P_0$, then denoting $\hat{u}_{1\mathrm{r}}(z) = \sqrt{\beta_0\omega^2}R(z)\cos(\psi(z))$ and $\hat{u}_{1\mathrm{i}}(z) = \sqrt{2\gamma_0 P_0 - \beta_0\omega^2}R(z)\sin(\psi(z))$, the system (7) reads as (25) and (26) with

$$q_0(\psi) = \kappa(\omega)\sin(2\psi), \qquad q_1(\psi, \mathbf{m}) = \kappa(\omega)\sin(2\psi)(r(\omega)m_\beta + s(\omega)m_\gamma),$$
$$h_0(\psi) = \kappa(\omega)\cos(2\psi), \qquad h_1(\psi, \mathbf{m}) = \kappa(\omega)s(\omega)(m_\gamma - m_\beta) + \kappa(\omega)\cos(2\psi)(r(\omega)m_\beta + s(\omega)m_\gamma),$$

where $\kappa(\omega)$ is the wave number defined by

$$\kappa(\omega) = \sqrt{-\beta_0\omega^2(\beta_0\omega^2 - 2\gamma_0 P_0)}. \tag{28}$$

5.2. Expression of the sample MI gain

Let us assume that the process $\mathbf{m}(z)$ is an ergodic Markov process with infinitesimal generator Q on a manifold M with invariant probability $\pi(d\mathbf{m})$. From Eq. (26), $(\psi, \mathbf{m})$ is a Markov process on the space $S^1 \times M$, where S^1 denotes the circumference of the unit circle with infinitesimal generator: $\mathcal{L} = Q + h(\psi, \mathbf{m})(\partial/\partial\psi)$ and with invariant measure $p(\psi, \mathbf{m})\, d\psi\, \pi(d\mathbf{m})$, where p can be obtained as the solution of $\mathcal{L}^* p = 0$. According to the theorem of Crauel [22], the long-time behavior of $R(z)$ can be expressed in terms of the Lyapunov exponent $G(\omega)$, which is given by

$$G(\omega) = 2\int_{S^1 \times M} q(\psi, \mathbf{m})p(\psi, \mathbf{m})\, d\psi\, \pi(d\mathbf{m}). \tag{29}$$

This result holds true under condition H1 [23] or H2 [24]:

(H1) M is a finite set and Q is a finite-dimensional matrix which generates a continuous parameter irreducible, time-reversible Markov chain.

(H2) M is a compact manifold. Q is a self-adjoint elliptic diffusion operator on M with zero an isolated, simple eigenvalue.

Note that the result can be greatly generalized. For instance, one can also work with the class of the ϕ-mixing processes with $\phi \in L^{1/2}$ (see [25, pp. 82–83]). The Lyapunov exponent $G(\omega)$ can be estimated in the case of small noise using the technique introduced by Pinsky [23] under H1 and Arnold et al. [24] under H2. In the following paper, we assume H2. As an example, we may think that $m_\gamma(z) = \arctan(U_1(z))$ and $m_\beta(z) = \arctan(U_2(z))$, where U_1 and U_2 are two (possibly correlated) Ornstein–Uhlenbeck processes.

We shall assume from now on that $\sigma \ll 1$ and we look for an expansion of $G(\omega)$ with respect to $\sigma \ll 1$. The strategy follows closely the one developed in [24]. We first divide the generator $\mathcal{L}$ into the sum $\mathcal{L} = \mathcal{L}_0 + \sigma\mathcal{L}_1$ with

$$\mathcal{L}_0 = Q + h_0(\psi)\frac{\partial}{\partial\psi}, \qquad \mathcal{L}_1 = h_1(\psi, \mathbf{m})\frac{\partial}{\partial\psi}.$$

As shown in [24], the probability density p can be expanded as $p = p_0 + \sigma p_1 + \sigma^2 p_2 + \cdots$, where p_0, p_1, and p_2 satisfy $\mathcal{L}_0^* p_0 = 0$ and $\mathcal{L}_0^* p_1 + \mathcal{L}_1^* p_0 = 0$, $\mathcal{L}_0^* p_2 + \mathcal{L}_1^* p_1 = 0, \ldots$. For once the expansion of p is known, it can be used in (29) to give the expansion of $G(\omega)$ at order 2 with respect to σ:

$$G(\omega) = 2\int_{S^1\times M} (q_0 p_0 + \sigma(q_1 p_0 + q_0 p_1) + \sigma^2(q_1 p_1 + q_0 p_2))(\psi, \mathbf{m})\,\mathrm{d}\psi\,\pi(\mathrm{d}\mathbf{m}). \tag{30}$$

5.3. Normal regime

Since h_0 is constant, p_0 is the density of the uniform distribution on $S^1 \times M : p_0 \equiv (2\pi)^{-1}$. Further, p_1 satisfies $\mathcal{L}_0^* p_1 = -\mathcal{L}_1^* p_0 = \partial_\psi(h_1 p_0)$ and is consequently given by

$$p_1(\psi, \mathbf{m}) = -\frac{1}{2\pi}\int_0^\infty \mathrm{d}z\,\mathbb{E}[\partial_\psi h_1(\psi + k(\omega)z, \mathbf{m}(z))/\mathbf{m}(0) = \mathbf{m}].$$

Substituting into (30), we obtain

$$G(\omega) = 2\sigma^2 k(\omega)^2 \alpha_1(\omega),$$

where $\alpha_1(\omega)$ is nonnegative and proportional to the power spectral density of the process $s(\omega)(m_\gamma - m_\beta)$ evaluated at the frequency $k(\omega)$:

$$\alpha_1(\omega) = s(\omega)^2 \int_0^\infty \mathrm{d}z\,\cos(2k(\omega)z)\mathbb{E}[(m_\gamma(0) - m_\beta(0))(m_\gamma(z) - m_\beta(z))]. \tag{31}$$

5.4. Anomalous regime

If $\beta_0\omega^2 \geq 2\gamma_0 P_0$, then we get the very same result as in the normal regime case. Let us assume that $2\gamma_0 P_0 - \beta_0\omega^2 > 0$, which is the region where the homogeneous MI gain is positive. The vanishing points of h_0 correspond to equilibrium points for the probability measure. Since only $\psi_0 = \pi/4$ is stable, we get that $p_0 = \delta_{\psi_0}$. Further, p_1 satisfies $\mathcal{L}_0^* p_1 = -\mathcal{L}_1^* p_0 = \partial_\psi(h_1 p_0)$, so the standard identity $\partial_\psi(h(\psi)\delta'_{\psi_0}(\psi)) = -\partial_\psi h(\psi_0)\delta'_{\psi_0}(\psi) + h(\psi_0)\delta''_{\psi_0}(\psi)$ yields that $p_1(\psi, \mathbf{m}) = r_1(\mathbf{m})\delta'_{\psi_0}(\psi)$, where

$$r_1(\mathbf{m}) = (Q - 2\kappa(\omega))^{-1} h_1(\mathbf{m}, \psi_0).$$

Since $\kappa(\omega) > 0$, the operator $(Q - 2\kappa(\omega))^{-1}$ is well defined as the resolvent of Q. The second-order term p_2 satisfies $\mathcal{L}_0^* p_2 = -\mathcal{L}_1^* p_1 = \partial_\psi(h_1(\psi, \mathbf{m})r_1(\mathbf{m})\delta'_{\psi_0}(\psi))$. Applying the identity $\partial_\psi(h(\psi)\delta''_{\psi_0}(\psi)) = \partial_\psi^2 h(\psi_0)\delta'_{\psi_0}(\psi) + (h - 2\partial_\psi h)(\psi_0)\delta''_{\psi_0}(\psi)$, we find p_2 in the form $p_2(\psi, \mathbf{m}) = r_{21}(\mathbf{m})\delta'_{\psi_0}(\psi) + r_{22}(\mathbf{m})\delta''_{\psi_0}(\psi)$, where

$$r_{21}(\mathbf{m}) = -(Q - 2\kappa(\omega))^{-1}(\partial_\psi h_1(\mathbf{m}, \psi_0) r_1(\mathbf{m})), \qquad r_{22}(\mathbf{m}) = (Q - 4\kappa(\omega))^{-1}(h_1(\mathbf{m}, \psi_0) r_1(\mathbf{m})).$$

An explicit representation of r_{22} is

$$\begin{aligned} r_{22}(\mathbf{m}) &= (Q - 4\kappa(\omega))^{-1}(h_1(\mathbf{m}, \psi_0)(Q - 2\kappa(\omega))^{-1} h_1(\mathbf{m}, \psi_0)) \\ &= \int_0^\infty \mathrm{d}s\,\mathrm{e}^{-4\kappa(\omega)s}\int_0^\infty \mathrm{d}z\,\mathrm{e}^{-2\kappa(\omega)z}\,\mathbb{E}[h_1(\mathbf{m}(s), \psi_0) h_1(\mathbf{m}(z+s), \psi_0)/\mathbf{m}(0) = \mathbf{m}]. \end{aligned}$$

Collecting the above results and using them in (29) establishes that

$$\begin{aligned}G(\omega)&=2\kappa(\omega)-8\sigma^2\kappa(\omega)\mathbb{E}[r_{22}(\mathbf{m}(0))]=2\kappa(\omega)-2\sigma^2\int_0^\infty \mathrm{d}z\,\mathrm{e}^{-2\kappa(\omega)z}\,\mathbb{E}[h_1(\mathbf{m}(0),\psi_0)h_1(\mathbf{m}(z),\psi_0)]\\&=2\kappa(\omega)-2\sigma^2\kappa(\omega)^2\alpha_2(\omega),\\ \alpha_2(\omega)&=s(\omega)^2\int_0^\infty \mathrm{d}z\,\mathrm{e}^{-2\kappa(\omega)z}\,\mathbb{E}[(m_\gamma(0)-m_\beta(0))(m_\gamma(z)-m_\beta(z))].\end{aligned}\tag{32}$$

Note that the coefficient of σ^2 is negative. This follows from its resolvent interpretation and the self-adjointness of Q, which implies that $\mathbb{E}[(m_\gamma(0)-m_\beta(0))(m_\gamma(z)-m_\beta(z))]$ is positive.

6. Fluctuations of the sample exponent growth

6.1. Central limit theorem for the fluctuations

The sample Lyapunov exponent $G(\omega)$ is the limit of

$$G(\omega)=2\lim_{z\to\infty}\frac{1}{z}\int_0^z q(\psi(z'),\mathbf{m}(z'))\,\mathrm{d}z'$$

with probability 1. It is interesting to obtain a central limit theorem for the corresponding fluctuations:

$$F(z):=\frac{1}{\sqrt{z}}\int_0^z 2q(\psi(z'),\mathbf{m}(z'))-G(\omega)\,\mathrm{d}z'.$$

Let us denote $\bar q(\psi,\mathbf{m}):=2q(\psi,\mathbf{m})-G(\omega)$. By the definition of $G(\omega)$, the random variable $\bar q(\psi,\mathbf{m}(0))$ has zero mean with respect to the invariant measure $p\pi(\mathrm{d}\mathbf{m})\,\mathrm{d}\psi$. Thus the equation $\mathcal{L}v=\bar q$ has a solution v which is bounded and unique up to an additive constant. In the following, we denote by v the solution which satisfies $\int vp(\psi,\mathbf{m})\pi(\mathrm{d}\mathbf{m})\,\mathrm{d}\psi=0$. The process M_v defined by

$$M_v(z):=v(\psi(z),\mathbf{m}(z))-v(\psi,\mathbf{m}(0))-\int_0^z\mathcal{L}v(\psi(z'),\mathbf{m}(z'))\,\mathrm{d}z'$$

is a martingale whole increasing process [25 Section 1.5]:

$$\langle M_v,M_v\rangle_z=\int_0^z(\mathcal{L}v^2-2v\mathcal{L}v)(\psi(z'),\mathbf{m}(z'))\,\mathrm{d}z'.$$

Note that the process R reads in terms of the martingale M_v as

$$R(z)^2=R_0^2\exp(G(\omega)z-M_v(z)+v(\psi(z),\mathbf{m}(z))-v(\psi,\mathbf{m}(0))),\tag{33}$$

and the fluctuations F of the sample exponent coefficient as

$$F(z)=z^{-1/2}(-M_v(z)-v(\psi(z),\mathbf{m}(z))-v(\psi,\mathbf{m}(0))).\tag{34}$$

Applying Theorem 2.1 [26], $\langle M_v,M_v\rangle$ satisfies

$$\frac{\langle M_v,M_v\rangle_z}{z}\overset{z\to\infty}{\to}V_\sigma:=-2\int_{S^1\times M}\bar q vp(\psi,\mathbf{m})\,\mathrm{d}\psi\,\pi(\mathrm{d}\mathbf{m}),\tag{35}$$

and the normalized process $z^{-1/2}M_v(z)$ converges in distribution to a Gaussian random variable with zero mean and variance V_σ. This proves the desired result that the normalized fluctuations $F(z)$ of the sample Lyapunov exponent obeys a Gaussian distribution with zero mean and variance V_σ. In the remainder of this section, we shall compute the expansion of V_σ at order 2 with respect to the noise level σ.

Since $\bar{q}$ expands as $\sigma\bar{q}_1+\sigma^2\bar{q}_2+\cdots$ with $\bar{q}_1=2q_1$, we can expand v in the form $\sigma v_1+\sigma^2 v_2+\cdots$, where v_1 is the solution of

$$Qv_1+h_0\frac{\partial v_1}{\partial\psi}=2q_1. \tag{36}$$

Collecting the terms with powers 2 in (35) establishes that $V_\sigma=\sigma^2 V_2+\mathrm{O}(\sigma^3)$ with

$$V_2=-4\int_{S^1\times M} q_1 v_1 p_0\,\mathrm{d}\pi(\mathbf{m})\,\mathrm{d}\psi. \tag{37}$$

6.2. Stable regime

If $\beta_0<0$ or $\beta_0>0$ and $\beta_0\omega^2>2\gamma_0 P_0$, then solving Eq. (36), we get

$$v_1(\psi,\mathbf{m})=-2k(\omega)s(\omega)\int_0^\infty \mathrm{d}z\,\mathbb{E}[\sin(2\psi+2k(\omega)z)(m_\gamma(z)-m_\beta(z))/\mathbf{m}(0)=\mathbf{m}].$$

Substituting into (35) yields $V_\sigma=V_2\sigma^2+\mathrm{O}(\sigma^3)$ with

$$V_2=4k(\omega)^2\alpha_1(\omega),$$

which is twice the value of $G(\omega)$.

6.3. Unstable regime

If $\beta_0>0$ and $\beta_0\omega^2<2\gamma_0 P_0$, then by the same method as above, we get that $V_\sigma=\sigma^2 V_2+\mathrm{O}(\sigma^3)$ with

$$V_2=8\kappa(\omega)^2\alpha_3(\omega), \tag{38}$$

$$\alpha_3(\omega)=\int_0^\infty \mathrm{d}z\,\mathbb{E}[(r(\omega)m_\beta(0)+s(\omega)m_\gamma(0))(r(\omega)m_\beta(z)+s(\omega)m_\gamma(z))]. \tag{39}$$

Note that $\alpha_3(\omega)$ is nonnegative since it is proportional to the power spectral density of the process $r(\omega)m_\beta(z)+s(\omega)m_\gamma(z)$.

7. The mean exponent growths

Since we know the sample exponent growth and its fluctuations, there remains only a few technical points to handle before establishing a closed form expression for the mean exponent growth. First note that the identity (33) involves that the mean Lyapunov exponent $G_{2n}(\omega)$ reads in terms of the martingale M_v as

$$G_{2n}(\omega)=G(\omega)+\lim_{z\to\infty}\frac{1}{nz}\ln\mathbb{E}[\exp(nM_v(z))].$$

Second V_σ is nothing else than the expectation of $f:=\mathcal{L}v^2-2v\mathcal{L}v$ with respect to the invariant measure $p\pi(\mathrm{d}\mathbf{m})\,\mathrm{d}\psi$. Denoting $\bar{f}(\psi,\mathbf{m}):=f(\psi,\mathbf{m})-V_\sigma$, the random variable $\bar{f}(\psi,\mathbf{m}(0))$ has zero mean so that the equation $\mathcal{L}w=\bar{f}$ has a unique solution which satisfies $\int wp(\psi,\mathbf{m})\pi(\mathrm{d}\mathbf{m})\,\mathrm{d}\psi=0$. Accordingly, the process M_w:

$$M_w:=w(\psi(z),\mathbf{m}(z))-w(\psi,\mathbf{m}(0))-\int_0^z\mathcal{L}w(\psi(z'),\mathbf{m}(z'))\,\mathrm{d}z'$$

is a martingale whole increasing process:

$$\langle M_w, M_w\rangle_z = \int_0^z (\mathcal{L}w^2 - 2w\mathcal{L}w)(\psi(z'), \mathbf{m}(z'))\,\mathrm{d}z'.$$

Note that v is of order σ, so f and w are of order σ^2 and $\langle M_w, M_w\rangle_z \le K\sigma^4 z$. Besides, the increasing process of the martingale M_v reads as

$$\langle M_v, M_v\rangle_z = V_\sigma z - M_w(z) + w(\psi(z), \mathbf{m}(z)) - w(\psi, \mathbf{m}(0)).$$

Thus, for any $n > 0$, the increasing process of the martingale $M_v - \frac{1}{2}nM_w$ reads as

$$\langle M_v - \tfrac{1}{2}nM_w, M_v - \tfrac{1}{2}nM_w\rangle_z = \langle M_v, M_v\rangle_z + \tfrac{1}{4}n^2\langle M_w, M_w\rangle_z - n\langle M_v, M_w\rangle_z = V_\sigma z - M_w(z) + \xi_\sigma(z),$$

where $|\xi_\sigma(z)| \le K(1+n\sigma^3 z+n^2\sigma^4 z)$. We substitute this identity into the expectation of the exponential martingale associated with $M_v - \frac{1}{2}nM_w$:

$$\mathbb{E}[\exp(n(M_v - \tfrac{1}{2}nM_w)(z) - \tfrac{1}{2}n^2\langle M_v - \tfrac{1}{2}nM_w, M_v - \tfrac{1}{2}nM_w\rangle_z)] = 1,$$

so that we get

$$\mathbb{E}[\exp(nM_v(z) - \tfrac{1}{2}n^2 V_\sigma z - \tfrac{1}{2}n^2\xi_\sigma(z))] = 1.$$

From the uniform bounds on ξ_σ, we obtain that

$$\mathbb{E}[\exp(nM_v(z))] \le \exp(\tfrac{1}{2}n^2 V_\sigma z + \tfrac{1}{2}n^2 K(1 + n\sigma^3 z + n^2\sigma^4 z)),$$
$$\mathbb{E}[\exp(nM_v(z))] \ge \exp(\tfrac{1}{2}n^2 V_\sigma z - \tfrac{1}{2}n^2 K(1 + n\sigma^3 z + n^2\sigma^4 z)).$$

Applying to these inequalities the operation $(nz)^{-1}\ln\{\cdot\}$ and taking the limit $z \to \infty$, we finally get the expansion at order 2 with respect to σ of $G_{2n}(\omega)$:

$$G_{2n}(\omega) = G(\omega) + \tfrac{1}{2}nV_\sigma + \mathrm{O}(n^2\sigma^3). \tag{40}$$

Since $V_\sigma = V_2\sigma^2 + \mathrm{O}(\sigma^3)$, the relative error in Eq. (40) is proportional to $n\sigma$. It is all the more important as the moment is higher. This fact is involved by the high sensitivity of the high-order moments to rare events.

8. Discussion

8.1. Stable regime

We assume in this section that either $\beta_0 < 0$ or $\beta_0 > 0$ and $\beta_0\omega^2 > 2\gamma_0 P_0$, which are the regions where there is no MI in the homogeneous framework. The sample and moment Lyapunov exponents have expansions at order 2 with respect to σ which are the ones of the random process:

$$I(z) = I_0 \exp(2\sigma\sqrt{k^2(\omega)\alpha_1(\omega)}W_z + 2\sigma^2 k^2(\omega)\alpha_1(\omega)z), \tag{41}$$

where W_z is a standard Brownian motion and α_1 is defined by (31). More exactly, the 2nth moment MI gain is given by

$$G_{2n}(\omega) = 2(n+1)\sigma^2 k^2(\omega)\alpha_1(\omega). \tag{42}$$

This formula demonstrates the empirical relation $G_{2n} = 2/(n+1)G_2$ which was conjectured in Section 4.2. Besides, the sample MI gain is

$$G(\omega) = 2\sigma^2 k^2(\omega)\alpha_1(\omega), \tag{43}$$

which is half the value of $G_2(\omega)$. Remember that the sample MI gain is the gain which is actually observed for a typical realization. This replies in the positive to the question whether randomness enhances MI process in the normal regime.

If the processes m_β and m_γ are independent white noises with delta-correlated correlation functions (14), then $\sigma^2\alpha_1(\omega) = \frac{1}{2}s(\omega)^2(\sigma_\gamma^2 + \sigma_\beta^2)$ for any ω and we get back the results corresponding to Section 4.1.1. This also justifies the choice of the Stratonovitch stochastic integral in Section 4: this configuration is the limit as $\sigma \to 0$ of the general configuration (2) and (3). This is due to the fact that after rescaling $z \to z/\sigma^2$, the processes m_β and m_γ appear in the forms $(1/\sigma)m_\beta(z/\sigma^2)$ and $(1/\sigma)m_\gamma(z/\sigma^2)$, whose distributions are close to white noises.

8.2. Unstable regime

We assume in this section that $\beta_0 > 0$ and $\beta_0\omega^2 < 2\gamma_0 P_0$, which is the region where the homogeneous MI gain is positive. The sample and moment Lyapunov exponents have expansions at order 2 with respect to σ, which are the ones of the random process:

$$I(z) = I_0 \exp(2\kappa(\omega)z + 2\sigma\sqrt{2\kappa^2(\omega)\alpha_3(\omega)}W_z - 2\sigma^2\kappa^2(\omega)\alpha_2(\omega)z), \tag{44}$$

where α_2 and α_3 are defined, respectively, by (32) and (39). The moment and sample MI gains are consequently

$$G_{2n}(\omega) = 2\kappa(\omega) + 4n\sigma^2\kappa(\omega)^2\alpha_3(\omega) - 2\sigma^2\kappa(\omega)^2\alpha_2(\omega), \tag{45}$$

$$G(\omega) = 2\kappa(\omega) - 2\sigma^2\kappa(\omega)^2\alpha_2(\omega). \tag{46}$$

Note that the sample MI gain spectrum $G(\omega)$ cannot be deduced from the mean MI gain spectrum $G_2(\omega)$ by a simple transform unlike the stable case. Indeed $\alpha_2(\omega)$ and $\alpha_3(\omega)$ are different from each other.

Let us examine the particular case $m_\gamma \equiv 0$. A striking point is that the MI peak is reduced due to the random fluctuations of the GVD, and this reduction is deterministic. This is mainly due to the fact that $r(\omega_{\mathrm{det,opt}} = \sqrt{\gamma_0 P_0/\beta_0}) = 0$:

$$G_{2n,\mathrm{opt}}|_{m_\gamma\equiv 0} = G_{\mathrm{opt}}|_{m_\gamma\equiv 0} = 2\gamma_0 P_0 - 2\sigma^2\gamma_0^2 P_0^2\alpha_4, \quad \alpha_4 = \int_0^\infty \mathrm{e}^{-2\gamma_0 P_0 z}\, \mathbb{E}[m_\beta(0)m_\beta(z)]\,\mathrm{d}z. \tag{47}$$

Furthermore, the optimal frequency is slightly below $\omega_{\mathrm{det,opt}}$ and is also deterministic:

$$\omega^2_{2n,\mathrm{opt}}|_{m_\gamma\equiv 0} = \omega^2_{\mathrm{opt}}|_{m_\gamma\equiv 0} = 2\beta_0^{-1}\gamma_0 P_0 - 2\sigma^2\beta_0^{-1}\gamma_0^2 P_0^2\alpha_4.$$

In the general case ($m_\gamma \neq 0$ and $m_\beta \neq 0$), the optimal frequency and the MI peak are random quantities. The frequency which corresponds to the maximal gain is for almost every realization of the processes (m_γ, m_β):

$$\omega^2_{\mathrm{opt}} = \beta_0^{-1}\gamma_0 P_0 - 2\sigma^2\beta_0^{-1}\gamma_0^2 P_0^2\alpha_4',$$
$$\alpha_4' = \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-2\gamma_0 P_0 z}\, \mathbb{E}[(m_\beta(z) - m_\gamma(z))(m_\beta(0) - m_\gamma(0))],$$

and the corresponding MI peak is

$$G_{\mathrm{opt}} = 2\gamma_0 P_0 - 2\sigma^2\gamma_0^2 P_0^2\alpha_4'. \tag{48}$$

The frequency which corresponds to the maximum of the moment MI gain G_{2n} is

$$\omega^2_{2n,\mathrm{opt}} = \beta_0^{-1}\gamma_0 P_0 + \sigma^2\beta_0^{-1}\gamma_0^2 P_0^2(2n\alpha_{\gamma\gamma} + n\alpha_{\gamma\beta} - 2\alpha_4'),$$
$$\alpha_{\gamma\gamma} = \int_0^\infty \mathbb{E}[m_\gamma(0)m_\gamma(z)]\,\mathrm{d}z,\quad \alpha_{\gamma\beta} = \int_0^\infty \mathbb{E}[m_\gamma(0)m_\beta(z) + m_\beta(0)m_\gamma(z)]\,\mathrm{d}z.$$

The corresponding MI peak is

$$G_{2n,\mathrm{opt}} = 2\gamma_0 P_0 + \sigma^2\gamma_0^2 P_0^2(4n\alpha_{\gamma\gamma} - 2\alpha_4'). \tag{49}$$

On the one hand, Eq. (49) shows that the fluctuations of the nonlinear coefficient involve an enhancement of the moment MI peak $G_{2n,\mathrm{opt}}$ for n larger than $\alpha_4'/2\alpha_{\gamma\gamma}$. On the one hand, Eq. (48) shows that the corrective term in σ^2 of the sample MI peak G_{opt} is nonpositive, which means that randomness always involves a reduction of the sample MI peak. This replies in the negative to the question whether randomness enhances MI process in the anomalous regime.

9. Numerical simulations

All of the above results stem from the linear stability analysis of plane wave solutions (4). To check the validity of the analytic predictions, we perform extensive numerical simulations of Eq. (1) with randomly varying coefficients $\beta(z)$ and $\gamma(z)$. The simulations are carried out solving (1) with the split-step Fourier method. The initial condition is set to $u_1 = \epsilon\,\mathrm{e}^{-\mathrm{i}\omega t} + \epsilon\,\mathrm{e}^{\mathrm{i}\omega t}$ with $\epsilon = 10^{-5}$. The number of sampling points in the time domain is 256, and the z step is a small fraction of the total propagation length ($\mathrm{d}z = L/300$, $L = 6$). As a model for the random processes m_β and m_γ, we choose step-wise constant functions which take independent and random values in $\{-\sigma, \sigma\}$ over elementary intervals with lengths l. Accordingly,

$$\int_0^\infty \mathbb{E}[m_\beta(0)m_\beta(z)]\cos(2kz)\,\mathrm{d}z = \sigma^2\frac{1-\cos(2kl)}{4k^2 l},\qquad \int_0^\infty \mathbb{E}[m_\beta(0)m_\beta(z)]\,\mathrm{e}^{-2\kappa z}\,\mathrm{d}z = \sigma^2\frac{\mathrm{e}^{-2\kappa l} + 2\kappa l - 1}{4\kappa^2 l}.$$

This configuration can be approximated by a white-noise model with $\sigma_\beta^2 = \sigma_\gamma^2 = l\sigma^2$. Note that this holds true only for frequency ω such that $2k(\omega)l \ll 1$, which is the case for all the configurations that are considered in this section.

Fig. 5 displays the comparison between numerical calculations (averaged over 100 simulations) and analytic gain curves in the anomalous dispersion regime with random nonlinear coefficients and constant GVD. Theory (left picture) predicts that the MI peak is enhanced and is random (in the sense that $G_{2n,\mathrm{opt}}$ increases with n), and that the spectral width is enhanced with respect to the $\sigma_\gamma = 0$ case (solid line). A good agreement between the theoretical predictions and the numerical simulations is observed (right picture).

Fig. 6 compares numerical calculations and analytic gain curves in the anomalous dispersion regime with random GVD and constant nonlinear coefficient. Theory (left picture) predicts that the MI peak is reduced and is deterministic (in the sense that $G_{2n,\mathrm{opt}}$ does not depend on n), and that the spectral width is enhanced with respect to the $\sigma_\beta = 0$ case (solid line). Once again the numerical simulations (right picture) confirm these predictions.

The results for the normal dispersion regime are shown in Figs. 7 and 8. The stability analysis predicts the extension of the domain of unstable side modes to the whole spectrum. This prediction is well confirmed by numerical simulations (here the average is over 1000 runs): as can be seen, the gain value agrees with the theoretical estimate. Here the MI gain is solely due to the presence of randomness. The presence of random-induced MI in the normal regime may be relevant to signal transmissions in dispersion managed links, where fibers of opposite

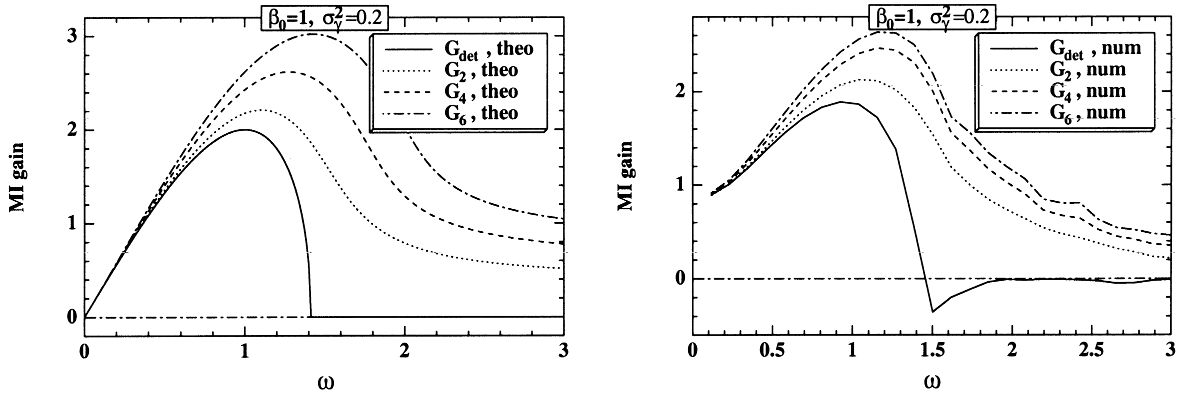

Fig. 5. MI gain curves for $\beta_0 = 1$, $\gamma_0 = 1$, $P_0 = 1$, homogeneous GVD and random nonlinear coefficient with $\sigma_\gamma^2 = 0.2$ ($l = 0.2$ and $\sigma = 1$). The solid lines correspond to the homogeneous case, while the dashed lines correspond to the mean Lyapunov exponents G_{2n}. The left picture corresponds to the theoretical formulae obtained with the white-noise approximations. The right picture corresponds to numerical simulations averaged over 100 runs.

dispersion are concatenated [27–29]. Another important feature is that the exponents G_{2n} strongly depend on n, which proves that the MI gain process is highly fluctuating.

In order to complete our comments, we point out that the value of the $2n$th moment MI gain G_{2n} is imposed by very rare events (the realizations of the processes) which correspond to very large deviations from the typical behavior G. Furthermore, the sets of events E_{2n} which control G_{2n} becomes smaller and smaller as n increases, and they correspond to larger and larger deviations. In fact, these sets are a decreasing sequence: $E_2 \supset E_4 \supset \cdots \supset E_{2n} \supset \cdots$. That is why in the numerical simulations, the mean value G_2 is well fitted, and so is G_4, but higher moments are not, because the few hundred runs that were performed have obviously not met some of the events of E_6.

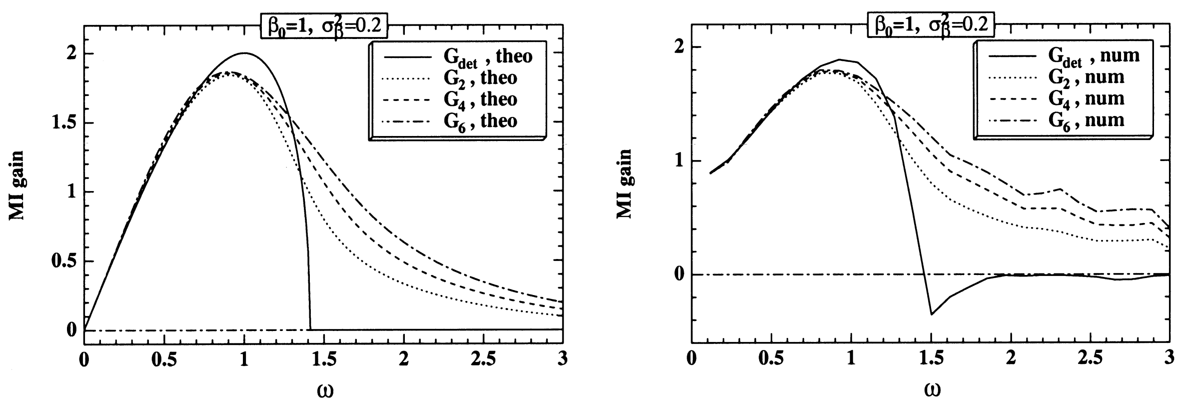

Fig. 6. MI gain curves for $\beta_0 = 1$, $\gamma_0 = 1$, $P_0 = 1$, homogeneous nonlinear coefficient and random GVD with $\sigma_\beta^2 = 0.2$ ($l = 0.2$ and $\sigma = 1$). The solid lines correspond to the homogeneous case, while the dashed lines correspond to the normalized exponential growth rates G_{2n}. The left picture corresponds to the theoretical formulae obtained with the white-noise approximations. The right picture corresponds to averaging over 100 numerical simulations.

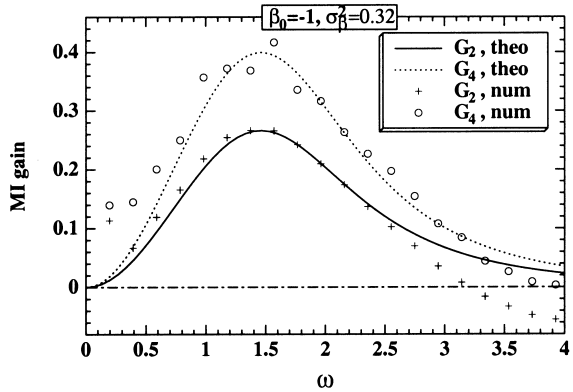

Fig. 7. MI gain curves for $\beta_0 = -1$, $\gamma_0 = 1$, $P_0 = 1$, homogeneous nonlinear coefficient and random GVD with $\sigma_\beta^2 = 0.32$ ($l = 0.02$ and $\sigma = 4$). The lines correspond to the theoretical formulae, while the crosses and circles correspond to the numerical results averaged over 1000 simulations.

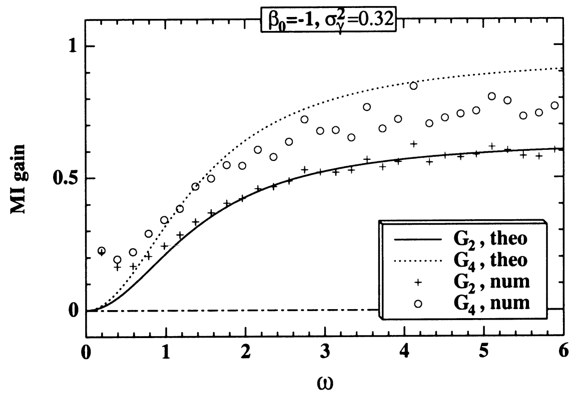

Fig. 8. MI gain curves for $\beta_0 = -1$, $\gamma_0 = 1$, $P_0 = 1$, homogeneous GVD and random nonlinear coefficient with $\sigma_\gamma^2 = 0.32$ ($l = 0.02$ and $\sigma = 4$). The lines correspond to the theoretical formulae, while the crosses and circles correspond to the numerical results averaged over 1000 simulations.

10. Conclusion

In this paper, we have investigated the MI of the continuum wave in nonlinear medium with random coefficients. Using the linear stability analysis and stochastic analysis, we obtain for the white-noise model of fluctuations of the coefficients, the system of equations for the low-order and high-order moments of the intensity of the modulation. Using this system, we obtain closed-form expressions for the MI gains. As a conclusion of this section, we have pointed out that the MI gain presents very large fluctuations, and that it is consequently difficult to predict a typical behavior.

We then analyze a more general configuration with a colored noise model so as to describe the complete statistical distribution of the MI gain. We have shown the following:

- The qualitative effect of randomness in the anomalous regime is to reduce the standard homogeneous MI gain peak and to enhance the MI spectrum. In the normal regime (where there is no instability with constant coefficients), all frequencies are made unstable.

- A striking point is that the statistical distribution of the intensity of the modulation can be expressed in terms of the log-normal statistics (see Eqs. (41) and (44)). The log-normal distribution has such a heavy tail that the moments of the intensity have very different behaviors. More exactly, the growth of the intensity of the modulation is governed by an expression of the kind $\exp(aW_z + bz)$, where $W_.$ is a standard Brownian motion. Accordingly, there are different behaviors for the mean case and for a typical case, because $W_z \sim \sqrt{z}$ with high probability, but $\mathbb{E}[\exp(aW_z)] = \exp(\frac{1}{2}a^2 z)$.
- The MI gain peak for almost every realization is described by a nonrandom quantity in case of GVD fluctuations, where only the Laplace transform of the autocorrelation function of the random process at a particular frequency occurs. In case of fluctuations of the nonlinear coefficient, we have put into evidence that the MI peak is reduced in probability, but is enhanced in mean, because there exist some rare events for which the MI peak is drastically increased, and these rare events impose the mean value.

We have checked the analytical predictions based on the linear stability analysis by numerical simulations of the full nonlinear Schrödinger equation. We find good agreement between predictions of theory and results of numerical simulations. Finally, we would like to add that the methodology we present here for the scalar nonlinear Schrödinger equation can be applied to the vector nonlinear Schrödinger equation that governs the propagation of electromagnetic waves in birefringent fibers [10].

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