PRICING PARISIAN OPTIONS USING LAPLACE TRANSFORMS

CÉLINE LABART AND JÉRÔME LELONG

Abstract. In this work, we propose to price Parisian options using Laplace transforms. Not only do we compute the Laplace transforms of all the different Parisian options, but we also explain how to invert them numerically. We prove the accuracy of the numerical inversion.

1. Introduction

The analysis of structured financial products often leads to the pricing of exotic options. For instance, consider a re-callable convertible bond. The holder typically wants to recall the bond if ever the underlying stock price is traded above or below a given level for too long. Such a contract can be modeled with the help of Parisian options. Parisian options are barrier options that are activated or canceled depending on the type of option if the underlying asset stays above or below the barrier long enough in a row. Parisian options are far less sensitive to influential agent on the market that standard barrier options. It is quite easy for an agent to push the price of a stock momentarily but not on a longer period so that it would affect the Parisian contract.

In this work, we study the pricing of European style Parisian options using Laplace transforms. Some other methods have already been proposed. On path dependent options, crude Monte Carlo techniques do usually not perform well. An improvement of this strategy using sharp large deviation estimates was proposed by Baldi et al. (2000). Techniques using a two dimensional partial differential equation have also drawn much attention, see for instance the works of Avellaneda and Wu (1999), Haber et al. (1999), or Wilmott (1998). The PDE approach is quite flexible and could even be used for American style Parisian option but the convergence is rather slow, which is badly suited for real time evaluation. In a not so different state of mind, tree methods based on the framework of Cox et al. (1979) were investigated by Costabile (2002). An original concept of implied barrier was developed by Anderluh and van der Weide (2004), the idea is to replace the Parisian option by a standard barrier option with a suitably shifted barrier. The idea of using Laplace transforms to price Parisian options was introduced by Chesney et al. (1997). Their work is based on Brownian excursion theory in general and in particular on the study of the Azéma martingale (see Azéma and Yor (1989)) and the Brownian meander. The prices are then computed by numerically inverting the Laplace transforms, an original way of doing so was proposed by Quittard-Pinon et al. (2004). Their idea is to approximate the Laplace transforms by negative power functions whose analytical inverse is well-known. Often, there is no upper bound for the error due to the inversion.

In this work, we give the formulae of the Laplace transforms of the prices of the different Parisian options ready to be implemented. We also derive the formulae for the prices at any time after the emission time. We prove an accuracy result for the numerical inversion of the Laplace transforms to find the prices back.

First, we define the Parisian contract and introduce some material related to the excursion theory. Then, we present a few parity relationships which enable to reduce the pricing of the eight different types of Parisian options to the pricing of the down and in call — when the barrier is smaller than the initial value — and the up and in call — when the barrier is greater than the initial value. The Laplace transforms of the prices of the two latter

Date: April 23, 2007.

options are computed in Sections 4 and 5. Section 6 is devoted to the pricing at any time after the emission time of the option. At this stage, we are able to compute the Laplace transforms of the prices of all the different Parisian options, we only need a method to accurately invert them. In Section 7, we study in details the numerical inversion of Laplace transforms as introduced by Abate and Whitt (Winter 1995) and prove an upper bound for the error. Finally, the last section is devoted to the comparison of our method with the enhanced Monte Carlo method of Baldi et al. (2000) whose implementation in PREMIA[1] has been used for the comparison. We have also implemented our method in PREMIA.

sec:defs-single

2. Definitions

2.1. **Some notations.** We consider a Brownian motion $W = \{W_t, t \geq 0\}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, which models a financial market. We assume that $\mathbb{Q}$ is the risk neutral measure and that $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of W. We denote by T the maturity time. In this context, we assume that the dynamics of an asset price is given by the process S

$$\forall t \in [0,T], \quad S_t = x\, \mathrm{e}^{(r-\delta-\sigma^2/2)t+\sigma W_t},$$

where $r > 0$ is the interest rate, $\delta > 0$ is the dividend rate, $\sigma > 0$ the volatility and $x > 0$ the initial value of the stock. The Cameron-Martin-Girsanov Theorem (see Karatzas and Shreve (1991)) enables to state the following proposition for a finite time horizon $[0,T]$ with $T > 0$.

prop:chgt-proba

Proposition 1. *Let $m = \frac{1}{\sigma}\left(r - \delta - \frac{\sigma^2}{2}\right)$ and $\mathbb{P}$ be a new probability, which makes $Z = \{Z_t = W_t + mt, 0 \leq t \leq T\}$ a $\mathbb{P}$-Brownian motion. The change of probability is given by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}_{|\mathcal{F}_T} = \mathrm{e}^{mZ_T - \frac{m^2}{2}T},$$

and under $\mathbb{P}$, the dynamics of S is given by

$$\forall t \in [0,T], \quad S_t = x\ \mathrm{e}^{\sigma Z_t}.$$

Remark 1. *Since the drift term linking W and Z is deterministic, $\mathcal{F}_t$ is also the natural filtration of Z.*

Before explaining what a Parisian option is, we introduce the notion of excursion.

Definition 1 (Excursion)**.** *For any $L > 0$ and $t > 0$, we define*

$$g^S_{L,t} = \sup\{u \leq t \,:\, S_u = L\} \qquad d^S_{L,t} = \inf\{u \geq t \,:\, S_u = L\}.$$

with the conventions $\sup \emptyset = 0$ and $\inf \emptyset = +\infty$. The trajectory of S between $g^S_{L,\,t}$ and $d^S_{L,\,t}$ is the excursion at level L, straddling time t.

Obviously, such an excursion can also be described in terms of the Brownian motion Z. For a given barrier L for the process S, we introduce the corresponding barrier b for Z defined by

$$b = \frac{1}{\sigma}\log\left(\frac{L}{x}\right).$$

Definition 2 (Stopping times T_b, T_b^- and T_b^+)**.** *Let $b \in \mathbb{R}$ and $t > 0$, we define the hitting time of level b by*

$$T_b(Z) = \inf\{u > 0 \,:\, Z_u = b\}.$$

In order to define $T_b^-(Z)$ and $T_b^+(Z)$, we introduce g_t^b and d_t^b

$$g_t^b = \sup\{u \leq t \,:\, Z_u = b\}, \quad d_t^b = \inf\{u \geq t \,:\, Z_u = b\}.$$

[1]PREMIA is a pricing software developed the MathFi team of INRIA Rocquencourt, see http://www.premia.fr.

Let $T_b^-(Z)$ denote the first time the Brownian motion Z makes an excursion longer than some time D below the level b

$$T_b^-(Z) = \inf\{t > 0 \,:\, (t - g_t^b)\, \mathbb{1}_{\{Z_t < b\}} \geq D\}. \tag{1}$$

eq:def_tb-

For the excursion above b, we define

$$T_b^+(Z) = \inf\{t > 0 \,:\, (t - g_t^b)\, \mathbb{1}_{\{Z_t > b\}} \geq D\}. \tag{2}$$

eq:def_tb+

When no confusion is possible, we write T_b, T_b^- and T_b^+ instead of $T_b(Z)$, $T_b^-(Z)$ and $T_b^+(Z)$.

Remark 2. *Note that $g_t^b = g_{L,t}^S$ and $d_t^b = d_{L,t}^S$. Moreover, we can also write*

$$T_b^-(Z) = \inf\{t > 0 \,:\, \forall s \in [t - D, t]\ Z_t < b\}.$$

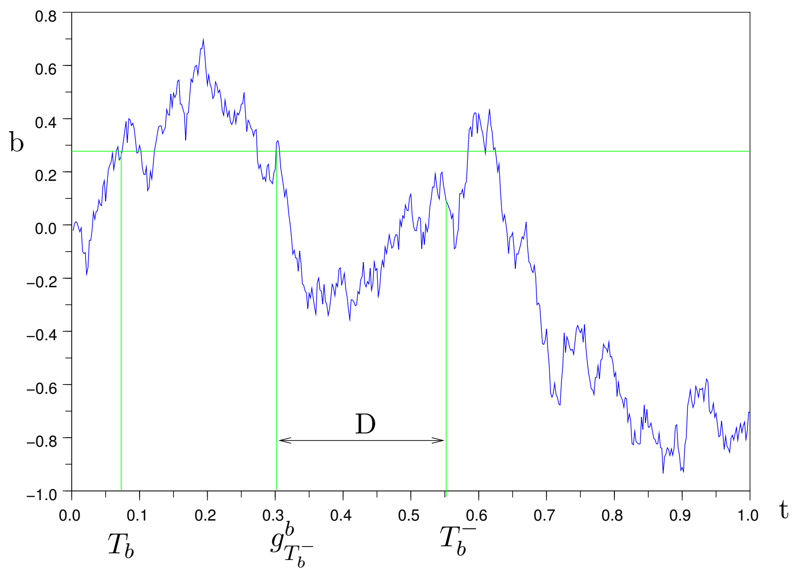

Figure 1. Excursion of Brownian Motion

fig:excursion

Definition 3 (Laplace transform)**.** *The Laplace Transform of a function f defined for all $t \geq 0$ is the following function $\hat{f}$*

$$\hat{f}(\lambda) = \int_0^{+\infty} e^{-\lambda t} f(t)dt.$$

Parisian options can be seen as barrier options where the condition involves the time spent in a row above or below a certain level and not only a hitting time. As for barrier options, which can be activated or canceled (depending whether they are in or out) when the asset S hits the barrier, Parisian options can be activated (in options) or canceled (out options) after S has spent more than a certain time in a excursion. Parisian options are defined in the following way

Definition 4 (Definition of θ, k and d)**.** *In the following we define*

$$\theta = \sqrt{2\lambda}, \quad k = \frac{1}{\sigma}\log\left(\frac{K}{x}\right), \quad d = \frac{b-k}{\sqrt{D}}.$$

Definition 5 (Parisian Options)**.** *A Parisian option is defined by three characteristics:*

- *Up or Down,*

- *In or Out,*
- *Call or Put.*

Combining the above characteristics together enables to distinguish eight types of Parisian options. For example, PDIC denotes a Parisian Down and In call, whereas PUOP denotes a Parisian Up and Out put.

In the following section, we present Parisian Down options.

2.2. **Parisian Down options.**

2.2.1. *Parisian Down and In options.* The owner of a Down and In option receives the payoff if and only if S makes an excursion below level L older than D before maturity time T. The price of a Down and In option at time 0 with payoff $\phi(S_T)$ is given by

$$e^{-rT}\,\mathbb{E}_{\mathbb{Q}}\left(\phi(S_T)\mathbb{1}_{\{T_b^- < T\}}\right) = e^{-(r+\frac{m^2}{2})T}\,\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{T_b^- < T\}}\phi(x\,e^{\sigma Z_T})\,e^{mZ_T}\right). \tag{3}$$ eq:DIO

For the sake of clearness, we introduce the following notation

Definition 6 (the star notation)**.** *For any function f, we define*

$$f^\star(T) = e^{(r+\frac{1}{2}m^2)T} f(T). \tag{4}$$ eq:star

From (3), we define the price of a Parisian Down and In call.

def:PDIC **Definition 7** (Parisian Down and In call)**.** *Let $PDIC(x,T;K,L;r,\delta)$ denote the value of a Parisian Down and In call. Then,*

$$PDIC(x,T;K,L;r,\delta) = e^{-(r+\frac{1}{2}m^2)T}\,\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{T_b^- < T\}}(x\,e^{\sigma Z_T} - K)^+\,e^{mZ_T}).$$

Using notation (4), we obtain

$$PDIC^\star(x,T;K,L;r,\delta) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{T_b^- < T\}}(x\,e^{\sigma Z_T} - K)^+\,e^{mZ_T}).$$

2.2.2. *Parisian Down and Out options.* A Down and Out Parisian option becomes worthless if S reaches L and remains constantly below level L for a time interval longer than D before maturity time T. The price of a Down and Out option at time 0 with payoff $\phi(S_T)$ is given by

$$e^{-rtT}\,\mathbb{E}_{\mathbb{Q}}\left(\phi(S_T)\mathbb{1}_{\{T_b^- > T\}}\right) = e^{-(r+\frac{m^2}{2})T}\,\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{T_b^- > T\}}\phi(x\,e^{\sigma Z_T})\,e^{mZ_T}\right). \tag{5}$$ eq:DOO

From (5), we define the price of a Parisian Down and Out call.

def:PDOC **Definition 8** (Parisian Down and Out call)**.** *Let $PDOC(x,T;K,L;r,\delta)$ denote the value of a Parisian Down and Out call. Then,*

$$PDOC(x,T;K,L;r,\delta) = e^{-(r+\frac{1}{2}m^2)T}\,\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{T_b^- > T\}}(x\,e^{\sigma Z_T} - K)^+\,e^{mZ_T}).$$

Using notation (4), we obtain

$$PDOC^\star(x,T;K,L;r,\delta) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{T_b^- > T\}}(x\,e^{\sigma Z_T} - K)^+\,e^{mZ_T}).$$

3. Relationship between prices

Parisian option prices cannot be computed directly. We are only able to give closed formulae for their Laplace transforms w.r.t. the maturity time T. As we have seen it in the above definitions, Parisian option prices depend on many parameters. The computation of the Laplace transform of one option price (say PDOC) requires to distinguish several cases, depending on the relative positions of x, L and K. The sign of b ($= \frac{1}{\sigma}\log(\frac{L}{x})$) is really important and even if computing the Laplace transform of $PDOC^\star$ (denoted $\widehat{PDOC}^\star$ in the following) in the case $b \leq 0$ is quite easy, the computation becomes more complex in the case $b > 0$. In Section 3.2 we explain that computing the value of $\widehat{PDOC}^\star$

when $b > 0$ can be reduced to computing the value of $\widehat{PDOC}^\star$ with $b = 0$. As we will see in Section 3.1, there also exists an in and out parity relationship between the prices, this means that we can deduce the value of $PDOC^\star$ from the value of $PDIC^\star$. The following scheme explains how to deduce the Laplace transforms of the different kinds of Parisian call options one from the others. Moreover, in Section 3.3, we state a call put parity relationship, which enables to deduce the Parisian put prices from the corresponding call price through the Black Scholes formula.

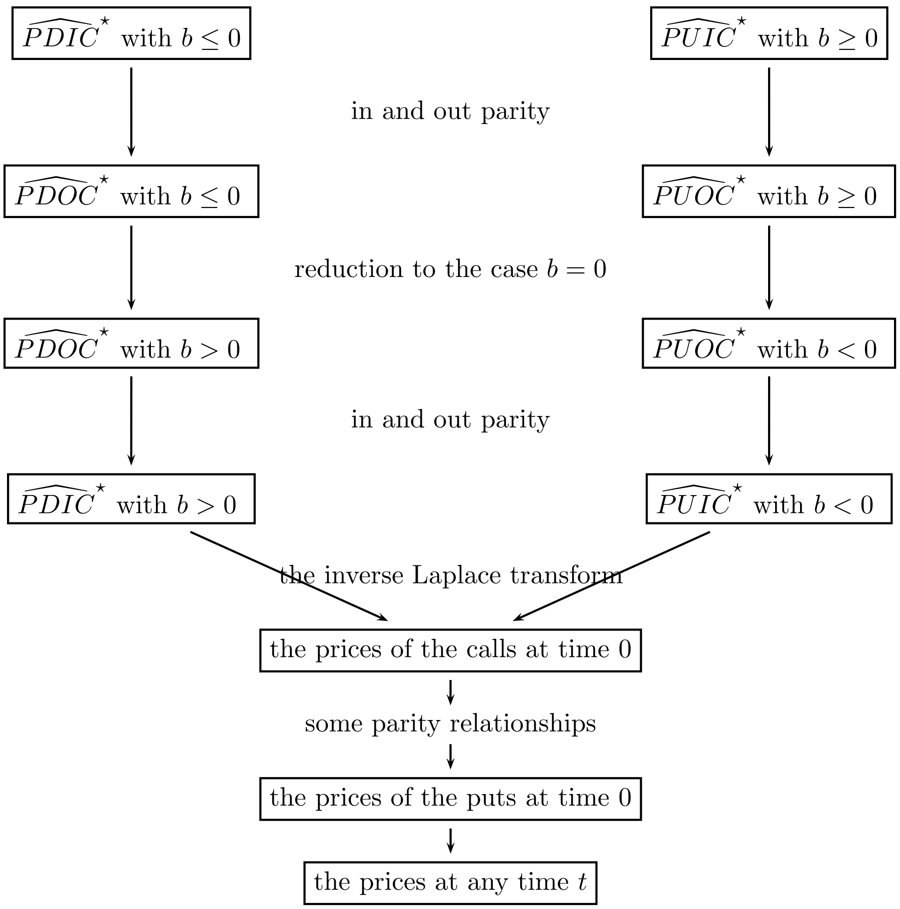

FIGURE 2. Deduction scheme of Parisian option prices fig:orga

in_out_parity

3.1. **In and Out parity.** This part is devoted to make precise the way we compute the value of $\widehat{PDOC}^\star$ from the value of $\widehat{PDIC}^\star$. The technique developed below remains valid for Parisian Up call options. We recall Definitions 7 and 8,

$$PDIC^\star(x, T; K, L; r, \delta) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{T_b^- < T\}}(x\, e^{\sigma Z_T} - K)^+ \, e^{m Z_T}).$$

$$PDOC^\star(x, T; K, L; r, \delta) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{T_b^- > T\}}(x\, e^{\sigma Z_T} - K)^+ \, e^{m Z_T}).$$

By adding the l.h.s. we get

$$PDIC^\star(x, T; K, L; r, \delta) + PDOC^\star(x, T; K, L; r, \delta) = \mathbb{E}_{\mathbb{P}}((x\, e^{\sigma Z_T} - K)^+ \, e^{m Z_T}). \quad (6)$$ eq:in-out-parity

def:BS **Definition 9.** *Let us define*

$$BSC^\star(x, T; K; r, \delta) = \mathbb{E}_{\mathbb{P}}((x\, e^{\sigma Z_T} - K)^+ \, e^{m Z_T}).$$

BSC is the price of a Black Scholes call option.

From (6), we get

$$\widehat{PDOC}^{\star}(x,\lambda;K,L;r,\delta) = \widehat{BSC}^{\star}(x,\lambda;K;r,\delta) - \widehat{PDIC}^{\star}(x,\lambda;K,L;r,\delta).$$

Then, if we get closed formulae for both $\widehat{PDIC}^{\star}$ and $\widehat{BSC}^{\star}$, we can easily deduce a closed formula for $\widehat{PDOC}^{\star}$. Since the pricing of a Parisian option can only be achieved through the numerical inversion of its Laplace transform, it makes sense to compute the Laplace transform of BSC even though it can also be accessed through the Black Scholes formula (Black and Scholes (1973,)).

The following proposition gives the value of $\widehat{BSC}^{\star}(x,\lambda;K;r,\delta)$

prop:bs

Proposition 2. *For $K \geq x$,*

$$\widehat{BSC}^{\star}(x,\lambda;K;r,\delta) = \frac{K}{\theta}\,\mathrm{e}^{(m-\theta)k}\left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta}\right).$$

For $K \leq x$,

$$\widehat{BSC}^{\star}(x,\lambda;K;r,\delta) = \frac{2K}{m^2-\theta^2} - \frac{2x}{(m+\sigma)^2-\theta^2} + \frac{K\,\mathrm{e}^{(m+\theta)k}}{\theta}\left(\frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta}\right). \quad (7)$$

Proof. From Definition 9

$$BSC^{\star}(x,T;K;r,\delta) = \int_{-\infty}^{+\infty} \mathrm{e}^{mz}(x\,\mathrm{e}^{\sigma z} - K)^{+} \frac{1}{\sqrt{2\pi T}}\,\mathrm{e}^{-\frac{z^2}{2T}}\,dz.$$

Then,

$$\widehat{BSC}^{\star}(x,\lambda;K;r,\delta) = \int_{-\infty}^{+\infty} \mathrm{e}^{mz}(x\,\mathrm{e}^{\sigma z} - K)^{+} \int_{0}^{+\infty} \frac{\mathrm{e}^{-\lambda t}}{\sqrt{2\pi t}}\,\mathrm{e}^{-\frac{z^2}{2t}}\,dt\,dz. \quad (8)$$ eq:BSC-1

The computation of the second integral on the right hand side is given in Appendix B. Combining (51) and (8), we find

$$\widehat{BSC}^{\star}(x,\lambda;K;r,\delta) = \int_{k}^{+\infty} \mathrm{e}^{mz}(x\,\mathrm{e}^{\sigma z} - K)\frac{\mathrm{e}^{-|z|\theta}}{\theta}dz. \quad (9)$$ eqs2

- In the case $K \geq x$, $k \geq 0$ and the result easily follows.
- In the case $K \leq x$, we split the integral in (9) into two parts

$$\widehat{BSC}^{\star}(x,\lambda;K;r,\delta) = \int_{k}^{0} \mathrm{e}^{mz}(x\,\mathrm{e}^{\sigma z} - K)\frac{\mathrm{e}^{z\theta}}{\theta}dz + \int_{0}^{\infty} \mathrm{e}^{mz}(x\,\mathrm{e}^{\sigma z} - K)\frac{\mathrm{e}^{-z\theta}}{\theta}dz,$$

and an easy computation yields the result.

■

:reduction_b_0

3.2. **Reduction to the case $b = 0$.** Assume that we know the value of $\widehat{PDOC}^{\star}$ with $b \leq 0$. This section aims at proving that computing $\widehat{PDOC}^{\star}$ with $b > 0$ boils down to computing the value of $\widehat{PDOC}^{\star}$ with $b = 0$, as suggested in Scheme 2. First, we state a Proposition which links $PDOC^{\star}$ with $b > 0$ to $PDOC^{\star}$ with $b = 0$.

oc_reduction_0

Proposition 3. *The price of a Parisian Down and Out call in the case $b > 0$ is given by*

$$PDOC^{\star}(x,T;K,L;r,\delta) = L\,\mathrm{e}^{mb}\int_{0}^{D} PDOC^{\star,0}(T-u;K/L;r,\delta)\mu_b(du) \quad (10)$$ eq:PDOC_0

where $\mu_b(du)$ is the law of T_b and

$$PDOC^{\star,0}(T;K;r,\delta) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{T_0^- \geq T\}}(\mathrm{e}^{\sigma Z_T} - K)^{+}\,\mathrm{e}^{mZ_T}\right).$$

Proof. First, we recall the value of $PDOC^\star$

$$PDOC^\star(x,T;K,L;r,\delta) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{T_b^- > T\}}(x\,\mathrm{e}^{\sigma Z_T} - K)^+\,\mathrm{e}^{mZ_T}).$$

Since Z starts from 0 and b is positive, $T_b < D$ on the set $\{T_b^- \geq T\}$. In fact, if T_b were strictly greater than D, it would mean that Z has not crossed b before D and then T_b^- would be equal to D, which is impossible since we are on the set $\{T_b^- \geq T\}$, and $T > D$. Therefore, we can write

$$PDOC^\star(x,T;K,L;r,\delta) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{T_b^- \geq T\}}\mathbb{1}_{\{T_b \leq D\}}(x\,\mathrm{e}^{\sigma Z_T} - K)^+\,\mathrm{e}^{mZ_T}).$$

Introducing Z_{T_b}, we can also write

$$PDOC^\star(x,T;K,L;r,\delta) =$$
$$\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{T_b \leq D\}}\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\{T_b^- - T_b \geq T - T_b\}}(x\,\mathrm{e}^{\sigma Z_T - Z_{T_b} + b} - K)^+\,\mathrm{e}^{m(Z_T - Z_{T_b} + b)} \mid \mathcal{F}_{T_b}]\right).$$

To compute the inner expectation in the previous formula, we rely on the strong Markov property. Let $B = \{B_t = Z_{T_b+t} - Z_{T_b}, t \geq 0\}$. B is independent of $\mathcal{F}_{T_b}$ and one can easily prove that $T_b^-(Z) - T_b(Z) = T_0^-(B)$ a.s. on the set $\{T_b^- \geq T\}$.
Hence, we find

$$PDOC^\star(x,T;K,L;r,\delta) = \mathbb{E}[\mathbb{1}_{\{T_b \leq D\}}\mathbb{E}[\mathbb{1}_{\{T_0^- \geq T-t\}}(x\,\mathrm{e}^{\sigma(B_{T-t}+b)} - K)^+\,\mathrm{e}^{m(B_{T-t}+b)}]_{|t=T_b}].$$

We get

$$PDOC^\star(x,T;K,L;r,\delta) = \int_0^D \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{T_0^- \geq T-u\}}(x\,\mathrm{e}^{\sigma(B_{T-u}+b)} - K)^+\,\mathrm{e}^{m(B_{T-u}+b)}\right)\mu_b(du),$$

where $\mu_b(du)$ is the law of T_b. As $b = \frac{1}{\sigma}\ln\left(\frac{L}{x}\right)$, we get

$$PDOC^\star(x,T;K,L;r,\delta) = L\,\mathrm{e}^{mb}\int_0^D \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{T_0^- \geq T-u\}}(\mathrm{e}^{\sigma B_{T-u}} - K/L)^+\,\mathrm{e}^{mB_{T-u}}\right)\mu_b(du),$$

and the result follows. ∎

Using this proposition leads to the following formula for the Laplace transform of $PDOC^\star(x,T;K,L;r,\delta)$.

Proposition 4. *The Laplace transform of $PDOC^\star$ when $b > 0$ is given by*

$$\widehat{PDOC}^\star(x,\lambda;K,L;r,\delta) = L\,\mathrm{e}^{mb}\int_0^D \mathrm{e}^{-\lambda u}\,\mu_b(du)\,\widehat{PDOC}^{\star,0}(\lambda;K/L;r,\delta),$$

where

$$\int_0^D \mathrm{e}^{-\lambda u}\,\mu_b(du) = \mathrm{e}^{-\theta b}\,\mathcal{N}\left(\sqrt{D}\theta - \frac{b}{\sqrt{D}}\right) + \mathrm{e}^{\theta b}\,\mathcal{N}\left(-\sqrt{D}\theta - \frac{b}{\sqrt{D}}\right).$$

Proof. From Proposition 3, we know the Laplace transform of $PDOC^\star(x,T;K,L;r,\delta)$ with respect to T

$$\begin{aligned}
\widehat{PDOC}^\star(x,\lambda;K,L;r,\delta) &= \int_0^{+\infty}\mathrm{e}^{-\lambda t}\;L\,\mathrm{e}^{mb}\int_0^D PDOC^{\star,0}(t-u;K/L;r,\delta)\mu_b(du)\mathbb{1}_{\{t>D\}}dt,\\
&= L\,\mathrm{e}^{mb}\int_0^D \mu_b(du)\int_D^{+\infty}\mathrm{e}^{-\lambda t}\,PDOC^{\star,0}(t-u;K/L;r,\delta)\,dt,\\
&\quad \text{we change variables } (v,u) = (t-u,u)\\
&= L\,\mathrm{e}^{mb}\int_0^D \mu_b(du)\,\mathrm{e}^{-\lambda u}\int_0^{+\infty}\mathrm{e}^{-\lambda v}\,PDOC^{\star,0}(v;K/L;r,\delta)\,dv,\\
&= L\,\mathrm{e}^{mb}\int_0^D \mu_b(du)\,\mathrm{e}^{-\lambda u}\;\widehat{PDOC}^{\star,0}(\lambda;K/L;r,\delta).
\end{aligned}$$

We refer the reader to Appendix A for the computation of $\int_0^D \mu_b(du)\, e^{-\lambda u}$. ■

call_put_parity

3.3. **call put parity.** In this part, we explain how to deduce the put prices from the call prices using a parity relationship.We state the following proposition

prop2

Proposition 5. *The following relationships hold*

$$PDOP(x, T; K, L, D, r, \delta) = xK\; PUOC\left(\frac{1}{x}, T; \frac{1}{K}, \frac{1}{L}, D, \delta, r\right),$$

$$PUOP(x, T; K, L, D, r, \delta) = xK\; PDOC\left(\frac{1}{x}, T; \frac{1}{K}, \frac{1}{L}, D, \delta, r\right),$$

$$PUIP(x, T; K, L, D, r, \delta) = xK\; PDIC\left(\frac{1}{x}, T; \frac{1}{K}, \frac{1}{L}, D, \delta, r\right),$$

$$PDIP(x, T; K, L, D, r, \delta) = xK\; PUIC\left(\frac{1}{x}, T; \frac{1}{K}, \frac{1}{L}, D, \delta, r\right).$$

Proof. Let us consider a Parisian Down and out put.

$$PDOP(x, T; K, L, D, r, \delta) = \mathbb{E}\left(e^{mZ_T}(K - x\, e^{\sigma Z_T})^+ \mathbb{1}_{\{T_b^- > T\}}\right) e^{-\left(r + \frac{m^2}{2}\right)T}.$$

One notices that the first time the Brownian motion Z makes below b an excursion longer than D is equal to the first time the Brownian motion $-Z$ makes above $-b$ an excursion longer than D. Therefore, introducing the new Brownian motion $W = -Z$ we can rewrite

$$\begin{aligned} PDOP(x, T; K, L, D, r, \delta) &= \mathbb{E}\left(e^{-mW_T}(K - x\, e^{-\sigma W_T})^+ \mathbb{1}_{\{T_{-b}^+ > T\}}\right) e^{-\left(r + \frac{m^2}{2}\right)T}, \\ &= xK\, \mathbb{E}\left(e^{-(m+\sigma)W_T}\left(\frac{1}{x} e^{\sigma W_T} - \frac{1}{K}\right)^+ \mathbb{1}_{\{T_{-b}^+ > T\}}\right) e^{-\left(r + \frac{m^2}{2}\right)T}. \end{aligned}$$

Let us introduce $m' = -(m + \sigma)$, $\delta' = r$, $r' = \delta$ and $b' = -b$. With these relations, we can easily check that $m' = \frac{1}{\sigma}\left(r' - \delta' - \frac{\sigma^2}{2}\right)$ and that $r' + \frac{m'^2}{2} = r + \frac{m^2}{2}$. Moreover, we notice that the barrier L' corresponding to $b' = -b$ is $\frac{1}{L}$.

Therefore, $\mathbb{E}\left(e^{-(m+\sigma)W_T}\left(\frac{e^{\sigma W_T}}{x} - \frac{1}{K}\right)^+ \mathbb{1}_{\{T_{-b}^+ > T\}}\right) e^{-\left(r + \frac{m^2}{2}\right)T}$ is in fact the price of an Up and Out call $PUOC\left(\frac{1}{x}, T; \frac{1}{K}, \frac{1}{L}, D, \delta, r\right)$. Finally, we come up with the following relation

$$PDOP(x, T; K, L, D, r, \delta) = xK\; PUOC\left(\frac{1}{x}, T; \frac{1}{K}, \frac{1}{L}, D, \delta, r\right).$$

The three other assertions in Proposition 5 can be proved in the same way. ■

sec:PDC

4. Valuation of Parisian calls

Looking at Scheme 2, we notice that we only need to compute $\widehat{PDIC}^\star$ with $b \leq 0$ and $\widehat{PUIC}^\star$ with $b \geq 0$. With these values we can deduce the values of all the others Parisian calls.

sec:pdic_b_leq_0

4.1. **The valuation of a Parisian Down and In call with $b \leq 0$.** Before computing the Laplace transform of $PDIC^\star$ in Section 4.1.2, we state some preliminary results in Section 4.1.1. We give a new expression for $PDIC^\star$ in Proposition 6 and we state a useful lemma for the computation of $\widehat{PDIC}^\star$ (Lemma 1).

sec:preliminary

4.1.1. *Preliminary results.*

prop:pdic_h_b

Proposition 6.

$$PDIC^\star(x,T;K,L;r,\delta) = \int_k^\infty e^{my}(xe^{\sigma y} - K)h_b^-(T,y)dy,$$

where

$$h_b^-(t,y) = \int_{-\infty}^b \frac{b-z}{D}\, \mathrm{e}^{-\frac{(z-b)^2}{2D}}\, \gamma(t, z-y)dz,$$

and

$$\gamma_-(t,x) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{T_b^- < t\}} \frac{\mathrm{e}^{-\frac{x^2}{2(t-T_b^-)}}}{\sqrt{2\pi(t-T_b^-)}} \right). \tag{11}$$

eq:gamma_t_x

Proof. Remember that the value of $PDIC^\star$ is given by

$$PDIC^\star(x,T;K,L;r,\delta) \;=\; \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{T_b^- < T\}}(x\,\mathrm{e}^{\sigma Z_T} - K)^+\,\mathrm{e}^{mZ_T}).$$

By conditioning with respect to $\mathcal{F}_{T_b^-}$, we can write

$$PDIC^\star(x,T;K,L;r,\delta) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{T_b^- < T\}}\mathbb{E}_{\mathbb{P}}[x\,\mathrm{e}^{\sigma(Z_T - Z_{T_b^-} + Z_{T_b^-})} - K)^+\,\mathrm{e}^{m(Z_T - Z_{T_b^-} + Z_{T_b^-})}|\mathcal{F}_{T_b^-}]).$$

First, we deal with on the conditional expectation. Let $B_t = Z_{t+T_b^-} - Z_{T_b^-}$ for $t \geq 0$. Relying on the strong Markov property, B is independent of $\mathcal{F}_{T_b^-}$. So, we obtain

$$\mathbb{E}_{\mathbb{P}}\left[(x\,\mathrm{e}^{\sigma(Z_T - Z_{T_b^-} + Z_{T_b^-})} - K)^+\,\mathrm{e}^{m(Z_T - Z_{T_b^-} + Z_{T_b^-})}|\mathcal{F}_{T_b^-}\right] =$$
$$\mathbb{E}_{\mathbb{P}}\left[(x\,\mathrm{e}^{\sigma(B_{T-\tau}+z)} - K)^+\,\mathrm{e}^{m(B_{T-\tau}+z)}\right]_{|z = Z_{T_b^-},\, \tau = T_b^-},$$

and

$$\mathbb{E}_{\mathbb{P}}\left[(x\,\mathrm{e}^{\sigma(B_{T-\tau}+z)} - K)^+\,\mathrm{e}^{m(B_{T-\tau}+z)}\right] = \frac{1}{\sqrt{2\pi(T-\tau)}}\left(\int_{-\infty}^\infty \mathrm{e}^{mu}(x\,\mathrm{e}^{\sigma u} - K)^+\,\mathrm{e}^{-\frac{(u-z)^2}{2(T-\tau)}}\,du\right).$$

Hence, we get

$$PDIC^\star(x,T;K,L;r,\delta) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{T_b^- < T\}}\mathcal{P}_{T-T_b^-}(f_x)(Z_{T_b^-})),$$

with

$$f_x(z) = \mathrm{e}^{mz}(x\,\mathrm{e}^{\sigma z} - K)^+, \tag{12}$$

eq:call-payoff

and

$$\mathcal{P}_t(f_x)(z) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^\infty f_x(u)\exp\left(-\frac{(u-z)^2}{2t}\right)du. \tag{13}$$

eq:transition-P

As explained by Chesney et al. (1997), the random variables $Z_{T_b^-}$ and T_b^- are independent. Denoting the law of $Z_{T_b^-}$ by $\nu^-(dz)$ leads to

$$\begin{aligned} PDIC^\star(x,T;K,L;r,\delta) &= \int_{-\infty}^\infty \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{T_b^- < T\}}\mathcal{P}_{T-T_b^-}(f_x)(z))\nu^-(dz), \\ &= \int_{-\infty}^\infty f_x(y)h_b^-(T,y)dy, \end{aligned}$$

where

$$h_b^-(t,y) = \int_{-\infty}^\infty \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{T_b^- < t\}}\frac{\exp\left(-\frac{(z-y)^2}{2(t-T_b^-)}\right)}{\sqrt{2\pi(t-T_b^-)}}\right)\nu^-(dz).$$

Using the expression of $\nu^-(dz)$ given in Appendix C, we know that

$$h_b^-(t,y) = \int_{-\infty}^{b} \frac{b-z}{D}\,\mathrm{e}^{-\frac{(z-b)^2}{2D}}\,\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{T_b^- < t\}} \frac{\mathrm{e}^{-\frac{(z-y)^2}{2(t-T_b^-)}}}{\sqrt{2\pi(t-T_b^-)}}\right) dz,$$

and the result follows. ■

Definition 10. *Let $\psi : \mathbb{R} \to \mathbb{R}$ denote*

$$\psi(z) \triangleq \int_0^{+\infty} x e^{-\frac{x^2}{2}+zx} dx = 1 + z\sqrt{2\pi} e^{\frac{z^2}{2}} \mathcal{N}(z). \quad (14)$$ psi

Remark 3. *For the numerical inversion of Laplace transforms, it is important to notice ψ is analytic on the complex plane.*

We can easily prove the following Lemma.

ation_K_lambda **Lemma 1.** *Let $K_{\lambda,D} : \mathbb{R} \to \mathbb{R}$ be defined as*

$$K_{\lambda,D}(a) = \int_0^{+\infty} v\,\mathrm{e}^{-\frac{v^2}{2D} - |a-v|\theta}\, dv. \quad (15)$$ eq:K_lambda_D

Then,

$$K_{\lambda,D}(a) = \begin{cases} e^{\theta a} D\psi(-\theta\sqrt{D}) & \text{if } a \le 0, \\ e^{-\theta a} D\psi(\theta\sqrt{D}) - D\theta\sqrt{2\pi D} e^{\lambda D} \Big\{ \mathcal{N}(\theta\sqrt{D} - \frac{a}{\sqrt{D}}) e^{-\theta a} + & \\ \qquad \mathcal{N}(-\theta\sqrt{D} - \frac{a}{\sqrt{D}}) e^{\theta a} \Big\} & \text{otherwise.} \end{cases}$$

laplace_pdic

4.1.2. *The Laplace transform of $PDIC^\star(x,T;K,L;r,\delta)$.*

thm:pdic **Theorem 1.** *The value of $\widehat{PDIC}^\star$ is given by the following formula*

$$\widehat{PDIC}^\star(x,\lambda;K,L;r,\delta) = \frac{\mathrm{e}^{\theta b}}{D\theta\psi(\theta\sqrt{D})} \int_k^\infty e^{my}(xe^{\sigma y} - K) K_{\lambda,D}(b-y). \quad (16)$$ eqs4

For any $\lambda > \frac{(m+\sigma)^2}{2}$ and for $K > L$, we get

$$\widehat{PDIC}^\star(x,\lambda;K,L;r,\delta) = \frac{\psi(-\theta\sqrt{D})\,\mathrm{e}^{2b\theta}}{\theta\psi(\theta\sqrt{D})} K\,\mathrm{e}^{(m-\theta)k} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta}\right), \quad (17)$$ eqs5

and for $K \le L$

$$\begin{aligned} \widehat{PDIC}^\star(x,\lambda;K,L) = {} & \frac{\mathrm{e}^{(m+\theta)b}}{\psi(\theta\sqrt{D})} \left(\frac{2K}{m^2-\theta^2}\left[\psi(m\sqrt{D}) - m\sqrt{2\pi D}\,\mathrm{e}^{\frac{Dm^2}{2}} \mathcal{N}(m\sqrt{D}+d)\right]\right. \\ & \left. - \frac{2L}{(m+\sigma)^2-\theta^2}\left[\psi((m+\sigma)\sqrt{D}) - (m+\sigma)\sqrt{2\pi D}\,\mathrm{e}^{\frac{D}{2}(m+\sigma)^2} \mathcal{N}\left((m+\sigma)\sqrt{D}+d\right)\right]\right) \\ & + \frac{K\,\mathrm{e}^{(m+\theta)k}}{\theta\psi(\theta\sqrt{D})}\left(\frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta}\right)\left[\psi(\theta\sqrt{D}) - \theta\sqrt{2\pi D}\,\mathrm{e}^{\lambda D} \mathcal{N}(\theta\sqrt{D}-d)\right] \\ & + \frac{\mathrm{e}^{\lambda D}\sqrt{2\pi D}}{\psi(\theta\sqrt{D})} K\,\mathrm{e}^{2b\theta}\,\mathrm{e}^{(m-\theta)k} \mathcal{N}(-d-\theta\sqrt{D})\left(\frac{1}{m+\sigma-\theta} - \frac{1}{m-\theta}\right). \quad (18) \end{aligned}$$ eqs6

Proof. (17) and (18) easily follow from (16):

- if $K > L$, $b - y < 0$ $\forall y \in [k,\infty]$. Then, using Lemma 1 and (16) yields

$$\widehat{PDIC}^\star(x,\lambda;K,L;r,\delta) = \frac{\psi(-\theta\sqrt{D})\,\mathrm{e}^{2b\theta}}{\theta\psi(\theta\sqrt{D})} \int_k^\infty \mathrm{e}^{(m-\theta)y}(x\,\mathrm{e}^{\sigma y} - K) dy,$$

  and the result easily follows.

- if $K < L$, $b-y$ is positive on $[k,b]$ and negative on $[b,\infty]$. We have to split the integral
$$I \triangleq \int_k^{+\infty} e^{my}(xe^{\sigma y}-K)K_{\lambda,D}(b-y)dy. \tag{19}$$
I

appearing in (16).
$$I = \int_k^b e^{my}(xe^{\sigma y}-K)K_{\lambda,D}(b-y)dy + \int_b^{+\infty} e^{my}(xe^{\sigma y}-K)K_{\lambda,D}(b-y)dy \triangleq I_1 + I_2. \tag{20}$$
eq:12

$$I_1 = D\psi(-\theta\sqrt{D})e^{\theta b}\int_b^{+\infty} e^{my}(xe^{\sigma y}-K)e^{-\theta y} = D\psi(-\theta\sqrt{D})e^{mb}\left(\frac{K}{m-\theta} - \frac{L}{m+\sigma-\theta}\right).$$

The integral I_2 can be split into three terms
$$\begin{aligned} I_{21} &= D\psi(\theta\sqrt{D})\int_k^b e^{my}(xe^{\sigma y}-K)e^{\theta(y-b)}dy, \\ I_{22} &= -D\theta\sqrt{2\pi D}e^{\lambda D}\int_k^b e^{my}(xe^{\sigma y}-K)e^{\theta(y-b)}\mathcal{N}(\theta\sqrt{D}-\frac{b-y}{\sqrt{D}})dy, \\ I_{23} &= -D\theta\sqrt{2\pi D}e^{\lambda D}\int_k^b e^{my}(xe^{\sigma y}-K)e^{\theta(b-y)}\mathcal{N}(-\theta\sqrt{D}-\frac{b-y}{\sqrt{D}})dy. \end{aligned}$$

An easy computation leads to
$$I_{21} = D\psi(\theta\sqrt{D})e^{-\theta b}\left\{e^{(m+\theta)b}\left[\frac{L}{m+\sigma+\theta} - \frac{K}{m+\theta}\right] + Ke^{(m+\theta)k}\left[\frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta}\right]\right\}.$$

I_{22} and I_{23} are computed in the following way: we change variables (we introduce $v = \theta\sqrt{D} - \frac{b-y}{\sqrt{D}}$ (for the valuation of I_{22})) and we use the following equality $\int_{a_1}^{a_2}\mathcal{N}(v)e^{bv}dv = \frac{1}{b}[\mathcal{N}(a_2)e^{a_2 b} - \mathcal{N}(a_1)e^{a_1 b} - e^{\frac{b^2}{2}}(\mathcal{N}(a_2-b) - \mathcal{N}(a_1-b))]$, for $a_1, a_2, b \in \mathbb{R}$, $b \neq 0$ and $a_1 \leq a_2$.

We refer to Proposition 10, for proving λ should be greater than $\frac{(m+\sigma)^2}{2}$. Let us prove (16). Using Proposition 6, we get
$$\widehat{PDIC}^{\star}(x,\lambda;K,L;r,\delta) = \int_k^{\infty} e^{my}(xe^{\sigma y}-K)\int_0^{\infty} \mathrm{e}^{-\lambda t}\, h_b^-(t,y)dtdy. \tag{21}$$
eq:pdic1

We would like to compute $\widehat{h_b^-}(\lambda,y) = \int_0^\infty \mathrm{e}^{-\lambda t}\, h_b^-(t,y)dt$. Using the definition of $h_b^-(t,y)$ in Proposition 6 yields
$$\widehat{h_b^-}(\lambda,y) = \int_{-\infty}^b \frac{b-z}{D}\,\mathrm{e}^{-\frac{(z-b)^2}{2D}}\int_0^\infty \mathrm{e}^{-\lambda t}\,\gamma_-(t,z-y)dt\; dz. \tag{22}$$
eq:hhat

So, we need to compute the Laplace transform of $\gamma_-(t,x)$.
$$\int_0^\infty \mathrm{e}^{-\lambda t}\,\gamma_-(t,x)dt = \mathbb{E}_{\mathbb{P}}\left(\int_{T_b^-}^\infty \mathrm{e}^{-\lambda t}\,\frac{\mathrm{e}^{-\frac{x^2}{2(t-T_b^-)}}}{\sqrt{2\pi(t-T_b^-)}}dt\right).$$

The change of variables $u = t - T_b^-$ gives
$$\int_0^\infty \mathrm{e}^{-\lambda t}\,\gamma_-(t,x)dt = \mathbb{E}_{\mathbb{P}}(\mathrm{e}^{-\lambda T_b^-})\int_0^\infty \mathrm{e}^{-\lambda u}\,\frac{\mathrm{e}^{-\frac{x^2}{2u}}}{\sqrt{2\pi u}}du.$$

Using results from Appendices A and B, we get
$$\int_0^\infty \mathrm{e}^{-\lambda t}\,\gamma_-(t,x)dt = \frac{\mathrm{e}^{-(|x|-b)\theta}}{\theta\psi(\theta\sqrt{D})}.$$

Thanks to (22), we can rewrite

$$\widehat{h_b^-}(\lambda, y) = \frac{e^{b\theta}}{D\theta\psi(\theta\sqrt{D})} \int_{-\infty}^{b} (b-z)\, e^{-\frac{(z-b)^2}{2D} - |z-y|\theta}\, dz.$$

By changing variables $v = b - z$, we obtain

$$\widehat{h_b^-}(\lambda, y) = \frac{e^{b\theta}}{D\theta\psi(\theta\sqrt{D})} \int_{0}^{\infty} v\, e^{-\frac{v^2}{2D} - |b-v-y|\theta}\, dv, \tag{23}$$ eq:hhat1

and (16) follows. ■

sec:PUC

5. The Parisian Up calls

This section is devoted to the computation of the Laplace transforms of the Parisian Up call prices. We will go exactly through the same points as in the previous section but dealing with an Up and In call with $b \geq 0$ instead of a Down and In call with $b \leq 0$.

5.1. **The valuation of a Parisian Up and In call with $b \geq 0$.** The owner of an Up and In option receives the payoff if S makes an excursion above the level L longer than D before the maturity time T, which is exactly the same as saying Brownian motion Z makes an excursion above b longer than D. Using the previous notations, the price of a Parisian Up and In call is given by

$$PUIC^\star(x, T; K, L; r, \delta) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{T_b^+ < T\}} (x\, e^{\sigma Z_T} - K)^+\, e^{m Z_T}), \tag{24}$$ eq:SP_20

where T_b^+ is defined by (2). The valuation of $\widehat{PUIC}^\star$ in the case $b \geq 0$ is similar to the valuation of $\widehat{PDIC}^\star$ in the case $b \leq 0$ (see previous Section). Before computing the Laplace transform of $PUIC^\star$ in Theorem 2, we give a new expression for $PUIC^\star$ in Proposition 7.

prop:puic_h_b **Proposition 7.**

$$PUIC^\star(x, T; K, L; r, \delta) = \int_k^\infty e^{my}(xe^{\sigma y} - K) h_b^+(T, y) dy,$$

where

$$h_b^+(t, y) = \int_b^\infty \frac{z-b}{D}\, e^{-\frac{(z-b)^2}{2D}}\, \gamma_+(t, z-y) dz,$$

and

$$\gamma_+(t, x) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{T_b^+ < t\}} \frac{e^{-\frac{x^2}{2(t - T_b^+)}}}{\sqrt{2\pi(t - T_b^+)}}\right). \tag{25}$$

The proof of Proposition 7 is the same as the proof of Proposition 6. We only need to replace T_b^- by T_b^+.

thm:puic **Theorem 2.** *The value of $\widehat{PUIC}^\star$ is given by the following formula*

$$\widehat{PUIC}^\star(x, \lambda; K, L; r, \delta) = \frac{e^{-\theta b}}{D\theta\psi(\theta\sqrt{D})} \int_k^\infty e^{my}(xe^{\sigma y} - K) K_{\lambda,D}(y-b). \tag{26}$$ eqs

For any $\lambda > \frac{(m+\sigma)^2}{2}$*, we get for* $K > L$

$$\widehat{PUIC}^{\star}(x,\lambda;K,L;r,\delta) = 2\,\mathrm{e}^{(m-\theta)b}\,\frac{\sqrt{2\pi D}}{\psi(\theta\sqrt{D})}\left[\frac{K}{m^2-\theta^2}\,\mathrm{e}^{\frac{Dm^2}{2}}\,m\mathcal{N}(m\sqrt{D}+d) - \frac{L}{(m+\sigma)^2-\theta^2}\,\mathrm{e}^{\frac{D(m+\sigma)^2}{2}}(m+\sigma)\mathcal{N}((m+\sigma)\sqrt{D}+d)\right] + \frac{\mathrm{e}^{-2b\theta}}{\psi(\theta\sqrt{D})}K\,\mathrm{e}^{(m+\theta)k}\,\mathrm{e}^{\lambda D}\,\sqrt{2\pi D}\mathcal{N}(d-\theta\sqrt{D})\left(\frac{1}{m+\sigma+\theta}-\frac{1}{m+\theta}\right) + \frac{\mathrm{e}^{(m-\theta)k}}{\theta\psi(\theta\sqrt{D})}K\left(\frac{1}{m-\theta}-\frac{1}{m+\sigma-\theta}\right)\left(\psi(\theta\sqrt{D})-\theta\sqrt{2\pi D}\,\mathrm{e}^{\lambda D}\,\mathcal{N}(d+\theta\sqrt{D})\right). \quad (27)$$

eq:PUIC_xLK

and for $K \leq L$

$$\widehat{PUIC}^{\star}(x,\lambda;K,L;r,\delta) = \frac{2\,\mathrm{e}^{(m-\theta)b}}{\psi(\theta\sqrt{D})}\left[\frac{K}{m^2-\theta^2}\psi(m\sqrt{D}) - \frac{L}{(m+\sigma)^2-\theta^2}\psi((m+\sigma)\sqrt{D})\right] + \frac{\mathrm{e}^{-2b\theta}\,\psi(-\theta\sqrt{D})}{\theta\psi(\theta\sqrt{D})}Ke^{(m+\theta)k}\left(\frac{1}{m+\theta}-\frac{1}{m+\theta+\sigma}\right). \quad (28)$$

eq:PUIC_KL_xL

Proof. (27) and (28) easily follow from (26):

- if $K > L$, $y - b > 0$ $\forall y \in [k,\infty[$. Then, using Lemma 1 and (26) yields

$$\widehat{PUIC}^{\star}(x,\lambda;K,L;r,\delta) = \frac{\mathrm{e}^{-\theta b}}{D\theta\psi(\theta\sqrt{D})}\int_k^\infty e^{my}(xe^{\sigma y}-K)K_{\lambda,D}(y-b)dy,$$

where

$$K_{\lambda,D}(y-b) = e^{-\theta(y-b)}D\psi(\theta\sqrt{D}) - D\theta\sqrt{2\pi D}e^{\lambda D}\left\{\mathcal{N}(\theta\sqrt{D}-\frac{y-b}{\sqrt{D}})e^{-\theta(y-b)} + \mathcal{N}(-\theta\sqrt{D}-\frac{y-b}{\sqrt{D}})e^{\theta(y-b)}\right\}.$$

As for the valuation of I_2, page 11, we compute the three terms

$$I_1 = D\psi(\theta\sqrt{D})\int_k^\infty e^{my}(xe^{\sigma y}-K)e^{-\theta(y-b)}dy,$$
$$I_2 = -D\theta\sqrt{2\pi D}e^{\lambda D}\int_k^\infty e^{my}(xe^{\sigma y}-K)e^{-\theta(y-b)}\mathcal{N}(\theta\sqrt{D}-\frac{y-b}{\sqrt{D}})dy,$$
$$I_3 = -D\theta\sqrt{2\pi D}e^{\lambda D}\int_k^b e^{my}(xe^{\sigma y}-K)e^{-\theta(y-b)}\mathcal{N}(-\theta\sqrt{D}-\frac{y-b}{\sqrt{D}})dy.$$

I_2 and I_3 are computed in the following way: we change variables (we introduce $v = \theta\sqrt{D} - \frac{y-b}{\sqrt{D}}$ (for the valuation of I_2)) and we use the following equality $\int_{-\infty}^a \mathcal{N}(v)e^{bv}dv = \frac{1}{b}[\mathcal{N}(a)e^{ab} - e^{\frac{b^2}{2}}(\mathcal{N}(a-b))]$, for $a, b \in \mathbb{R}$, $b \geq 0$.

- if $K < L$, $y - b$ is negative on $[k,b]$ and positive on $[b,\infty[$. We have to split the integral

$$I \triangleq \int_k^{+\infty} e^{my}(xe^{\sigma y}-K)K_{\lambda,D}(y-b)dy. \quad (29)$$

Ibis

appearing in (26).

$$I = \int_k^b e^{my}(xe^{\sigma y}-K)K_{\lambda,D}(y-b)dy + \int_b^{+\infty} e^{my}(xe^{\sigma y}-K)K_{\lambda,D}(y-b)dy \triangleq I_1 + I_2.$$

An easy computation gives us

$$I_1 = D\psi(-\theta\sqrt{D})e^{-\theta b}\int_k^b e^{my}(xe^{\sigma y}-K)e^{\theta y}dy = D\psi(-\theta\sqrt{D}) \left[e^{-\theta b}Ke^{(m+\theta)k}\left(\frac{1}{m+\theta}-\frac{1}{m+\sigma+\theta}\right)+e^{mb}\left(\frac{L}{m+\sigma+\theta}-\frac{K}{m+\theta}\right)\right].$$

The integral I_2 can be split into three terms

$$I_{21} = D\psi(\theta\sqrt{D})e^{\theta b}\int_b^\infty e^{my}(xe^{\sigma y}-K)e^{-\theta y}dy,$$
$$I_{22} = -D\theta\sqrt{2\pi D}e^{\lambda D}e^{\theta b}\int_b^\infty e^{my}(xe^{\sigma y}-K)e^{-\theta y}\mathcal{N}(\theta\sqrt{D}+\frac{b-y}{\sqrt{D}})dy,$$
$$I_{23} = -D\theta\sqrt{2\pi D}e^{\lambda D}e^{-\theta b}\int_b^\infty e^{my}(xe^{\sigma y}-K)e^{\theta y}\mathcal{N}(-\theta\sqrt{D}+\frac{b-y}{\sqrt{D}})dy.$$

An easy computation leads to

$$I_{21} = D\psi(\theta\sqrt{D})e^{mb}\left(\frac{K}{m-\theta}-\frac{L}{m+\sigma-\theta}\right).$$

I_{22} and I_{23} are computed in the following way: we change variables (we introduce $v=\theta\sqrt{D}+\frac{b-y}{\sqrt{D}}$ (for the valuation of I_{22})) and we use the equality $\int_{a_1}^{a_2}\mathcal{N}(v)e^{bv}dv = \frac{1}{b}[\mathcal{N}(a_2)e^{a_2 b}-\mathcal{N}(a_1)e^{a_1 b}-e^{\frac{b^2}{2}}(\mathcal{N}(a_2-b)-\mathcal{N}(a_1-b))]$, for $a_1, a_2, b\in\mathbb{R}$, $b\neq 0$ and $a_1\leq a_2$, as we did for the valuation of the PDIC, when $K<L$.

Let us prove (26). Using Proposition 7 yields

$$\widehat{PUIC}^\star(x,\lambda;K,L;r,\delta) = \int_k^\infty e^{my}(xe^{\sigma y}-K)\int_0^\infty \mathrm{e}^{-\lambda t}\, h_b^+(t,y)dtdy. \quad (30) \quad \text{eq:puic1}$$

Following the proof of Theorem 1, we get

$$\widehat{h_b^+}(\lambda,y) = \int_b^{+\infty}\frac{z-b}{D}\,\mathrm{e}^{-\frac{(z-b)^2}{2D}}\int_0^\infty \mathrm{e}^{-\lambda t}\,\gamma_+(t,z-y)dt\,dz. \quad (31) \quad \text{eq:hhatpuic}$$

Using results from Appendices C and B, we find

$$\int_0^\infty \mathrm{e}^{-\lambda t}\,\gamma_+(t,x)dt = \frac{\mathrm{e}^{-(|x|+b)\theta}}{\theta\psi(\theta\sqrt{D})}. \quad (32) \quad \text{eq:SP\_30}$$

Thanks to (31), we can rewrite

$$\widehat{h_b^+}(\lambda,y) = \frac{\mathrm{e}^{-b\theta}}{D\theta\psi(\theta\sqrt{D})}\int_0^\infty x\,\mathrm{e}^{-\frac{x^2}{2D}-|b+x-y|\theta}\,dx. \quad (33) \quad \text{eq:hhat+}$$

and (26) follows. ■

6. Prices at any time t

sec:price_t

So far, we have explained how to compute the prices at time 0 of the different Parisian options by numerically inverting their Laplace transforms w.r.t the maturity time. From a practical point of view, the real prize is to be able to hedge these options. This requires to compute the option prices at any given time t between 0 and the maturity time T. In this part, we explain how to deduce the prices at any time $t>0$ from the prices at time 0. In the following, we have chosen to focus on the Down and In call but the formula we obtain can easily be extended to the other options by means of parity relationships. We assume in the following computations that the relevant excursion has not occurred yet, otherwise the Parisian option has been turned into the corresponding vanilla option and its price at time t is of common knowledge.

6.1. **Down and In call.** We introduce the r.v. D_t to count the time already spent in the excursion below b straddling time t

$$D_t = \begin{cases} t - g_t^b & \text{if } S_t \le b, \\ 0 & \text{if } S_t > b. \end{cases} \tag{34}$$

eq:def_D_t

Note that D_t is $\mathcal{F}_t$-measurable.
Let $PDIC(t, S_t, D_t, T; K, L, D, r, \delta)$ be the price of a Down and In call at time t. We know that

$$PDIC(t, S_t, D_t, T; K, L, D, r, \delta) = \mathrm{e}^{-r(T-t)} \, \mathbb{E}_{\mathbb{Q}} \left((x \, \mathrm{e}^{\sigma(W_T + mT)} - K)^+ \, \mathbb{1}_{\{T_b^- \le T\}} | \mathcal{F}_t \right). \tag{35}$$

eq:pdic_price_t

rice_t_general

Proposition 8. *On the set* $\{T_b^- > t\}$,

$$\begin{aligned} & PDIC(S_t, t, D_t, T; K, L, D, r, \delta) \\ = \;& \mathrm{e}^{-\left(r + \frac{m^2}{2}\right)T'} \Bigg\{ \mathbb{1}_{\{S_t > L\}} \mathbb{E} \left(\mathrm{e}^{m Z'_{T'}} (x \, \mathrm{e}^{\sigma Z'_{T'}} - K)^+ \, \mathbb{1}_{\{T'^-_{b'} \le T'\}} \right)_{|x = S_t} \\ & + \mathbb{1}_{\{S_t \le L\}} \mathbb{1}_{\{D - D_t \le T'\}} \mathbb{E} \left(\mathrm{e}^{m Z'_{T'}} (x \, \mathrm{e}^{\sigma Z'_{T'}} - K)^+ \, \mathbb{1}_{\{T'_{b'} \ge D - d\}} \right)_{|x = S_t, d = D_t} \\ & + \mathbb{1}_{\{S_t \le L\}} \mathbb{E} \left(\mathrm{e}^{m Z'_{T'}} (x \, \mathrm{e}^{\sigma Z'_{T'}} - K)^+ \, \mathbb{1}_{\{T'_{b'} \le D - d\}} \mathbb{1}_{\{T'^-_{b'} \le T'\}} \right)_{|x = S_t, d = D_t} \Bigg\}. \end{aligned} \tag{36}$$

eq:pdic_price_t_g

where Z' *is a* $\mathbb{P}-$*Brownian motion independent of* $\mathcal{F}_t$ *and*

$$T' = T - t, \quad b' = \frac{1}{\sigma} \ln \left(\frac{L}{S_t} \right), \quad T'^-_{b'} = T^-_{b'}(Z'), \quad T'_{b'} = T_{b'}(Z'). \tag{37}$$

eq:data_t

Proof. We can change the probability measure as we did at the beginning to make $Z = \{W_t + mt; t \ge 0\}$ a $\mathbb{P}-$Brownian motion ($\mathbb{P}$ is defined in Proposition 1). $\mathbb{E}$ denotes $\mathbb{E}_{\mathbb{P}}$. Then, by changing the probability in Equation (35) we can write

$$\begin{aligned} & PDIC(t, S_t, D_t, T; K, L, D, r, \delta) \\ & \quad = \mathrm{e}^{-r(T-t)} \, \frac{\mathbb{E} \left(\mathrm{e}^{m Z_T - \frac{1}{2} m^2 T} (x \, \mathrm{e}^{\sigma Z_T} - K)^+ \, \mathbb{1}_{\{T_b^- \le T\}} | \mathcal{F}_t \right)}{\mathrm{e}^{m Z_t - \frac{1}{2} m^2 t}}, \\ & \quad = \mathrm{e}^{-r(T-t)} \, \frac{\mathbb{E} \left(\mathrm{e}^{m Z_t} \, \mathrm{e}^{m(Z_T - Z_t) - \frac{1}{2} m^2 T} (x \, \mathrm{e}^{\sigma Z_T} - K)^+ \, \mathbb{1}_{\{T_b^- \le T\}} | \mathcal{F}_t \right)}{\mathrm{e}^{m Z_t - \frac{1}{2} m^2 t}}, \\ & \quad = \mathrm{e}^{-(r + \frac{m^2}{2})(T-t)} \, \mathbb{E} \left(\mathrm{e}^{m(Z_T - Z_t)} (S_t \, \mathrm{e}^{\sigma(Z_T - Z_t)} - K)^+ \, \mathbb{1}_{\{T_b^- \le T\}} | \mathcal{F}_t \right). \end{aligned}$$

We introduce $Z'_s = Z_{t+s} - Z_t$ for all $s \ge 0$. Z' is is a $\mathbb{P}-$Brownian motion independent of $\mathcal{F}_t$.

$$PDIC(t, S_t, D_t, T; K, L, D, r, \delta) = \mathrm{e}^{-(r + \frac{m^2}{2})T'} \, \mathbb{E} \left(\mathrm{e}^{m Z'_{T'}} (S_t \, \mathrm{e}^{\sigma Z'_{T'}} - K)^+ \, \mathbb{1}_{\{T_b^- \le T\}} | \mathcal{F}_t \right).$$

The indicator can be split up in several parts describing the different possible evolutions of Z' (see Figure 3). It is quite obvious that on the set $\{T_b^- > t\}$, the indicator $\mathbb{1}_{\{T_b^- \le T\}}$ can be rewritten as follows

$$\mathbb{1}_{\{T_b^- \le T\}} = \mathbb{1}_{\{Z_t > b\}} \mathbb{1}_{\{T'^-_{b'} \le T'\}} + \mathbb{1}_{\{Z_t \le b\}} \left(\mathbb{1}_{\{T'_{b'} \ge D - D_t\}} \mathbb{1}_{\{D - D_t \le T'\}} + \mathbb{1}_{\{T'_{b'} < D - D_t\}} \mathbb{1}_{\{T'^-_{b'} \le T'\}} \right).$$

To find Equation (36), it is sufficient to notice that both T'_b and T'^-_b are independent of Z', whereas S_t and D_t are $\mathcal{F}_t-$measurable. ■

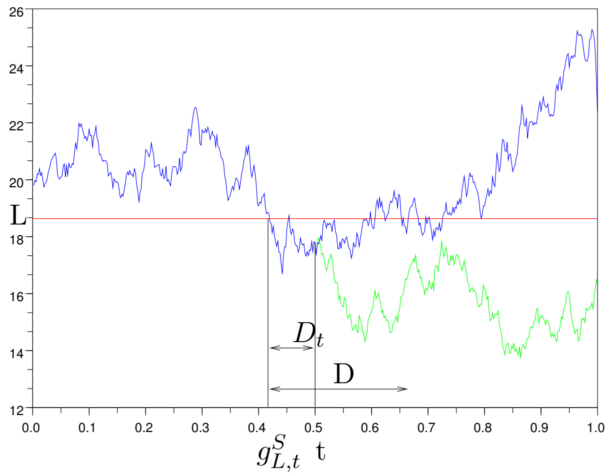

FIGURE 3. Possible evolutions of an asset price fig:asset_t

In the sequel, we use the following notation based on Proposition 8

$$PDIC(S_t, t, D_t, T; K, L, D, r, \delta) \triangleq \mathrm{e}^{-\left(r+\frac{m^2}{2}\right)T'} \left\{\mathbb{1}_{\{S_t>L\}} E_1(S_t, T') \right. \\ \left. + \mathbb{1}_{\{S_t \le L\}} E_2(S_t, D_t, T') + \mathbb{1}_{\{S_t \le L\}} E_3(S_t, D_t, T')\right\}. \quad (38)$$

eq:pdic_price_t_s

From Equation (36), we notice that E_1 is the star price of a Parisian Down and In call,

$$E_1(x, T') = PDIC^\star(x, T'; K, L; r, \delta). \quad (39)$$

eq:pdic_price_t_E

Proposition 9. *On the set $\{T_b^- > t\}$, the price of a Down and In call at time t is given by*

$$PDIC^\star(t, S_t, D_t, T; K, L, D, r, \delta) = \mathbb{1}_{\{Z_t>b\}} PDIC^\star(S_t, T-t, K, L, D, r, \delta) \\ + \mathbb{1}_{\{Z_t \le b\}} (\mathbb{1}_{\{D-D_t \le T-t\}} BSC^\star(S_t, T-t, K, r, \delta) + g(S_t, D_t, T-t)) \quad (40)$$

eq:pdic_price_t_f

where the function g is characterized by its Laplace transform

$$\widehat{g}(S_t, D_t, \lambda) \quad = \quad \mathrm{e}^{mb'} \int_0^{D-D_t} \mu_{b'} \, \mathrm{e}^{-\lambda u} \, du \left(L\, \widehat{PDIC}^{\star,0}(\lambda; \frac{K}{L}; r, \delta) \right. \\ \left. - \widehat{BSC}^\star(L, \lambda, K, r, \delta) \right).$$

Proof. Let us go back to Equation (38). First, we deal with E_2 and after with E_3 since the value of E_1 is already known (see (39)) and gives the first term on the r.h.s of (40).

Step 1 : Laplace transform of E_2.

$$\begin{aligned} E_2(x, d, t) &= \mathbb{E}\left(\mathrm{e}^{mZ_t}(x\,\mathrm{e}^{\sigma Z_t} - K)^+ \mathbb{1}_{\{T'_{b'} \ge D-d\}} \mathbb{1}_{\{D-d \le t\}}\right) \\ &= \mathbb{1}_{\{D-d \le t\}} BSC^\star(x, t; K, r, \delta) - \mathbb{1}_{\{D-d \le t\}} \mathbb{E}\left(\mathrm{e}^{mZ_t}(x\,\mathrm{e}^{\sigma Z_t} - K)^+ \mathbb{1}_{\{T'_{b'} \le D-d\}}\right) \\ &\triangleq E_{21}(x, d, t) - E_{22}(x, d, t). \end{aligned}$$

The term E_{21} corresponds to the first half of the second term on the r.h.s of (40). By conditioning w.r.t $\mathcal{F}_{T'_{b'}}$ and introducing $X_u = Z_{u+T'_{b'}} - b'$, which is a Brownian motion independent of $\mathcal{F}_{T'_{b'}}$, we get

$$\begin{aligned} E_{22}(x,d,t) &= \mathbb{1}_{\{D-d\leq t\}}\mathbb{E}\left(\mathbb{1}_{\{T'_{b'}\leq D-d\}}\mathbb{E}\left(\mathrm{e}^{mX_{t-\tau}}\,\mathrm{e}^{mb'}(x\,\mathrm{e}^{+\sigma b'}\,\mathrm{e}^{\sigma X_{t-\tau}}-K)^+\right)|_{\tau=T'_{b'}}\right), \\ &= \mathbb{1}_{\{D-d\leq t\}}\int_0^{D-d}\mathrm{e}^{mb'}\,\mathbb{E}(\mathrm{e}^{mX_{t-u}}(x\,\mathrm{e}^{\sigma b'}\,\mathrm{e}^{\sigma X_{t-u}}-K)^+)\mu_{b'}(u)du \end{aligned}$$

where $\mu_{b'}$ is the density function of the hitting time $T'_{b'}$.
Taking the Laplace transform in the previous equality leads to

$$\widehat{E_{22}}(x,d,\lambda) = \int_{D-d}^{\infty}d\tau\,\mathrm{e}^{-\lambda\tau}\int_0^{D-d}du\,\mathrm{e}^{mb'}\,\mathbb{E}(\mathrm{e}^{mX_{\tau-u}}(x\,\mathrm{e}^{\sigma b'}\,\mathrm{e}^{\sigma X_{\tau-u}}-K)^+)\mu_{b'}(u).$$

The change of variables $(u,\xi)=(u,\tau-u)$ enables to separate the two dimensional integral

$$\widehat{E_{22}}(x,d,\lambda) = \mathrm{e}^{mb'}\int_0^{\infty}\mathrm{e}^{-\lambda\xi}\,\mathbb{E}(\mathrm{e}^{mX_\xi}(x\,\mathrm{e}^{\sigma b'}\,\mathrm{e}^{\sigma X_\xi}-K)^+)d\xi\int_0^{D-d}\mathrm{e}^{-\lambda u}\,\mu_{b'}(u)du. \quad (41)$$

eq:E22_2_integral

The first integral in (41) is in fact the Laplace transform of the price of a Black Scholes call and it has already been computed (see 2). The value of the second integral is given in Appendix A.

$$\widehat{E_{22}}(S_t,D_t,\lambda) = \mathrm{e}^{mb'}\,\widehat{BSC^{\star}}(L,\lambda;K;r,\delta)\int_0^{D-D_t}\mathrm{e}^{-\lambda u}\,\mu_{b'}(u)du. \quad (42)$$

eq:E22_final

Step 2: Laplace transform of E_3. From Equation (38),

$$E_3(x,d,t)=\mathbb{E}\left(\mathrm{e}^{mZ'_t}(x\,\mathrm{e}^{\sigma Z'_t}-K)^+\,\mathbb{1}_{\{T'_{b'}\leq D-d\}}\mathbb{1}_{\{T'^-_{b'}\leq t\}}\right)$$

To compute E_3, we condition w.r.t $\mathcal{F}_{T'_{b'}}$ and introduce $X_u=Z'_{u+T'_{b'}}-b'$. X is a Brownian motion independent of $\mathcal{F}_{T'_{b'}}$ so we get

$$\begin{aligned} E_3(x,d,t) &= \mathbb{E}\left(\mathbb{E}\left(\mathrm{e}^{mZ'_t}(x\,\mathrm{e}^{\sigma Z'_t}-K)^+\,\mathbb{1}_{\{T'_{b'}\leq D-d\}}\mathbb{1}_{\{T'^-_{b'}\leq t\}}|\mathcal{F}_{T'_{b'}}\right)\right), \\ &= \mathrm{e}^{mb'}\,\mathbb{E}\left(\mathbb{1}_{\{T'_{b'}\leq D-d\}}\mathbb{E}\left(\mathrm{e}^{mX_{t-T'_{b'}}}(x\,\mathrm{e}^{\sigma b'}\,\mathrm{e}^{\sigma X_{t-T'_{b'}}}-K)^+\,\mathbb{1}_{\{T'^-_{b'}\leq t\}}|\mathcal{F}_{T'_{b'}}\right)\right). \end{aligned}$$

Moreover on the set $\{T'_{b'}\leq D-d\}$, $T'^-_{b'}(Z')=T'_{b'}(Z')+T^-_0(X)$ a.s.. Hence, we find

$$\begin{aligned} E_3(x,d,t) &= \mathrm{e}^{mb'}\,\mathbb{E}\left(\mathbb{1}_{\{T'_{b'}\leq D-d\}}\mathbb{E}\left(\mathrm{e}^{mX_{t-T'_{b'}}}(x\,\mathrm{e}^{\sigma b'}\,\mathrm{e}^{\sigma X_{t-T'_{b'}}}-K)^+\,\mathbb{1}_{\{T^-_0\leq t-T'_{b'}\}}|\mathcal{F}_{T'_{b'}}\right)\right) \\ &= \mathrm{e}^{mb'}\,\mathbb{E}\left(\mathbb{1}_{\{\tau\leq D-d\}}\mathbb{E}\left(\mathrm{e}^{mX_{t-\tau}}(x\,\mathrm{e}^{\sigma b'}\,\mathrm{e}^{\sigma X_{t-\tau}}-K)^+\,\mathbb{1}_{\{T^-_0\leq t-\tau\}}\right)_{|\tau=T'_{b'}}\right) \\ &= \mathrm{e}^{mb'}\int_0^{D-d}\mathbb{E}\left(\mathrm{e}^{mX_{t-\tau}}(x\,\mathrm{e}^{\sigma b'}\,\mathrm{e}^{\sigma X_{t-\tau}}-K)^+\,\mathbb{1}_{\{T^-_0\leq t-\tau\}}\right)\mu_{b'}(\tau)d\tau. \end{aligned}$$

We can now compute the Laplace transform of E_3 w.r.t t.

$$\widehat{E_3}(x,d,\lambda) = \mathrm{e}^{mb'}\int_0^{\infty}du\,\mathrm{e}^{-\lambda u}\int_0^{D-d}d\tau\mathbb{E}\left(\mathrm{e}^{mX_{u-\tau}}(x\,\mathrm{e}^{\sigma b'}\,\mathrm{e}^{\sigma X_{u-\tau}}-K)^+\,\mathbb{1}_{\{T^-_0\leq u-\tau\}}\right)\mu_{b'}(\tau).$$

Once again, the change of variables $(\tau,\xi)=(\tau,u-\tau)$ enables to separate the two dimensional integral

$$\begin{aligned} \widehat{E_3}(x,d,\lambda) &= \mathrm{e}^{mb'}\int_0^{\infty}d\xi\,\mathrm{e}^{-\lambda\xi}\int_0^{D-d}\mathrm{e}^{-\lambda\tau}\,\mathbb{E}\left(\mathrm{e}^{mX_\xi}(x\,\mathrm{e}^{\sigma b'}\,\mathrm{e}^{\sigma X_\xi}-K)^+\,\mathbb{1}_{\{T^-_0\leq\xi\}}\right)\mu_{b'}(\tau) \\ &= \mathrm{e}^{mb'}\int_0^{\infty}\mathrm{e}^{-\lambda\xi}\,d\xi\int_0^{D-d}\mathrm{e}^{-\lambda\tau}\,x\,\mathrm{e}^{\sigma b'}\,PDIC^{\star,0}(\xi;K\,\mathrm{e}^{-\sigma b'}/x;r,\delta)\mu_{b'}(\tau) \\ &= x\,\mathrm{e}^{(m+\sigma)b'}\,\widehat{PDIC}^{\star,0}(\lambda;K\,\mathrm{e}^{-\sigma b'}/x;r,\delta)\int_0^{D-d}\mathrm{e}^{-\lambda\tau}\,\mu_{b'}(\tau). \end{aligned}$$

Finally, we get

$$\widehat{E_3}(S_t, D_t, \lambda) \;=\; L\,\mathrm{e}^{mb'}\,\widehat{PDIC}^{\star,0}(\lambda, K/L, r, \delta)\int_0^{D-D_t} \mathrm{e}^{-\lambda\tau}\,\mu_{b'}(\tau). \qquad (43)$$ eq:E3_final

Noticing that $E_3(S_t, D_t, \lambda) - E_{22}(S_t, D_t, \lambda) = \widehat{g}(S_t, D_t, \lambda)$ ends the proof. ■

6.2. **Other Parisian option.** The price at time t of an Up and In call can be computed closely following what has been done for the Down and In call and it is sufficient to replace $PDIC$ by $PUIC$ in the above formula. All the other Parisian option prices can be deduced using either an In and Out parity or a call put parity relationships.

sec:inverse

7. The inversion of Laplace transforms

This section is devoted to the numerical inversion of the Laplace transforms computed previously. We recall that the Laplace transforms are computed with respect to the maturity time. We explain how to recover a function from its Laplace transform using a contour integral. The real problem is how to numerically evaluate this complex integral. This is done in two separate steps involving two different errors. First, as explained in Section 7.2 we replace the integral by a series. The first step creates a discretisation error, which is handled by Proposition 11. Secondly, one has to compute a non-finite series. This can be achieved by simply truncating the series but it leads to a tremendously slow convergence. Here, we prefer to use the Euler acceleration as presented in Section 7.3. Proposition 12 states an upper-bound for the error due to the accelerated computation of the non finite series. Theorem 1 gives a bound for the global error.

7.1. **Analytical prolongations.** Because the Laplace inversion is performed in the complex plane, we have to extend the expressions obtained the Laplace transforms computed so far to the complex plane. To do so, we have to introduce the analytic prolongation of the cumulative normal distribution function on the complex plane. From Proposition 10, it is quite easy to show that the expressions obtained for a real value of the Laplace parameter are still valid for a complex one with the function $\mathcal{N}$ defined by Lemma 2.

ce-abscissa-cv **Proposition 10** (abscissa of convergence)**.** *The abscissa of convergence of the Laplace transforms of the star prices of Parisian options is smaller than $\frac{(m+\sigma)^2}{2}$. All these Laplace transform are analytic on the complex half plane $\{z \in \mathbb{C} : \mathcal{R}\mathrm{e}\,(z) > \frac{(m+\sigma)^2}{2}\}$.*

Proof. It is sufficient to notice that the star price of a Parisian option is bounded by $\mathbb{E}(\mathrm{e}^{mZ_T}(\mathrm{e}^{\sigma W_T} + K))$.

$$\mathbb{E}(\mathrm{e}^{mZ_T}(\mathrm{e}^{\sigma W_T} + K)) \le K\,\mathrm{e}^{\frac{m^2}{2}T} + x\,\mathrm{e}^{\frac{(m+\sigma)^2}{2}T} = \mathcal{O}(\mathrm{e}^{\frac{(m+\sigma)^2}{2}T}).$$

Hence, Widder (1941, Theorem 2.1) yields that the abscissa of convergence of the Laplace transforms of the star prices is smaller that $\frac{(m+\sigma)^2}{2}$. The second part of the proposition ensues from Widder (1941, Theorem 5.a). ■

lem:cn-anal **Lemma 2** (Analytical prolongation of $\mathcal{N}$)**.** *The unique analytical prolongation of the cumulative normal distribution function on the complex plane is defined by*

$$\mathcal{N}(x + iy) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-\frac{(v+iy)^2}{2}}\,dv. \qquad (44)$$ eq:cdfnor_anal

Proof. It is sufficient to notice that the function defined above is holomorphic on the complex plane (and hence analytical) and that it coincides with the cumulative normal distribution function on the real axis. ■

With the definition of $\mathcal{N}$ given by Equation (44), it is clear that all the expressions obtained so far for the Laplace transforms are also valid for complex values of λ satisfying $\mathcal{R}\mathrm{e}\,(\lambda) > \frac{(m+\sigma)^2}{2}$ since their are analytic on the complex half plane $\{z \in \mathbb{C} : \mathcal{R}\mathrm{e}\,(z) > \frac{(m+\sigma)^2}{2}\}$.

four-seri-repr

7.2. **The Fourier series representation.** Thanks to Widder (1941, Theorem 9.2), we know how to recover a function from its Laplace transform.

thm:inversion

Theorem 3. *Let f be a continuous function defined on $\mathbb{R}^+$ and α a positive number. If the function $f(t)\,\mathrm{e}^{-\alpha t}$ is integrable. Then, given the Laplace transform $\hat{f}$, f can be recovered from the contour integral*

$$f(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \mathrm{e}^{st}\,\hat{f}(s)ds, \quad t > 0. \tag{45}$$

eqn:inverse

The variable α has to be chosen greater than the abscissa of convergence of $\hat{f}$. In our case, α must be chosen strictly greater than $(m+\sigma)^2/2$.

For any real valued function satisfying the hypotheses of Theorem 3, we introduce a trapezoidal discretisation of Equation (45) of step π/t.

$$f_{\pi/t}(t) = \frac{e^{\alpha t}}{2t}\widehat{f}(\alpha) + \frac{e^{\alpha t}}{t}\sum_{k=1}^{\infty}(-1)^k\,\mathcal{R}\mathrm{e}\left(\widehat{f}\left(\alpha + i\frac{k\pi}{t}\right)\right). \tag{46}$$

eqn:fh

_discret_error

Proposition 11. *If f is a continuous bounded function satisfying $f(t) = 0$ for $t < 0$, we have*

$$\left|e_{\pi/t}(t)\right| \triangleq \left|f(t) - f_{\pi/t}(t)\right| \leq \|f\|_{\infty}\,\frac{e^{-2\alpha t}}{1 - e^{-2\alpha t}}. \tag{47}$$

eq:discret_error

A proof of Proposition 11 can be found in Labart and Lelong (2006).

Remark 4. *For the upper bound in Proposition 11 to be smaller than $10^{-8}\,\|f\|_{\infty}$, one has to choose $2\alpha t = 18.4$. In fact, this bound holds for any choice of the discretisation step h satisfying $h < 2\pi/t$.*

Simply truncating the summation in the definition of $f_{\pi/t}$ to compute the trapezoidal integral is far too rough to provide a fast and accurate numerical inversion. One way to improve the convergence of the series is to use the Euler summation.

uler-summation

7.3. **The Euler summation.** To improve the convergence of a series S, we use the Euler summation technique as described by Abate et al. (1999), which consists in computing the binomial average of q terms from the p-th term of the series S. The binomial average obviously converges to S as p goes to infinity. The following proposition describes the convergence rate of the binomial average to the infinite series $f_{\pi/t}(t)$ when p goes to ∞.

prop:Euler

Proposition 12. *Let f be a function of class $\mathcal{C}^{q+4}$ such that there exists $\epsilon > 0$ s.t. $\forall k \leq q+4$, $f^{(k)}(s) = \mathcal{O}(e^{(\alpha-\epsilon)s})$, where α is the abscissa of convergence of $\widehat{f}$. We define $s_p(t)$ as the approximation of $f_{\pi/t}(t)$ when truncating the non-finite series in (46) to p terms*

$$s_p(t) = \frac{e^{\alpha t}}{2t}\widehat{f}(\alpha) + \frac{e^{\alpha t}}{t}\sum_{k=1}^{p}(-1)^k\,\mathcal{R}\mathrm{e}\left(\widehat{f}\left(\alpha + i\frac{\pi k}{t}\right)\right), \tag{48}$$

eq:5

and $E(q,p,t) = \sum_{k=0}^{q} C_q^k 2^{-q} s_{p+k}(t)$. Then,

$$\left|f_{\pi/t}(t) - E(q,p,t)\right| \leq \frac{te^{\alpha t}\,|f'(0) - \alpha f(0)|}{\pi^2}\frac{(p+1)!\,q!}{2^{q-2}\,(p+q+2)!} + \mathcal{O}\left(\frac{1}{p^{q+3}}\right)$$

when p goes to infinity.

Using Propositions 11 and 12, we get the following result concerning the global error on the numerical computation of the price of a Parisian call option.

r:global_error **Corollary 1.** *Let f be the price of a Parisian call option. Using the notations of Proposition 12, we have*

$$|f(t) - E(q,p,t)| \leq S_0 \, \frac{e^{-2\alpha t}}{1 - e^{-2\alpha t}} + \frac{e^{\alpha t} t \, |f'(0) - \alpha f(0)| (p+1)! \, q!}{\pi^2 2^{q-2} \, (p+q+2)!} + \mathcal{O}\left(\frac{1}{p^{q+3}}\right) \qquad (49)$$ eq:price-inv-erro

where α is defined in Theorem 3.

We refer the reader to Labart and Lelong (2006) for a proof of Theorem 1 and Proposition 12.
For $2\alpha t = 18.4$ and $q = p = 15$, the global error is bounded by $S_0 10^{-8} + t \, |f'(0) - \alpha f(0)| \, 10^{-11}$. As one can see, the method we use to invert Laplace transforms provides a very good accuracy with few computations.

Remark 5. *Considering the case of call options in Theorem 1 is sufficient since put prices are computed using parity relations and their accuracy is hung up to the one of call prices.*

8. A few graphs

In this section, we do a few numerical experiments with the method we have studied so far and compare it with the enhanced Monte Carlo method of Baldi et al. (2000).
First, we consider an example of delta hedging of a Parisian up and out call with the characteristics defined in Table 1. We simulate an asset path and try to hedge along this trajectory. For this purpose, we use the formulae to derive the price of Parisian options at any time strictly positive. The delta simply ensues from a finite difference scheme. The discrete delta hedging proves quite efficient as we can see it on Figure 4. In our example, the hedging portfolio can be rebalanced three times a day. One ca see that there are huge variations in the hedging portfolio when the option is about to be activated or canceled. This phenomena introduces some hedging error because the hedging is discontinuous.

$S_0 = 100$	$K = 100$	$T = 1$	$L = 110$
$D = 20$ day	$\sigma = 0.2$	$r = 0.025$	$\delta = 0$

tab:hedge Table 1. example of PUOC for hedging

$S_0 = 100$	$K = 100$	$T = 1$	$L = 90$
$\sigma = 0.2$	$r = 0.025$	$\delta = 0$	

tab:cmp Table 2. example of PDIC

Now, we would like to compare the prices obtained with our method with the prices given the Monte Carlo method of Baldi et al. (2000). The Monte Carlo computation uses 10000 samples and 250 discretisation steps between 0 and T. Figure 5 shows the evolution of the prices of down and in call as defined in Table 2 computed either with the invert Laplace transform method or the enhanced Monte Carlo method. The evolution of the prices provided by our method is much smoother than the ones given by Monte Carlo. As one can see, the accuracy of the Monte Carlo method has nothing to do with the accuracy of our method. Let us recall that our prices are accurate up to 10^-6 (when $S_0 = 100$) as stated in Theorem 1. Concerning the computational costs of the two methods, the invert Laplace transform method runs a thousand times faster than the corrected Monte Carlo.

9. Conclusion

In this work, we provide all the Laplace transforms of the different Parisian option prices, be it explicit formulae or parity relationships. We also explain how to invert these formulae to compute the prices and the detailed study of the inversion algorithm enables to prove

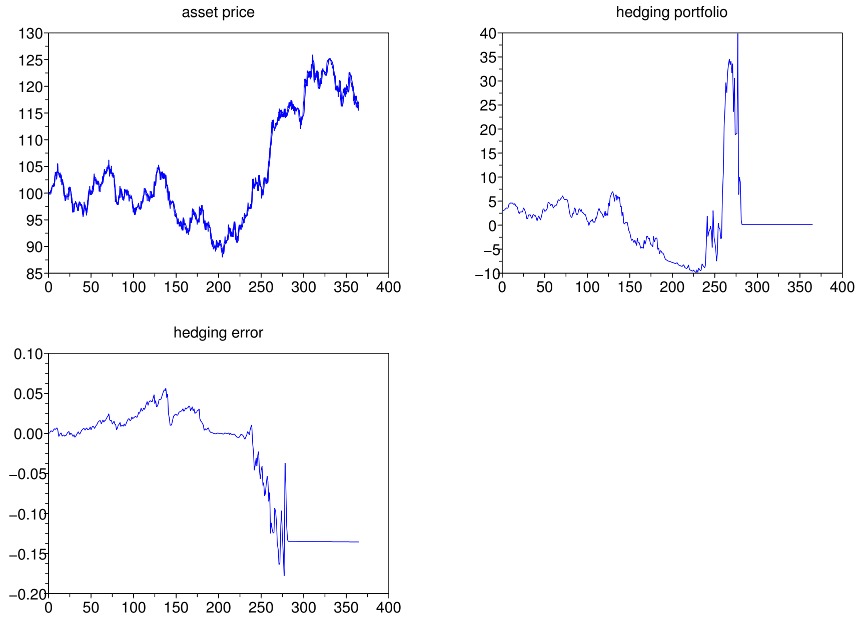

FIGURE 4. Example of delta hedging fig:hedge

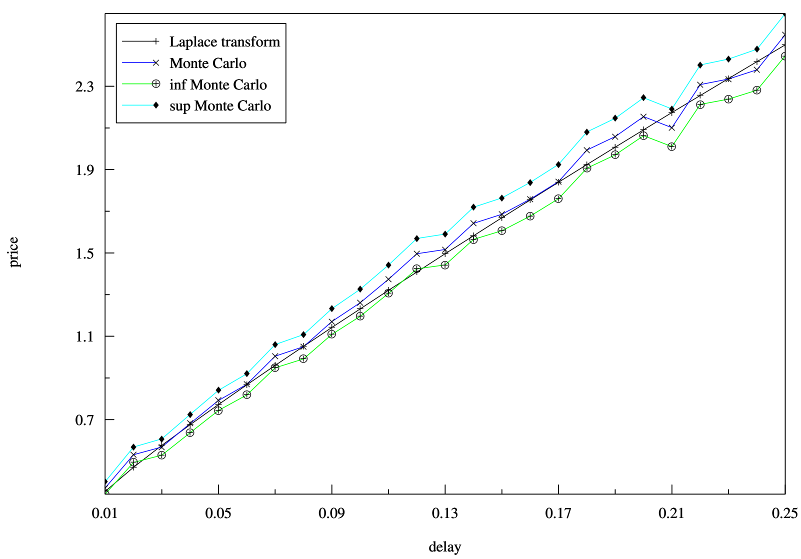

FIGURE 5. Comparison with improved Monte Carlo method fig:cmp_mc

the accuracy and then the efficiency of the method. The efficiency is confirmed by the comparison with the enhanced Monte Carlo, which in fact is already very efficient as Parisian options are known to be very difficult to price.

Appendix A. The Laplace transform of μ_b in the case $b > 0$

sec:mub

We already know that $\mu_b(u) = \frac{|b|}{\sqrt{2\pi u^3}} e^{\left(\frac{-b^2}{2u}\right)}$. We use the notation $\theta = \sqrt{2\lambda}$.

$$\int_0^D e^{-\lambda u} \mu_b(du) = \int_0^D e^{-\frac{\theta^2}{2}u} \frac{b}{\sqrt{2\pi u^3}} e^{\frac{-b^2}{2u}} du$$

The change of variable $t = \sqrt{\frac{b}{\theta}} \frac{1}{\sqrt{u}}$ leads to

$$\begin{aligned}
\int_0^D e^{-\lambda u} \mu_b(u) du &= \int_{\frac{\sqrt{b}}{\sqrt{\theta D}}}^{+\infty} \sqrt{\frac{2b\theta}{\pi}} \exp\left(\frac{-\theta b}{2}\left(\frac{1}{t^2} + t^2\right)\right) dt, \\
&= \int_{\frac{\sqrt{b}}{\sqrt{\theta D}}}^{+\infty} \sqrt{\frac{2b\theta}{\pi}} \exp\left(\frac{-\theta b}{2}\left(\frac{1}{t} - t\right)^2\right) e^{-\theta b} dt, \\
&\quad \text{a new change of variable } v = \tfrac{1}{t} - t \text{ gives} \\
&= \sqrt{\frac{b\theta}{2\pi}} e^{-\theta b} \int_{-\infty}^{\sqrt{\frac{\theta D}{b}} - \frac{\sqrt{b}}{\sqrt{\theta D}}} e^{\frac{-\theta b}{2} v^2} \left(1 - \frac{v}{\sqrt{v^2 + 4}}\right) dv, \\
&\quad \text{we set } u = \sqrt{\theta b} v \\
&= \frac{1}{\sqrt{2\pi}} e^{-\theta b} \int_{-\infty}^{\theta\sqrt{D} - \frac{b}{\sqrt{D}}} e^{-u^2/2} \left(1 - \frac{u}{\sqrt{u^2 + 4\theta b}}\right) du.
\end{aligned}$$

A last change of variable $v = \sqrt{u^2 + 4\theta b}$ ends the computation

$$\hat{\mu_b}(\lambda) = e^{-\theta b} \mathcal{N}\left(\theta\sqrt{D} - \frac{b}{\sqrt{D}}\right) + e^{\theta b} \mathcal{N}\left(-\theta\sqrt{D} - \frac{b}{\sqrt{D}}\right).$$

If we let D go to infinity, we can deduce the Laplace transform of T_b, for any real b

$$\mathbb{E}[e^{-\lambda T_b}] = e^{-\sqrt{2\lambda}|b|}. \tag{50}$$

eq:laplace-Tb

Appendix B. The valuation of $\int_0^{+\infty} \mathrm{e}^{-\lambda u} \frac{\mathrm{e}^{-\frac{x^2}{2u}}}{\sqrt{2\pi u}} du$

sec:integrale

Once again we introduce $\theta = \sqrt{2\lambda}$.
The change of variable $u = \frac{|x| t^2}{\theta}$ straightly gives the new expression

$$\begin{aligned}
\int_0^{+\infty} \mathrm{e}^{-\lambda u} \frac{\mathrm{e}^{-\frac{x^2}{2u}}}{\sqrt{2\pi u}} du &= \int_0^{+\infty} \sqrt{\frac{2 \mid x \mid}{\pi\theta}} \exp\left(-\frac{\theta \mid x \mid}{2}\left(\frac{1}{t^2} + t^2\right)\right) dt, \\
&= \sqrt{\frac{2 \mid x \mid}{\pi\theta}} e^{-\theta\,|x|} \int_0^{+\infty} \exp\left(-\frac{\theta \mid x \mid}{2}\left(\frac{1}{t} - t\right)^2\right) dt.
\end{aligned}$$

Once again, we can use the change of variable $s = u - \frac{1}{u}$, which maps $[0, +\infty[$ into $]-\infty, +\infty[$ and we have $du = \frac{ds}{2}\left(1 + \frac{s}{\sqrt{s^2+4}}\right)$. The second term is odd, so its integral over $\mathbb{R}$ cancels and we get

$$\sqrt{\frac{|x|}{2\pi\theta}}\, \mathrm{e}^{-\theta\,|x|} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{\theta\,|x|}{2} s^2}\, ds.$$

Finally, we obtain

$$\int_0^{+\infty} e^{-\lambda u} \frac{\mathrm{e}^{-\frac{x^2}{2u}}}{\sqrt{2\pi u}} du = \frac{1}{\theta} e^{-\theta\,|x|}. \tag{51}$$

eq:parisian-int-2

sec:excursion

Appendix C. Some results around Brownian meanders

In this part, we recall some useful results about Brownian motion and excursion. We are interested in the law of $(T_b^-, Z_{T_b^-})$ and $(T_b^+, Z_{T_b^+})$. Such results can be found in Azéma and Yor (1989); Revuz and Yor (1999); Chung (1976).
In the following, we consider a standard Brownian motion Z.

C.1. **Case** $b = 0$**.**

$$\mathbb{P}(Z_{T^-} \in dx) = -\frac{x}{D}\, \mathrm{e}^{-\frac{x^2}{2D}}\, \mathbb{1}_{\{x<0\}} dx \quad \text{and} \quad \mathbb{P}(Z_{T^+} \in dx) = \frac{x}{D}\, \mathrm{e}^{-\frac{x^2}{2D}}\, \mathbb{1}_{\{x>0\}} dx. \tag{52}$$

eq:law-Z-t-minus

$$\mathbb{E}\left(\mathrm{e}^{-\frac{1}{2}\lambda^2 T^-}\right) = \frac{1}{\psi(\lambda\sqrt{D})} \quad \text{and} \quad \mathbb{E}\left(\mathrm{e}^{-\frac{1}{2}\lambda^2 T^+}\right) = \frac{1}{\psi(\lambda\sqrt{D})}. \tag{53}$$

eq:law-t-minus

This kind of formula goes back to the work of Wendel (1964).

C.2. **Case** $b < 0$**.** This case can be reduced to the previous one with the help of the stopping time T_b. By introducing a new Brownian motion $W = \{W_t = Z_{T_b+t} - b; t \geq 0\}$ independent of $\mathcal{F}_{T_b}$, we can write $T_b^- = T_b + T^-(W)$. T_b and $T_0^-(W)$ are independent, hence we find

$$\mathbb{E}(\mathrm{e}^{-\frac{1}{2}\lambda^2 T_b^-}) = \mathbb{E}(\mathrm{e}^{-\frac{1}{2}\lambda^2 T_b})\mathbb{E}(\mathrm{e}^{-\frac{1}{2}\lambda^2 T_0^-(W)}).$$

As $\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b)) = \exp(-|b|\lambda)$, we get

$$\mathbb{E}\left(\mathrm{e}^{-\frac{1}{2}\lambda^2 T_b^-}\right) = \frac{\mathrm{e}^{b\lambda}}{\psi(\lambda\sqrt{D})}. \tag{54}$$

eq:law-tb-minus

Concerning the law of $Z_{T_b^-}$, we have

$$\mathbb{P}(Z_{T_b^-} \in dx) = \mathbb{P}(W_{T_b^-(Z)-T_b(Z)} \in dx - b) = \mathbb{P}(W_{T_0^-} \in dx - b)$$

Finally, we obtain

$$\nu^-(dx) = \mathbb{P}(Z_{T_b^-} \in dx) = \frac{b-x}{D}\, \mathrm{e}^{-\frac{(x-b)^2}{2D}}\, \mathbb{1}_{\{x<b\}} dx. \tag{55}$$

eq:law-Z-tb-minus

C.3. **Case** $b > 0$**.** Following closely the above reasoning, we find

$$\mathbb{E}\left(\mathrm{e}^{-\frac{1}{2}\lambda^2 T_b^+}\right) = \frac{\mathrm{e}^{-b\lambda}}{\psi(\lambda\sqrt{D})}. \tag{56}$$

eq:law-tb-plus

$$\nu^+(dx) = \mathbb{P}(Z_{T_b^+} \in dx) = \frac{x-b}{D}\, \mathrm{e}^{-\frac{(x-b)^2}{2D}}\, \mathbb{1}_{\{x>b\}} dx. \tag{57}$$

eq:law-Z-tb-plus

References

aw J. Abate and W. Whitt. Numerical inversion of laplace transforms of probability distributions. *ORSA Journal of Computing*, 7(1), Winter 1995.

c_probab_model J. Abate, L. Choudhury, and G. Whitt. An introduction to numerical transform inversion and its application to probability models. *Computing Probability*, pages 257 – 323, 1999.

MR2233425 J. Anderluh and H. van der Weide. Parisian options—the implied barrier concept. In *Computational science—ICCS 2004. Part IV*, volume 3039 of *Lecture Notes in Comput. Sci.*, pages 851–858. Springer, Berlin, 2004.

avellaneda_99 M. Avellaneda and L. Wu. Pricing parisian-style options with a lattice method. *International Journal of Theoretical and Applied Finance*, 2(1):1–16, 1999.

MR1022900 J. Azéma and M. Yor. Étude d'une martingale remarquable. In *Séminaire de Probabilités, XXIII*, volume 1372 of *Lecture Notes in Math.*, pages 88–130. Springer, Berlin, 1989.

lucia_00 P. Baldi, L. Caramellino, and M. G. Iovino. Pricing complex barrier options with general features using sharp large deviation estimates. In *Monte Carlo and quasi-Monte Carlo methods 1998 (Claremont, CA)*, pages 149–162. Springer, Berlin, 2000.

_corpor_liabil F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–654, 1973,.

cjy M. Chesney, M. Jeanblanc-Picqué, and M. Yor. Brownian excursions and Parisian barrier options. *Adv. in Appl. Probab.*, 29(1):165–184, 1997. ISSN 0001-8678.

MR0467948 K. L. Chung. Excursions in Brownian motion. *Ark. Mat.*, 14(2):155–177, 1976. ISSN 0004-2080.

MR1954893 M. Costabile. A combinatorial approach for pricing Parisian options. *Decis. Econ. Finance*, 25(2):111–125, 2002. ISSN 1593-8883.

_option_pricin J. Cox, S. Ross, and M. Rubinstein. Option pricing: A simplified approach. *Journal of Financial Economics*, (7):229–263, 1979.

n_paris_option R. Haber, P. Schonbucker, and P. Wilmott. Pricing parisian options. *Journal of Derivatives*, (6):71–79, 1999.

MR1121940 I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991. ISBN 0-387-97655-8.

_option_laplac C. Labart and J. Lelong. Pricing double barrier parisian options using laplace transforms. Technical report, ENPC, `http://cermics.enpc.fr/reports/CERMICS-2006/CERMICS-2006-328.pdf`, November 2006.

non04:_evaluat F. Quittard-Pinon, C. Bernard, and O. Le Courtois. Evaluation numérique d'options parisiennes. *Banque & Marchés*, 2004.

MR1725357 D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999. ISBN 3-540-64325-7.

MR0171319 J. G. Wendel. Zero-free intervals of semi-stable Markov processes. *Math. Scand.*, 14:21–34, 1964. ISSN 0025-5521.

MR0005923 D. V. Widder. *The Laplace Transform*. Princeton Mathematical Series, v. 6. Princeton University Press, Princeton, N. J., 1941.

wilmott98 P. Wilmott. *Derivatives*. University Edition, 1998.

CMAP, ECOLE POLYTECHNIQUE, CNRS, ROUTE DE SACLAY 91128 PALAISEAU, FRANCE
E-mail address: `labart@cmap.polytechnique.fr`

CERMICS, ECOLE DES PONTS, PARISTECH, 6-8 AVENUE BLAISE PASCAL, CHAMPS SUR MARNE 77455 MARNE LA VALLÉE, FRANCE
E-mail address: `lelong@cermics.enpc.fr`