Sinai's walk via stochastic calculus

by

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Summary. Sinai's walk $(S_n,\ n \ge 0)$ is a recurrent nearest-neighbour random walk on $\mathbb{Z}$ in random environment, and is reputed for its exotic slow movement: $S_n \approx (\log n)^2$, as n goes to infinity. The present paper summarizes the approach via stochastic calculus in the study of Sinai's walk. The main tool is the Ray–Knight theorem which describes the local time process of Brownian motion stopped at some special random times. The method is very powerful. For example, it allows to (i) establish all the possible Lévy classes for Sinai's walk; (ii) determine the escape rate of favourite sites. It is interesting to mention that the latter problem remains open for the usual random walk. A number of unanswered questions, which concern various asymptotic properties of Sinai's walk, are listed at the end of the paper.

Keywords. Random walk in random environment, diffusion in a random potential, Ray–Knight theorem.

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1 Introduction

Let $\omega = (\omega_i,\ i \in \mathbb{Z})$ be an i.i.d. collection of random variables taking values in $(0,1)$, such that the support of the law of ω_0 is contained in $(0,1)$. Upon any realization of $(\omega_i,\ i \in \mathbb{Z})$ (which we call an **environment**), we can define a Markov chain $(S_n,\ n \ge 0)$ by $S_0 = 0$ and

$$\mathbb{P}\left(S_{n+1} = y \,|\, S_n = x,\ \{\omega_i\}_{i\in\mathbb{Z}}\right) = \begin{cases} \omega_x & \text{if } y = x+1, \\ 1-\omega_x & \text{if } y = x-1, \\ 0 & \text{otherwise.} \end{cases}$$

The process $(S_n,\ n \geq 0)$ is a so-called **random walk in random environment**, or simply RWRE, and its origin can be traced back at least to Temkin [71]. One of the interesting features of the model of RWRE is that it exhibits some phenomena which are not shared by the usual random walk.

Throughout the paper, we are interested in the asymptotic behaviours of RWRE under the absolute probability $\mathbb{P}$ ("annealed" properties).

Solomon [59] (see also Kozlov [39] for a particular case) completely solved the transience/recurrence problem for RWRE. He proved that $(S_n,\ n \geq 0)$ is recurrent if and only if

$$\mathbb{E}\left[\log \frac{1-\omega_0}{\omega_0}\right] = 0. \tag{1.1}$$

In the transient case, i.e., when (1.1) fails, Solomon [59] determined the speed of the walk; his law of large numbers was later developed into a central limit theorem by Kesten et al. [35].

The very delicate recurrent case was analyzed in a celebrated paper of Sinai [58], and is the main subject of this paper. To avoid the (trivial) non-random environment case, we assume

$$\sigma^2 := \mathrm{var}\left[\log \frac{1-\omega_0}{\omega_0}\right] > 0. \tag{1.2}$$

Here is Sinai's theorem for recurrent RWRE.

Theorem 1.1 (Sinai [58]) *Under* (1.1) *and* (1.2)*, we have*

$$\frac{\sigma^2}{(\log n)^2} S_n \xrightarrow{\text{law}} b_\infty, \tag{1.3}$$

where "$\xrightarrow{\text{law}}$" *denotes convergence in law.*

The density function of the limit distribution in (1.3) was later determined independently by Kesten [34] and Golosov [18]:

$$\mathbb{P}\left[b_\infty \in \mathrm{d}x\right] = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2\pi^2}{8}\,|x|\right) \mathrm{d}x. \tag{1.4}$$

It is interesting to note that in the recurrent case, the magnitude order $(\log n)^2$ of the walk is much smaller than $n^{1/2}$, the magnitude order of the usual random walk.

The present paper aims to give an overview of the recurrent walk, by the method of stochastic calculus. Throughout the paper, we assume (1.1)–(1.2), and the recurrent walk (S_n) is referred to as **Sinai's walk**. The method of stochastic calculus does not directly apply to Sinai's walk, but rather to a continuous-time model which is related to Sinai's walk via an invariance principle (Theorem 3.1 in Section 3).

We refer to the books of Révész [52] and Hughes [25], as well as the lecture notes of Zeitouni [75], for background and general properties of RWRE. See also Section 6 for discussions of some related topics and further references.

The rest of the paper is organized as follows. In Section 2, we introduce the continuous-time model in question, and explain in Section 3 why this is related to Sinai's walk. Section 4 is the heart of the paper, where we outline the idea of using stochastic calculus to study Sinai's walk via a continuous-time model. Our main tool will be the Ray–Knight theorem for Brownian local times. In Section 5, we are interested in a different continuous-time model (which corresponds to transient RWRE), and explain how stochastic calculus can be used for such a process. Section 6 is devoted to a brief description of some important related topics which we do not study in the present paper, together with a few references. Finally, in Section 7, we mention some open problems for Sinai's walk.

2 A continuous-time model

Let $V = (V(x),\ x \in \mathbb{R})$ be a stochastic process defined on $\mathbb{R}$ such that $V(0) = 0$. A diffusion process $X_V = (X_V(t),\ t \geq 0)$ in random potential V, is an informal solution to the stochastic differential equation

$$\begin{cases} \mathrm{d}X_V(t) = \mathrm{d}\beta(t) - \frac{1}{2} V'(X_V(t))\,\mathrm{d}t \\ X_V(0) = 0 \end{cases}, \tag{2.1}$$

where $(\beta(t), t \geq 0)$ is a standard Brownian motion independent of V. We write a subscript for X_V to insist on the dependence of X_V upon the potential V, and will later on look at various examples of V.

Strictly speaking, the process X_V should be considered as a diffusion whose conditional generator given V is

$$\frac{1}{2}\,\mathrm{e}^{V(x)} \frac{\mathrm{d}}{\mathrm{d}x} \left(\mathrm{e}^{-V(x)} \frac{\mathrm{d}}{\mathrm{d}x} \right).$$

It is now clear that the potential V does not need to be differentiable.

We now proceed to get an analytical and more treatable expression for X_V. Assume that V is a càdlàg process (right-continuous with limits on the left) and is locally bounded. For each realization of the potential V, if we write

$$A_x := \int_0^x \mathrm{e}^{V(y)}\,\mathrm{d}y, \qquad x \in \mathbb{R}, \tag{2.2}$$

which is a scale function of X_V, then $(A_{X_V(t)},\ t \geq 0)$ is a continuous local martingale; there exists therefore a standard Brownian motion $(B(t);\, t \geq 0)$ independent of V, such that

$$A_{X_V(t)} = B(\Theta(t)), \tag{2.3}$$

where $\Theta(t) := \langle A_{X_V} \rangle (t)$ denotes the increasing process associated with the local martingale A_{X_V}. Note that in case $\Theta(\infty) < \infty$, the representation (2.3) involves only $(B(u);\, u < \Theta(\infty))$.

We define the first hitting times of B:

$$\sigma(r) = \inf\{t > 0 :\ B(t) > r\}, \qquad r > 0. \tag{2.4}$$

Then $\Theta(\infty) = \sigma(A_\infty)$ (which can be finite or infinite). Writing T for the inverse of Θ:

$$T(r) := \Theta^{-1}(r) = \int_0^r \mathrm{e}^{-2V(A^{-1}_{B(u)})}\, \mathrm{d}u, \qquad 0 \leq r < \Theta(\infty), \tag{2.5}$$

and we get the following representation for X (cf. Brox [9]):

$$X_V(t) = A^{-1}_{B(T^{-1}(t))}, \qquad t \geq 0. \tag{2.6}$$

Here, A^{-1} and T^{-1} are the inverses of A and T, respectively.

The reason why X_V is of interest to us is that we can embed Sinai's walk into X_V when its random potential V has a particular form. This is done in the next section.

We close this section by presenting the following theorem of Brox [9] which is a continuous-time analogue of Sinai's theorem recalled in Theorem 1.1.

Theorem 2.1 (Brox [9]) *If* $(V(x),\, x \in \mathbb{R})$ *is a Brownian motion, then*

$$\frac{X_V(t)}{(\log t)^2} \xrightarrow{\text{law}} b_\infty, \tag{2.7}$$

where the law of b_∞ *is defined in* (1.4).

To the best of my knowledge, there has been so far no rigorous argument showing that $\frac{X_V(t)}{(\log t)^2}$ (when V is Brownian potential) and $\frac{\sigma^2}{(\log n)^2} S_n$ (for Sinai's walk) should *a priori* have the same limit distribution. See Subsection 7.1 for a few comments upon this.

3 Embedding Sinai's walk into a diffusion

Just as the usual random walk can be pathwise constructed from a Brownian motion via Skorokhod's embedding, Schumacher [54] (see also Solomon [59]) made the important observation that it is possible to embed Sinai's walk (S_n) into a diffusion process in a carefully chosen random potential. To define this particular random potential, we start from a Sinai's walk (S_n) whose environment $\omega = (\omega_i,\ i \in \mathbb{Z})$ satisfies (1.1)–(1.2). Let

$$V(x) := \sum_{i=1}^{n} \log \frac{1-\omega_i}{\omega_i}, \qquad \text{if } x \in [n, n+1) \text{ for } n \in \mathbb{Z}, \tag{3.1}$$

with the conventions $\sum_{i=1}^{0} x_i := 0$ and $\sum_{i=1}^{n} x_i := x_{-1} + \cdots + x_n$ for negative n. In words, $\{V(x);\ x \in \mathbb{R}\}$ is a step function with $V(0) = 0$, which is flat on each interval $[n, n+1)$, with jumps $V(n) - V(n-) = \log \frac{1-\omega_n}{\omega_n}$ (for $n \in \mathbb{Z}$).

Now that our potential V is chosen, we can define a diffusion process X_V via (2.1), or more rigorously, via (2.6). It is easy to check that $\limsup_{t\to\infty} X_V(t) = \infty$ and $\liminf_{t\to\infty} X_V(t) = -\infty$, $\mathbb{P}$-a.s., so that we can define the sequence $(\mu_n,\ n \geq 0)$ by induction: $\mu_0 := 0$ and

$$\mu_{n+1} := \inf\Big\{ t > \mu_n :\ |X_V(t) - X_V(\mu_n)| = 1 \Big\}, \qquad n = 0, 1, 2, \cdots \tag{3.2}$$

Let $A_x := \int_0^x \mathrm{e}^{V(y)}\, \mathrm{d}y$ be as in (2.2). As we have already pointed out, for any given V, A is a scale function of the diffusion X_V, so that for each $n \geq 0$,

$$\mathbb{P}\Big[X_V(\mu_{n+1}) = i+1 \ \Big|\ X_V(\mu_n) = i;\ (\omega_k)_{k\in\mathbb{Z}} \Big] = \frac{A_i - A_{i-1}}{A_{i+1} - A_{i-1}} = \omega_i \ ,$$

which means that $(X_V(\mu_n),\ n \geq 0)$ is distributed as $(S_n,\ n \geq 0)$. We arrive thus at:

Theorem 3.1 (Schumacher [54]) *Let (S_n) be Sinai's walk whose environment satisfies* (1.1)–(1.2)*, and let V be as in* (3.1)*. If X_V is a diffusion process in potential V, and if $(\mu_n,\ n \geq 0)$ is defined as in* (3.2)*, then for each realization of $(\omega_i,\ i \in \mathbb{Z})$, $(X_V(\mu_n),\ n \geq 0)$ and $(S_n,\ n \geq 0)$ have the same distribution.*

Theorem 3.1 says that $(X_V(\mu_n),\ n \geq 0)$ is Sinai's walk. One may like to get rid of the random clock (μ_n). This can be done via the following property.

Proposition 3.2 *Assume* (1.1)–(1.2)*. Under $\mathbb{P}$, $(\mu_{n+1} - \mu_n,\ n \geq 0)$ is a sequence of iid random variables, each distributed as* $\inf\{t > 0 :\ |B(t)| = 1\}$*, where B is a standard Brownian motion.*

Since μ_1 has finite exponential moments such that $\mathbb{E}[\mu_1] = 1$, the law of large numbers says that $\mathbb{P}$-almost surely $\mu_n \sim n$, $n \to \infty$. For more precision, one can use the iterated logarithm law to see that $\mu_n = n + \mathcal{O}(\sqrt{n \log\log n}\,)$, $\mathbb{P}$-almost surely. (This is the reason for which one writes sometimes $S_n \approx X_V(n)$, even though the true meaning resides in Theorem

3.1). In view of Theorem 3.1 and Proposition 3.2, if we can prove that $\frac{X_V(n)}{(\log n)^2}$ converges in law to b_∞ when the potential V is defined in (3.1), then we will have proved Sinai's theorem stated in (1.3).

The next section is devoted to showing how to handle the process X_V, via the study of the process of its first hitting times. It will then be clear why stochastic calculus is a good tool.

4 Stochastic calculus for Sinai's walk

Let V be càdlàg and locally bounded, and let X_V be a diffusion process in potential V as defined in (2.6). We introduce the process of first hitting times of X_V:

$$H(r) := \inf\{t:\ X_V(t) > r\}, \qquad r > 0. \tag{4.1}$$

Observe that $H(r)$ is related to X_V via the following trivial relationship:

$$H(r) < t \iff \sup_{0\le s\le t} X_V(s) > r. \tag{4.2}$$

We now proceed to get a convenient formulation for $H(r)$. By (2.6), we have $H(r) = T(\sigma(S(r)))$, where $\sigma(r) = \inf\{t > 0:\ B(t) > r\}$ as in (2.4). According to the definition of T in (2.5), this leads to:

$$H(r) = \int_0^{\sigma(A_r)} \mathrm{e}^{-2V(A^{-1}_{B(u)})}\,\mathrm{d}u.$$

Given V, $H(r)$ is an additive functional of the Brownian motion B, and it is natural to introduce $(L(t,x),\ t\ge 0,\ x\in\mathbb{R})$, the local time of B. That is, for all Borel function $f\ge 0$,

$$\int_0^t f(B(s))ds = \int_{\mathbb{R}} f(x)L(t,x)\,\mathrm{d}x.$$

It is known (Trotter [72]) that L admits a jointly continuous version which is again denoted by L. Going back to $H(r)$, we see that

$$\begin{align} H(r) &= \int_{-\infty}^{A_r} \mathrm{e}^{-2V(A_x^{-1})} L(\sigma(A_r),x)\,\mathrm{d}x \tag{4.3}\\ &= \int_{-\infty}^{r} \mathrm{e}^{-V(u)} L(\sigma(A_r),A_u)\,\mathrm{d}u \notag\\ &:= H_+(r) + H_-(r), \tag{4.4}\end{align}$$

where

$$\begin{align} H_+(r) &:= \int_0^r \mathrm{e}^{-V(u)} L(\sigma(A_r),A_u)\,\mathrm{d}u, \tag{4.5}\\ H_-(r) &:= \int_{-\infty}^0 \mathrm{e}^{-V(u)} L(\sigma(A_r),A_u)\,\mathrm{d}u. \tag{4.6}\end{align}$$

At this stage, we can apply the Ray–Knight theorem (Ray [50], Knight [37]), recalled as follows.

Fact 4.1 (Ray–Knight theorem) *Fix $a > 0$. The process $(L(\sigma(a), a-x);\ x \geq 0)$ is continuous inhomogeneous Markovian, starting from 0. It is a 2-dimensional squared Bessel process for $x \in [0,a]$, and becomes a squared Bessel process of dimension 0 for $x \in [a,\infty)$.*

Now define

$$Z_+(s) := \frac{L(\sigma(A_r), A_r - sA_r)}{A_r}, \qquad 0 \leq s \leq 1, \tag{4.7}$$

$$Z_-(t) := \frac{L(\sigma(A_r), -tA_r)}{A_r}, \qquad t \geq 0. \tag{4.8}$$

By scaling, for any bounded Borel functionals f and g,

$$\begin{aligned}
& \mathbb{E}[\, f(Z_+(s),\, s\in[0,1],\ Z_-(t),\, t\in\mathbb{R}_+)\, g(V(x),\, x\in\mathbb{R})\,] \\
= \ & \mathbb{E}[\, g(V(x),\, x\in\mathbb{R})\, \mathbb{E}\{ f(Z_+(s),\, s\in[0,1],\ Z_-(t),\, t\in\mathbb{R}_+) \,|\, V\}\,] \\
= \ & \mathbb{E}[\, g(V(x),\, x\in\mathbb{R})\, \mathbb{E}\{ f(L(\sigma(1), 1-s),\, s\in[0,1],\ L(\sigma(1),-t),\, t\in\mathbb{R}_+)\}\,] \\
= \ & \mathbb{E}[\, g(V(x),\, x\in\mathbb{R})\,]\ \mathbb{E}\{ f(L(\sigma(1), 1-s),\, s\in[0,1],\ L(\sigma(1),-t),\, t\in\mathbb{R}_+)\}\,.
\end{aligned}$$

Therefore, (Z_+, Z_-) is independent of V; moreover, according to Fact 4.1, Z_+ is a 2-dimensional squared Bessel process on $[0,1]$ with $Z_+(0) = 0$, and Z_- is a 0-dimensional squared Bessel process with $Z_-(0) = Z_+(1)$.

By definition, we have

$$\begin{aligned}
H_+(r) &= \int_0^r \mathrm{e}^{-V(u)} A_r\, Z_+\left(\frac{A_r - A_u}{A_r}\right) \mathrm{d}u, \\
H_-(r) &= \int_{-\infty}^0 \mathrm{e}^{-V(u)} A_r\, Z_-\left(\frac{|A_u|}{A_r}\right) \mathrm{d}u.
\end{aligned}$$

We have to study the asymptotic behaviours of $H_+(r)$ and $H_-(r)$, as $r \to \infty$. Unfortunately, quite a lot of technical difficulties are involved, so we only present some **heuristic** ideas. From now on, we assume that the random potential V is defined in (3.1). In this case, V is a partial sum process of iid mean-zero bounded random variables, and can thus be approximated by a Brownian motion: indeed, according to a well known theorem of Komlós–Major–Tusnády [38], there exists a coupling for V and a standard Brownian motion $(W(x),\ x \in \mathbb{R})$ such that as $t \to \infty$,

$$\sup_{|x|\leq t} |V(x) - \sigma W(x)| = \mathcal{O}(\log t), \qquad \text{a.s.,} \tag{4.9}$$

where $\sigma^2 := \mathrm{var}(\log \frac{1-\omega_0}{\omega_0}) \in \mathbb{R}_+^*$ as is defined in (1.2).

We start with estimating $H_+(r)$. Loosely speaking, $Z_+(t)$ behaves somewhat like t for $t \in [0,1]$, so that

$$H_+(r) \approx \int_0^r \mathrm{e}^{-V(u)} (A_r - A_u)\, \mathrm{d}u = \int_0^r \mathrm{d}u \int_u^r \mathrm{d}v\ \mathrm{e}^{V(v)-V(u)}.$$

In the spirit of Laplace's method, if we work in the logarithmic scale, then we would have

$$\log H_+(r) \approx \sup_{0\le u\le v\le r} (V(v) - V(u)). \tag{4.10}$$

Heuristics for $H_-(r)$ now. To make all the processes indexed by $\mathbb{R}_+$, we write $\widetilde{V}(t) := V(-t)$ and $\widetilde{A}_t := |A_{-t}|$ for $t \ge 0$. Note that $t \mapsto \widetilde{V}(t)$ and $t \mapsto \widetilde{A}_t$ have similar properties as, and are independent of, $t \mapsto V(t)$ and $t \mapsto A_t$, $t \ge 0$. We can now write

$$\begin{aligned} H_-(r) &= A_r \int_0^\infty \mathrm{e}^{-\widetilde{V}(t)} Z_-\left(\frac{\widetilde{A}_t}{A_r}\right) \mathrm{d}t \\ &= A_r \int_0^{\zeta(r)} \mathrm{e}^{-\widetilde{V}(t)} Z_-\left(\frac{\widetilde{A}_t}{A_r}\right) \mathrm{d}t, \end{aligned}$$

where

$$\zeta(r) := \inf\left\{ t \ge 0 :\ Z_-\left(\frac{\widetilde{A}_t}{A_r}\right) = 0 \right\}.$$

For $t \in (0, \zeta(r))$, $Z_-(\frac{\widetilde{A}_t}{A_r})$ behaves loosely speaking like a constant (when t is in the neighbourhood of $\zeta(r)$, $Z_-(\frac{\widetilde{A}_t}{A_r})$ should be close to 0, but this essentially does not affect the asymptotic behaviour of $H_-(r)$ in the logarithmic scale), so that

$$\log H_-(r) \approx \log\left(A_r \int_0^{\zeta(r)} \mathrm{e}^{-\widetilde{V}(t)}\, \mathrm{d}t \right).$$

By Laplace's method, this would yield

$$\begin{aligned} \log H_-(r) &\approx \log\left(A_r \sup_{0\le t\le \zeta(r)} \mathrm{e}^{-\widetilde{V}(t)} \right) \\ &\approx \log\left(\exp\left(\sup_{0\le s\le r} V(s) \right) \sup_{0\le t\le\zeta(r)} \mathrm{e}^{-\widetilde{V}(t)} \right) \\ &= \sup_{0\le s\le r} V(s) - \inf_{0\le t\le \zeta(r)} \widetilde{V}(t). \end{aligned} \tag{4.11}$$

To see how $\zeta(r)$ behaves, we note that by Laplace's method, $\log \widetilde{A}_t \approx \log \sup_{s\in[0,t]} \widetilde{V}(s)$ and $\log A_r \approx \log \sup_{s\in[0,r]} V(s)$ (for large t and r), whereas the life time of Z_- is of constant order of magnitude. Accordingly,

$$\zeta(r) \approx \inf \left\{ t \ge 0 : \sup_{s\in[0,t]} \widetilde{V}(s) > \sup_{s\in[0,r]} V(s) \right\}.$$

Plugging this into (4.11) — and in the spirit of Laplace's method again — yields that

$$\log H_-(r) \approx \gamma \left(\sup_{0\le s\le r} V(s) \right), \tag{4.12}$$

where $\gamma(t) := t - \inf_{0\le s\le \inf\{u:\, \widetilde{V}(u)>t\}} \widetilde{V}(s)$. Assembling (4.4), (4.10) and (4.12) gives

$$\log H(r) \approx \max \left\{ \sup_{0\le u\le v\le r} (V(v) - V(u)),\ \gamma \left(\sup_{0\le s\le r} V(s) \right) \right\}. \tag{4.13}$$

Note that the process γ is independent of $\sup_{0\le u\le v\le r}(V(v) - V(u))$ and $\sup_{0\le s\le r} V(s)$. According to (4.9), $(V(t),\ t\ge 0)$ and $(\widetilde{V}(t),\ t\ge 0)$ behave asymptotically like (constant multiples of) independent Brownian motions; hence it is not very hard to analyze the asymptotic properties of the process on the right hand side of (4.13), and to get via (4.2) the corresponding properties of $\sup_{0\le s\le t} X_V(s)$. Since $\sup_{0\le s\le t} X_V(s)$ and $X_V(t)$ have the same upper functions (in the sense of P. Lévy), we arrive at the following integral test. It is formulated only for Sinai's walk, but it has an obvious analogue for any diffusion process whose random potential is asymptotically like a Brownian motion (in the sense of (4.9)). All the technical details, which have been omitted here for the sake of clarity, can be found in [22].

Theorem 4.2 ([22]) *Let* $(S_n,\ n\ge 0)$ *be Sinai's walk satisfying* (1.1) *and* (1.2). *For any sequence of positive non-decreasing numbers* $(a_n)_{n\ge 1}$, *we have*

$$\mathbb{P}\left[S_n > (\log n)^2 a_n \ \text{ i.o.} \right] = \begin{cases} 0 \\ 1 \end{cases} \iff \sum_{n\ge 2} \frac{a_n}{n\log n} \exp\left(-\frac{\pi^2\sigma^2}{8} a_n \right) \begin{cases} < \infty \\ = \infty \end{cases}, \tag{4.14}$$

*where "*i.o.*" stands for "infinitely often", and* $\sigma > 0$ *is the constant in* (1.2). *In particular,*

$$\limsup_{n\to\infty} \frac{S_n}{(\log n)^2 \ \log\log\log n} = \frac{8}{\pi^2\sigma^2}, \qquad \mathbb{P}\text{-a.s.} \tag{4.15}$$

We can replace S_n *by either* $\max_{0\le k\le n} |S_k|$ *or* $\max_{0\le k\le n} S_k$ *in* (4.14)–(4.15).

Theorem 4.2 provides a rather accurate image of the "how large" asymptotic behaviours of Sinai's walk. The corresponding "how small" questions can also be addressed, except that the answers are different depending on whether the maximum is "one-sided" or "two-sided".

Theorem 4.3 ([22]) *Assume* (1.1) *and* (1.2). *If* $(a_n)_{n\ge 1}$ *is positive non-decreasing, then*

$$\mathbb{P}\left[\max_{0\le k\le n}|S_k| \le \frac{(\log n)^2}{a_n} \quad \text{i.o.}\right] = \begin{cases} 0 \\ 1 \end{cases} \iff \sum_{n\ge 2} \frac{\sqrt{a_n}}{n\log n}\exp\left(-\frac{a_n}{\sigma^2}\right)\begin{cases} < \infty \\ = \infty \end{cases}.$$

As a consequence,

$$\liminf_{n\to\infty} \frac{\log\log\log n}{(\log n)^2}\max_{0\le k\le n}|S_k| = \frac{1}{\sigma^2}, \qquad \mathbb{P}\text{-a.s.}$$

Theorem 4.4 ([22]) *Assume* (1.1) *and* (1.2). *If* $(a_n)_{n\ge 1}$ *is positive non-decreasing, then*

$$\mathbb{P}\left[\max_{0\le k\le n} S_k \le \frac{(\log n)^2}{a_n} \quad \text{i.o.}\right] = \begin{cases} 0 \\ 1 \end{cases} \iff \sum_{n} \frac{1}{n\sqrt{a_n}\,\log n}\begin{cases} < \infty \\ = \infty \end{cases}.$$

Therefore, with $\mathbb{P}$*-probability one,*

$$\liminf_{n\to\infty} \frac{(\log\log n)^a}{(\log n)^2}\max_{0\le k\le n} S_k = \begin{cases} 0 & \textit{if } a \le 2, \\ \infty & \textit{otherwise.} \end{cases}$$

Theorems 4.2, 4.3 and 4.4 together characterize all the possible Lévy classes of Sinai's walk. It is however possible to apply a refined version of the argument to obtain other asymptotic properties of (S_n). We cite two examples here. The first property is an almost sure central limit theorem.

Theorem 4.5 (Hu [21]) *Assume* (1.1)–(1.2). *Fix* $c \in (0, \pi^2\sigma^2/8)$. $\mathbb{P}$*-almost surely for any function* $f : \mathbb{R} \to \mathbb{R}$ *which is almost everywhere continuous such that* $\sup_{x\in\mathbb{R}}(|f(x)|\mathrm{e}^{-c|x|}) < \infty$, *we have*

$$\lim_{n\to\infty} \frac{1}{\log\log n}\sum_{k=3}^{n} \frac{1}{k\log k} f\left(\frac{S_k}{(\log k)^2}\right) = \mathbb{E}\left[f\left(\frac{b_\infty}{\sigma^2}\right)\right],$$

where the law of b_∞ *is defined in* (1.4).

The second application of the method is for the favourite sites of Sinai's walk. Let

$$\xi(n,x) := \#\{0 \le k \le n :\ S_k = x\}, \qquad x \in \mathbb{Z}. \tag{4.16}$$

In words, $\xi(n,x)$ records the number of visits of the walk at position x in the first n steps. Define

$$\mathbb{F}(n) := \left\{x \ge 0 :\ \xi(n,x) = \max_{y\ge 0}\xi(n,y)\right\}.$$

Following Erdős and Révész [14], any element of the set $\mathbb{F}(n)$ is called a favourite site (in $\mathbb{Z}_+$) of the walk at time n, since it is a site where the walk has spent the most time. For the usual recurrent random walk (in non-random environment), it was an open problem of Erdős and Révész [14] to know whether $0 \in \mathbb{F}(n)$ for infinitely many n. Somewhat surprisingly, Bass and Griffin [3] obtained a negative answer. They proved that $\mathbb{P}\{0 \in \mathbb{F}(n) \text{ i.o.}\} = 0$.

Let $F(n)$ denote any element of $\mathbb{F}(n)$. Bass and Griffin [3] proved that for the usual recurrent random walk, almost surely,

$$\liminf_{n\to\infty} \frac{(\log n)^a}{n^{1/2}} F(n) = \begin{cases} 0 & \text{if } a < 1, \\ \infty & \text{if } a > 11 \text{ (eleven)}. \end{cases}$$

The exact rate of escape of $F(n)$, however, remains unknown.

We ask the same question for Sinai's walk. A refined version of the method of stochastic calculus shows that for Sinai's walk, the process of favourite sites is again transient with $\mathbb{P}$-probability one, and moreover, one can characterize its rate of escape via an integral test.

Theorem 4.6 ([23]) *Assume* (1.1) *and* (1.2). *For any non-decreasing sequence* $a_n > 1$, *we have*

$$\mathbb{P}\left[F(n) \ge \frac{(\log n)^2}{a_n} \text{ i.o.} \right] = \begin{cases} 0 \\ 1 \end{cases} \iff \sum_n \frac{\log a_n}{n\sqrt{a_n}\,\log n} \begin{cases} = \infty \\ < \infty \end{cases}.$$

In particular, with $\mathbb{P}$*-probability one,*

$$\liminf_{n\to\infty} \frac{(\log\log n)^a}{(\log n)^2} F(n) = \begin{cases} 0 & \textit{if } a \le 2, \\ \infty & \textit{otherwise.} \end{cases}$$

The problem of favourite sites is the **only** problem I am aware of so far, which is solved for RWRE, but which still remains open for the usual random walk.

5 Stochastic calculus for drifted Brownian potential

The main concern of the paper is Sinai's walk on $\mathbb{Z}$. We saw in the previous sections how to obtain precise asymptotic properties of Sinai's walk via stochastic calculus, by means of the Ray–Knight theorem. A basic attitude was that we first apply stochastic calculus to a diffusion process whose random potential is partial sum of iid mean-zero random variables.

This section is devoted to the study of a closely related model, namely, diffusion process whose random potential is a drifted Brownian motion. More precisely, let $V := (V(x),\ x \in \mathbb{R})$ be such that

$$V(x) := W(x) - \frac{\kappa}{2} x, \qquad x \in \mathbb{R}, \tag{5.1}$$

where $\kappa > 0$ is a fixed constant, and W is a standard Brownian motion on $\mathbb{R}$. We consider $(X_V(t),\ t \geq 0)$, a diffusion process in random potential V, as defined in Section 2. The process X_V is a kind of continuous-time analogue of transient RWRE studied in Kesten et al. [35]. We outline the method of stochastic calculus for studying such a process. Since $\kappa > 0$, we have $\lim_{t\to\infty} X_V(t) = \infty$, $\mathbb{P}$-a.s., and it turns out that $X_V(t)$ and $\sup_{0\leq s\leq t} X_V(s)$ have similar asymptotic behaviours (see for example [31] for a rigorous justification). We therefore only have to study $\sup_{0\leq s\leq t} X_V(s)$. Define

$$H(r) := \inf\{t \geq 0 :\ X_V(t) > r\}, \qquad r > 0. \tag{5.2}$$

Again, $H(r)$ and $\sup_{0\leq s\leq t} X_V(s)$ are related to each other via (4.2). In the rest of the section, we briefly describe how to obtain asymptotics of $H(r)$ when $r \to \infty$.

There are at least two different approaches, using either so-called "Kotani's lemma" (Fact 5.1 below), or Lamperti's representation theorem for exponential functionals of Lévy processes. For the sake of clarity, the two approaches are presented in distinct subsections.

Just as for Sinai's and Brox's theorems (Theorems 1.1 and 2.1, respectively) in the recurrent case, it is not clear how to rigorously relate the results described in this section to those obtained by Kesten et al. [35] for transient RWRE.

5.1 Stochastic calculus via Kotani's lemma

A good tool, referred to as "Kotani's lemma" in Kawazu and Tanaka [31], reduces the study of $H(r)$ to that of a linear diffusion process.

Fact 5.1 ("Kotani's lemma"; see Kawazu and Tanaka [31]) *Let* $V = (V(x),\ x \in \mathbb{R})$ *be the potential defined in* (5.1)*, and let* H *be as in* (5.2)*. For all* $\lambda > 0$ *and* $r > 0$*,*

$$\mathbb{E}\{\mathrm{e}^{-\lambda H(r)} \,|\, V\} = \exp\left(-\int_0^r U_\lambda(s)\,\mathrm{d}s\right), \qquad \text{a.s.,} \tag{5.3}$$

where U_λ *is the unique stationary positive solution of the stochastic differential equation*

$$\mathrm{d}U_\lambda(t) = U_\lambda(t)\,\mathrm{d}B(t) + \left(2\lambda + \frac{1-\kappa}{2}\,U_\lambda(t) - U_\lambda^2(t)\right)\mathrm{d}t.$$

By taking expectations on both sides of (5.3), we see that in order to get the distributional asymptotic behaviour of $H(r)$, we only need to study the Laplace transform of $\alpha_\lambda(r) := \int_0^r U_\lambda(s)\,\mathrm{d}s$. This is done by techniques for linear diffusions. Indeed, consider $Y_\lambda(t) :=$

$f_\lambda(U_\lambda(\alpha_\lambda^{-1}(t)))$, $t > 0$, where α_λ^{-1} is the inverse of $r \to \alpha_\lambda(r)$, and f_λ is the deterministic function defined by

$$f_\lambda(x) := (2\lambda)^{-\kappa} \int_{2\lambda}^{x} y^{\kappa-1} \exp\left(2y + \frac{4\lambda}{y}\right) \mathrm{d}y, \qquad x \in \mathbb{R}.$$

If $\mathbb{P}^x$ denotes the probability under which the diffusion process U_λ starts from x, then

$$\mathbb{E}^x \left[\exp\left(-\int_0^r U_\lambda(s)\,\mathrm{d}s\right)\right] = \mathbb{E}^x\left[\mathrm{e}^{-\alpha_\lambda(r)}\right] = \mathbb{E}\left[\mathrm{e}^{-H_Y(0)} \,|\, Y_\lambda(0) = f_\lambda(x)\right], \tag{5.4}$$

where $H_Y(0) := \inf\{t \geq 0 : \ Y_\lambda(t) = 0\}$. Since Y_λ is a linear diffusion process in natural scale and with speed measure $m_\lambda(\mathrm{d}x) = 2(f_\lambda^{-1}(x))^{-2\kappa+1} \exp\{-8\lambda f_\lambda^{-1}(x) - 4/f_\lambda^{-1}(x)\}\,\mathrm{d}x$ (here, f_λ^{-1} stands for the inverse function of f_λ), one-dimensional diffusion theory gives the value of $\mathbb{E}\left[\mathrm{e}^{-H_Y(0)} \,|\, Y_\lambda(0) = f_\lambda(x)\right]$ in terms of the solution of a Sturm–Liouville equation. In view of (5.4), this will yield the Laplace transform of $\int_0^r U_\lambda(s)\,\mathrm{d}s$, and will allow to establish asymptotics of $H(r)$ stated as follows. We refer to Kawazu and Tanaka [31] for full details of the argument.

Theorem 5.2 (Kawazu and Tanaka [31]) *Let V and H be as in Fact* 5.1.

(i) *If $0 < \kappa < 1$, then $H(r)/r^{1/\kappa}$ converges weakly (when $r \to \infty$) to a completely asymmetric stable distribution of index κ.*

(ii) *If $\kappa = 1$, then $H(r)/(r \log r)$ converges in probability to 4.*

(iii) *If $\kappa > 1$, then $H(r)/r$ converges $\mathbb{P}$-almost surely to $4/(\kappa - 1)$.*

It is possible (see Kawazu and Tanaka [32]) to refine the method to obtain the rates of convergence in Parts (ii) and (iii) of Theorem 5.2, thus providing a complete analogue of what Kesten et al. [35] did for transient RWRE. We observe that although the theorem is formulated for $H(r)$, the first hitting times of X_V, a simple argument allows to obtain the corresponding results for X_V itself, just as in the case of transient RWRE. This method works also for the case $\kappa = 0$: it yields weak convergence for $\frac{H(r)}{\exp(\sqrt{r})}$ (or equivalently, for $\frac{\sup_{s\in[0,t]} X_V(s)}{(\log t)^2}$), but unlike the transient case, this does not allow to recover Brox's theorem (see Theorem 2.1) for $\frac{X_V(t)}{(\log t)^2}$.

It is worth pointing out that the approach via Kotani's lemma can be exploited to prove various asymptotic results. For example, Taleb [67] established large deviation results for X_V when V is defined in (5.1) including the case $\kappa = 0$ (the Brox case), in both quenched and annealed settings (and thanks to the power of stochastic calculus, the results obtained in [67] contain slightly more explicit constants than the corresponding known results for RWRE).

Recently, Kawazu et al. [28] obtained weak convergence results when the random potential is a one-sided Brownian motion, i.e., $V(x) = W(x)\,\mathbf{1}_{\mathbb{R}_+}(x)$, $x \in \mathbb{R}$, where W is a Brownian motion (on $\mathbb{R}_+$).

5.2 Stochastic calculus via Lamperti's representation

Recall that we are interested in a diffusion process X_V whose random potential V is a drifted Brownian motion, defined by (5.1). While Kotani's lemma provides information for X_V (via its first hitting times) under the conditional probability given the potential V, the second approach we present here, which is based on Lamperti's representation theorem for exponential functionals of Lévy processes, applies under the abosolute probability $\mathbb{P}$.

The starting point is the same as in the study of Sinai's walk. By (4.3),

$$H(r) = \left(\int_{-\infty}^{0} + \int_{0}^{A_r} \right) \mathrm{e}^{-2V(A_x^{-1})} L(\sigma(A_r), x)\, \mathrm{d}x,$$

where as before, $A_x = \int_0^x \mathrm{e}^{V(y)}\,\mathrm{d}y$, $x \in \mathbb{R}$, $L(\,\cdot\,,\,\cdot\,)$ is the local time of a Brownian motion B independent of V, and $\sigma(r) := \inf\{t \geq 0 :\; B(t) > r\}$. We do not have to worry about the first term $\int_{-\infty}^{0} \cdots \,\mathrm{d}x$ on the right hand side: it is bounded by $\int_{-\infty}^{0} \mathrm{e}^{-V(u)}\,\mathrm{d}u \times \sup_{x\leq 0} L(\sigma(A_\infty), x)$ which is $\mathbb{P}$-almost surely finite (noting that $\int_{-\infty}^{0} \mathrm{e}^{-V(u)}\,\mathrm{d}u < \infty$ and $A_\infty < \infty$, a.s.). All we have to care for now is the second term, namely,

$$H_+(r) := \int_0^{A_r} \mathrm{e}^{-2V(A_x^{-1})} L(\sigma(A_r), x)\, \mathrm{d}x. \tag{5.5}$$

It is time to apply Lamperti's general theorem for exponential functionals of Lévy processes. The theorem bears a particularly simple form in our setting.

Fact 5.3 (Lamperti [41]) *Let $V(x) := W(x) - \frac{\kappa}{2} x$ be as in (5.1). There exists a Bessel process $\varrho = (\varrho(t),\ t \geq 0)$ of dimension $(2 - 2\kappa)$, starting from $\varrho(0) = 2$, such that*

$$\mathrm{e}^{V(t)/2} = \frac{1}{2}\,\varrho(A_t), \qquad t \geq 0, \tag{5.6}$$

where $A_t := \int_0^t \mathrm{e}^{V(y)}\,\mathrm{d}y$ as before.

Let ϱ be the Bessel process of dimension $(2 - 2\kappa)$ and starting from 2 as in Fact 5.3 (for an account of general properties of Bessel processes, see Chapter XI of Revuz and Yor [53]). According to Williams [74]'s time reversal theorem for Bessel processes, if we write

$$R(t) := \varrho(A_\infty - t), \qquad 0 \leq t \leq A_\infty,$$

then R is a transient Bessel process of dimension $(2+2\kappa)$, starting from 0. Moreover, A_∞ is the last exit time of R from 2, that is, $A_\infty = \sup\{t \ge 0 : \; R(t) = 2\}$. Therefore, (5.6) can be rewritten as

$$\mathrm{e}^{V(A_x^{-1})/2} = \frac{1}{2}\, R(A_\infty - x), \qquad 0 \le x < A_\infty.$$

Plugging this into (5.5) yields that

$$\begin{aligned} H_+(r) &= 16 \int_0^{A_r} \frac{1}{R^4(A_\infty - x)} L(\sigma(A_r), x)\, \mathrm{d}x \\ &= 16 A_r \int_0^1 \frac{1}{R^4(A_\infty - A_r + sA_r)} L(\sigma(A_r), A_r - sA_r)\, \mathrm{d}s, \end{aligned}$$

the second identity following from a change of variables $x = (1-s)A_r$. Let

$$Z_+(s) := \frac{L(\sigma(A_r), A_r - sA_r)}{A_r}, \qquad 0 \le s \le 1,$$

which according to the Ray–Knight theorem (Fact 4.1) is a two-dimensional squared Bessel process, starting from 0, and which is independent of V (thus of A and R). This gives us

$$H_+(r) = 16 A_r^2 \int_0^1 \frac{Z_+(s)}{R^4(A_\infty - A_r + sA_r)}\, \mathrm{d}s.$$

Without loss of generality, we can assume our probability space to be so rich that Z_+ is defined on $\mathbb{R}_+$ (instead of on $[0,1]$ only). Let

$$Z(t) := A_r\, Z_+\left(\frac{t}{A_r}\right), \qquad t \ge 0,$$

which by scaling is also a two-dimensional squared Bessel process starting from 0, independent of the processes A and R. By writing $D_r := A_\infty - A_r$, we arrive at:

$$\begin{aligned} H_+(r) &= 16 \int_0^{A_r} \frac{Z(t)}{R^4(D_r + t)}\, \mathrm{d}t \\ &= 16 \left(\int_0^1 + \int_1^{A_\infty - D_r} \right) \frac{Z(t)}{R^4(D_r + t)}\, \mathrm{d}t \\ &= 16\, I(D_r) + \text{ a bounded term,} \end{aligned}$$

where for all $\delta > 0$,

$$I(\delta) := \int_0^1 \frac{Z(t)}{R^4(\delta + t)}\, \mathrm{d}t = \delta \int_0^{1/\delta} \frac{Z(v\delta)}{R^4(\delta(1+v))}\, \mathrm{d}t.$$

We know the asymptotic behaviour of D_r when $r \to \infty$: by the iterated logarithm law, $\log D_r = -\frac{\kappa}{2} r + \mathcal{O}(\sqrt{r \log\log r}\,)$, $\mathbb{P}$-almost surely. So all we need to study is the asymptotics of $I(\delta)$ when $\delta \to 0$.

By scaling, for each fixed $\delta > 0$, $I(\delta)$ is distributed as $\int_0^{1/\delta} \frac{Z(v)}{R^4(1+v)}\,\mathrm{d}v$. When $\delta \to 0$, this random variable behaves essentially like $\int_0^{1/\delta} \frac{Z(v)}{R^4(v)}\,\mathrm{d}v$. It is now time to apply a theorem of Warren and Yor [73] which states that

$$\int_0^t \frac{Z(v)}{R^4(v)}\,\mathrm{d}v = \int_0^{\Lambda(t)} \frac{Y(s)}{(1-Y(s))^2}\,\mathrm{d}s, \qquad t \ge 0,$$

where $\Lambda(t) := \int_0^t \frac{1}{Z(s)+R^2(s)}\,\mathrm{d}s$, and Y is a Jacobi process (starting from 0) of dimension $(2, 2+2\kappa)$, in the sense that it is a $[0,1]$-valued diffusion process with generator $(\mathcal{G}f)(x) = 2(x-x^2)f''(x) + (2-(4+2\kappa)x)f'(x)$. (For more details about the Jacobi process, we refer to Chap. 15 of Karlin and Taylor [27].)

By the additivity of (squared) Bessel processes, $s \mapsto Z(s) + R^2(s)$ is a squared Bessel process of dimension $(4+2\kappa)$, so an application of the ergodic theorem on path space immediately gives that when $t \to \infty$,

$$\frac{\Lambda(t)}{\log t} \to \frac{1}{2+2\kappa}, \qquad \text{a.s.}$$

There is a general recipe for studying the asymptotics of $\int_0^u f(Y(s))\,\mathrm{d}s$ when u is large: (i) transferring Y into a diffusion in its natural scale, (ii) writing it as a Brownian time change, (iii) applying usual techniques for additive functionals of Brownian motion (via occupation time formula and local time). In our case (i.e., $f(x) = \frac{x}{(1-x)^2}$), we also use an important result of Biane and Yor [4] which relates Brownian local times to stable distributions, and this is how we recover the stable limits in Theorem 5.2 (as well as in the rates of convergence in Parts (ii) and (iii) of Theorem 5.2). For technical details, see Hu et al. [24].

An interesting feature of this method is that it also yields explicit values of all the underlying constants in Theorem 5.2 and in its rates of convergence. On the other hand, the method fails in the case $\kappa = 0$.

The method deals with convergence of $H(r)$ (and $X_V(t)$). It is possible to refine the argument to establish some very delicate probability estimates for deviations of $H(r)$, see Taleb [68].

6 Related topics

This paper focuses on the study of Sinai's walk (by means of stochastic calculus). Let us briefly mention some related topics and a few references for these topics.

The references of Kozlov [40] and Molchanov [49] contain detailed analysis of results, prior to 1985 and to 1992 respectively, concerning various models of RWRE.

Among the known results of nearest-neighbour RWRE on $\mathbb{Z}$, the aspect of large deviations has probably received the most recent attention (and has possibly been the best understood), initiated by Greven and den Hollander [19]. For updated developments, we refer to Comets et al. [10] and the survey paper by Gantert and Zeitouni [16].

A natural extension of Sinai's walk is the model of RWRE which does not necessarily execute nearest-neighbour jumps. Key [36] established some recurrence/transience criteria for such RWRE. See Andjel [1], Lëtchikov [43] [44] [45], Derriennic [12] and Brémont [7] for various asymptotic results of such random walks.

Another interesting extension is to study RWRE on a strip. The walk has some higher-dimensional flavour, though it is more convenient to see it as a one-dimensional non-nearest-neighbour walk. For recurrence/transience criteria for such a walk, see Bolthausen and Goldsheid [5], Keane and Rolles [33].

It is very hard to study RWRE in higher dimensions. Here are a few references: Kalikow [26], Lawler [42], Astrauskas [2], Bricmont and Kupiainen [8]. Some remarkable recent progresses on multi-dimensional RWRE can be found in Zerner [76] [77], Sznitman [60] [61] [62] [63], Sznitman and Zerner [66], Zerner and Merkl [78], Shen [56], Bolthausen and Sznitman [6]. We refer to Sznitman [64] [65] for surveys on this exciting topic. Let us also mention a beautiful multi-dimensional extension of Sinai's theorem by Durrett [13] in the recurrent case.

In order to get a better understanding of RWRE in higher dimensions, it is interesting to study random walk on a Galton–Watson tree. Loosely speaking, an infinite Galton–Watson tree is an infinite dimensional lattice network, so that results for random walk on a Galton–Watson tree can be thought of as for $\mathbb{Z}^\infty$-valued RWRE. See Dembo et al. [11] for the latest developments and Gantert [15] for a survey on this.

A few words about the continuous-time model of diffusion process in a random potential. We saw in Section 3 the reason why it can be considered as a companion of RWRE. There are a number of refined versions of Brox's result (Theorem 2.1), called localization results, which have been obtained by Kawazu et al. [30], Hu [20] (for a survey of results prior to 1995, see Tanaka [70]), motivated by the article of Golosov [17] which dealt with the case of RWRE. Let us also mention some multi-dimensional extensions of the continuous-time model, accomplished by Tanaka [69], Mathieu [47] [48]. Unfortunately, the methods of stochastic calculus described in this paper fail to work in the multi-dimensional setting.

The Saint Flour lecture notes by Zeitouni [75] provide an authoritative overview of RWRE, and cover some other interesting aspects (for example, the aging property) of Sinai's walk which are not discussed in the present paper.

7 Open questions for Sinai's walk

This final section consists of some questions about Sinai's walk. All these questions remain unanswered so far, to the best of my knowledge. As before, we write $(S_n,\ n \geq 0)$ to denote Sinai's walk on $\mathbb{Z}$, and assume that conditions (1.1)–(1.2) are fulfilled.

7.1 Invariance principle?

Theorem 3.1 confirms that Sinai's walk enjoys the same asymptotic properties as a diffusion process in a particular random potential (defined in (3.1)). This particular potential is the partial sum of certain iid mean-zero bounded random variables, and thus can be approximated via (4.9) by (a constant multiple of) Brownian motion, with satisfying precision. However, two diffusion processes whose respective random potentials are close to each other do not necessarily possess similar asymptotic properties. The question is open about whether (in some probability space) there are a Sinai's walk (S_n) and a diffusion process $(X_V(t))$ in a Brownian potential such that S_n is reasonably close to $X_V(n)$. More precisely, we ask:

Question 7.1 *Does there exist a coupling for Sinai's walk* (S_n) *and diffusion process* $(X_V(t))$ *in a Brownian potential, such that when* n *goes to infinity,*

$$\text{(7.1)} \qquad \frac{\sigma^2 S_n - X_V(n)}{(\log n)^2} \quad \textit{converges to } 0 \textit{ in probability ?}$$

It is proved by Seignourel [55] that the answer to (7.1) is in the affirmative if one is allowed to change the random environment at each step of the walk.

7.2 How recurrent is Sinai's walk?

Sinai's walk is recurrent. A quantity to measure "how recurrent a process is" is the local time $\xi(n,x) := \#\{0 \leq k \leq n : \ S_k = x\}$, already introduced in (4.16). Let

$$\text{(7.2)} \qquad \xi^*(n) := \max_{x \in \mathbb{Z}} \xi(n,x)$$

which denotes the maximal number of visits the walk can pay to any site in the first n steps. Quite remarkably, Révész [51] [52] proved the following:

$$\text{(7.3)} \qquad \limsup_{n \to \infty} \frac{\xi^*(n)}{n} > 0, \qquad \mathbb{P}\text{-a.s.}$$

Loosely speaking, with $\mathbb{P}$-probability one, Sinai's walk spends a positive proportion of time in a special site. This is in complete contrast with the case of usual random walk.

Property (7.3) was proved by Révész [51] [52] when the random environment takes only two values. The latter condition, however, can be removed, see [57].

Concerning other aspects of the asymptotic behaviours of $\xi^*(n)$, we have two conjectures.

Conjecture 7.2 *When n goes to infinity, $\xi^*(n)/n$ converges weakly.*

Conjecture 7.3 (Révész [52]) *We have,*

$$\liminf_{n\to\infty} \frac{\xi^*(n)}{n/\log\log n} = 0, \qquad \mathbb{P}\text{-a.s.} \tag{7.4}$$

$$\liminf_{n\to\infty} \frac{\xi^*(n)}{n/(\log\log n)^3} = \infty, \qquad \mathbb{P}\text{-a.s.} \tag{7.5}$$

Conjecture 7.3 is borrowed from Chapter 29 of Révész [52]. We make the trivial observation that (7.5) is equivalent to: $\lim_{n\to\infty} \frac{\xi^*(n)}{n/(\log\log n)^3} = \infty$, $\mathbb{P}$-a.s.

7.3 Intersection local time

Let $\xi(n,x)$ be as in (4.16). The intersection local time associated to (S_n) is

$$I(n) := \sum_{x\in\mathbb{Z}} \xi^2(n,x) = \sum_{i=0}^{n}\sum_{j=0}^{n} \mathbf{1}_{\{S_i=S_j\}}.$$

In words, it records the frequency with which Sinai's walk intersects with its own trajectories. Clearly, $I(n)$ is closely related to $\xi^*(n)$ (see (7.2)). Let us mention that $I(n)$ also appears naturally in the model, studied by Lyons and Schramm [46], of random walk (say (Y_n)) in random environment with a Gaussian random scenery: indeed, for each n, Y_n is distributed as $\sqrt{I(n)}\,\mathcal{N}$, where $\mathcal{N}$ denotes a standard Gaussian random variable independent of $I(n)$.

Conjecture 7.2 has a companion for intersection local time, stated as follows.

Conjecture 7.4 *When n goes to infinity, $I(n)/n^2$ converges weakly.*

7.4 Towards higher dimensions: a naive attempt

As we said in Section 6, it is very delicate to study higher-dimensional RWRE. Little is known so far about recurrent RWRE in higher dimensions. For example, Kalikow [26] considered an innocent-looking example of two-dimensional RWRE, and asked whether this RWRE is recurrent (Problem 4 in [26]). The problem remains unsolved after twenty years.

Loosely speaking, the first component of the $\mathbb{Z}^2$-valued RWRE considered by Kalikow [26] somewhat looks like (but is much more complicated than) Sinai's walk, whereas the second basically behaves like a usual simple symmetric random walk. If one were able to estimate the quenched probability that Kalikow's random walk hits 0 at time n, then there might be some hope to prove the (conjectured) recurrence of the walk. As a possible first step to do this, we ask the following question:

Question 7.5 *For Sinai's walk* (S_n)*, find an asymptotic estimate of* $P^\omega(S_n = 0)$*, where* P^ω *denotes the* (*quenched*) *probability given environment.*

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